Math 314 Problem Set #2 Due week of Jan. 28, 2019, in lab.

Read Hoffman and Kunze, Chapter 2, Sections 2 and 3.

1. a) Suppose that F is a field, and $a, b \in F$. Prove that if $ab = 0$, then either $a = 0$ or $b = 0$. (Hint: Multiply on the left by a suitable element.)

b) Suppose that V is a vector space over the field of scalars F, and that $c \in F$. Prove that $cO = O$. (Hint: Use the same strategy as was used to prove that $0v = O$ for $v \in V$.)

c) Suppose that V is a vector space over the field of scalars F, and that $c \in F$ and $v \in V$. Prove that if $cv = O$, then either $c = 0$ or $v = O$. (Hint: See parts (a) and (b).)

2. For each of the following, determine whether W is a subspace of the vector space V over the real numbers. Explain your assertions. For each W that is *not* a subspace, explicitly describe the span of W in V.

a) $V = \mathbb{R}^2$, $W = \{(x, y) \in V \mid xy \geq 0\}$.

b) $V = \mathbb{R}^2$, $W = \{(x, y) \in V \mid x^2 - 2xy + y^2 = 0\}$.

c) $V = \mathbb{R}^4$, $W = \{(x, y, z, t) \in V \mid x + y + 2z = 0,\ y + z + 2t = 0\}$.

d) V is the set of solutions to the differential equation $f''(x) + f'(x) - 2f(x) = 0$, and W is the set of solutions to the differential equation $f'(x) = f(x)$.

e) V is the set of differential real-valued functions on $\mathbb{R}$, c is a fixed real number, and $W = W_c$ is the set of $f \in V$ such that $f(1) = 0$ and $f'(2) = c$. (Your answer should depend on the value of c.)

f) V is the set of convergent sequences $a_1, a_2, a_3, \ldots$ of real numbers, and W is the set of sequences $a_1, a_2, a_3, \ldots$ of real numbers such that the series $\sum_{i=1}^{\infty} a_i$ converges.

3. Prove that the functions e^x, e^{2x}, e^{3x} are linearly independent in the real vector space V consisting of all the real-valued differentiable functions on $\mathbb{R}$. (Hint: If not, then take a linear relation and differentiate twice. What three relations do you get? What happens if you set $x = 0$ in those relations?)

4. Let S be a linearly independent subset of a vector space V. Let $v \in V$ with $v \notin S$, and let $S' = S \cup \{v\}$. Prove the following: S' is linearly independent if and only if v is not in the span of S.

5. Let $A = (a_{ij})$ be an $n \times n$ lower triangular matrix; i.e., $a_{ij} = 0$ for $i < j$. Suppose that B is an $n \times n$ matrix such that AB is the $n \times n$ identity matrix. Prove that

(i) B is also lower triangular.

(ii) the diagonal elements of A are all non-zero.

(Hint: First do this with $n = 2$, and then do $n = 3$, in order to see the pattern.)

6. Let V be a vector space. Suppose that W_1, W_2 are subspaces of V, with the property that neither is contained in the other. Prove that the union $W_1 \cup W_2$ is *not* a subspace of V. (Hint: Take an element $w_1 \in W_1$ that is not in W_2, and vice versa. What happens if you add those two elements? Where does the sum lie?)