Math 314 Problem Set #3 Due week of Feb. 4, 2019, in lab.

In Hoffman and Kunze, Chapter 2, review Section 3 and read Section 4.

1. This problem concerns vectors in $\mathbb{R}^3$.

a) Express $u = (7, -1, 5)$ as a linear combination of $v = (3, -1, 2)$ and $w = (1, 1, 1)$.

b) Show that $z = (0, 0, 1)$ is *not* a linear combination of $v$ and $w$.

c) Can $z$ be expressed as a linear combination of the vectors $u, v, w$? (Hint: You don't need to do any computations for this part, once you've done the previous two parts. Note that you are not asked to find a linear combination, but just to determine if one exists.)

d) Can the vector $(\pi, e, \sqrt{2})$ be expressed as a linear combination of the vectors $v, w, z$? (Same hint as in the previous part.)

2. a) Find all real numbers $\alpha$ such that the vectors $(\alpha, 1, 0)$, $(1, \alpha, 1)$, $(0, 1, \alpha)$ are linearly independent in $\mathbb{R}^3$. (Hint: A linear dependency relation gives a homogeneous system of three equations in three unknowns, if one writes out the coordinates. When does such a system have a non-trivial solution?)

b) Does your answer change if instead you work over the rational vector space $\mathbb{Q}^3$ or over the complex vector space $\mathbb{C}^3$ (and allow $\alpha$ to be in $\mathbb{Q}$ or $\mathbb{C}$ respectively)?

3. Show that if a vector space $V$ is not finite dimensional, then there is an infinite linearly independent subset $\{v_1, v_2, v_3, \ldots\}$ of $V$. (Hint: Does $V$ have a finite spanning set? Use your answer to find appropriate vectors $v_1, v_2, v_3, \ldots$ successively.)

4. Let $v_0, v_1, \ldots, v_n$ be vectors in a vector space $V$, such that $\{v_1, \ldots, v_n\}$ is a basis of $V$, and such that $v_0$ is not in the span of $v_1, \ldots, v_{n-1}$. Prove that $\{v_0, v_1, \ldots, v_{n-1}\}$ is a basis of $V$.

5. Let $v_1, v_2, v_3, v_4$ be vectors in $\mathbb{R}^4$. Suppose that the set $S = \{v_1, v_2\}$ is linearly independent, and that the set $S' = \{v_3, v_4\}$ is also linearly independent. Suppose also that $\mathrm{span}(S) \cap \mathrm{span}(S') = O$. Prove that $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$.