Math 314 Problem Set #6 Due week of Feb. 25, 2019, in lab.

In Hoffman and Kunze, read Chapter 3, Section 4.

1. From Hoffman and Kunze, Chapter 3, do these problems: pages 83-84, #3, 5; pages 85-86, #7; pages 95-97, # 1(a), 5, 8.

2. Suppose that A and B are similar $n \times n$ matrices (i.e. $B = C^{-1}AC$ for some invertible $n \times n$ matrix C).

a) Show that A^n and B^n are similar.

b) Show that if $f(x)$ is a polynomial, then $f(A)$ and $f(B)$ are similar.

c) Show that if $f(x)$ is a polynomial such that $f(A) = 0$, then $f(B) = 0$.

3. Let $T : V \to V$ be a linear transformation, where V is a finite dimensional vector space. Let T^n denote $T \circ T \circ \cdots \circ T$ (with n factors of T) and let $r_n = \text{rank}(T_n)$. Prove that $r_{n+1} \leq r_n$ for all n, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant. (Hint: See Problem Set 5, #3.)

4. Let n be a non-negative integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $\mathcal{P}_n$ be the vector space of real polynomials $f(x)$ of degree $\leq n$. Define $F : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.

a) Show that F is an isomorphism. [Hint: $\dim(\mathcal{P}_n) =$? kernel$(F) =$?]

b) *Explicitly* find $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ (where $e_1, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{R}^{n+1}$) in the case $n = 2$ and $\alpha_j = j$ for $j = 1, 2, 3$. [Hint: Where does $(x - a)(x - b)$ vanish?]

c) In general, deduce that $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ form a basis of $\mathcal{P}_n$. In the special case done in (b), express the polynomial x as a linear combination of these basis elements.

5. Let $\mathcal{P}_n$ be the vector space of real polynomials $f(x)$ of degree at most n. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $f(x) \mapsto f(x) - (x - 1)f'(x)$, where $f'(x)$ is the derivative of $f(x)$.

a) Show that T is a linear transformation.

b) Find the kernel of T. (Hint: Differential equations.)

c) Find the matrix of T with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$.

d) Find the matrix of T with respect to the basis $\{f_1, f_2, f_3\}$ of $\mathcal{P}_2$, where $f_i = F^{-1}(e_i)$ as in problem 4.