Chapter 5

Determinants

Recall: 2×2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$\det A = 0 \iff A$ is singular

(not invertible)

(T_A has non-0 kernel)

$\det A \neq 0 \iff A$ invertible

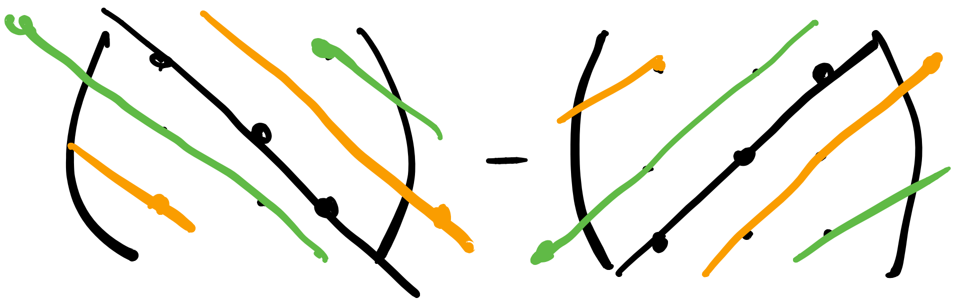
(T_A inv)

$\iff$ rows are lin ind

$\iff$ cols - - - - -

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

3x3 det



Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det = (2+0+6) - (0+0+8) = 8-8=0$$

So singular.

Rows lin dep & cols.

$$2R_1 - R_2 - 3R_3 = 0$$

$$C_1 - 2C_2 + C_3 = 0$$

Or: evaluate det by expanding along a row or col & using minors/cofactors.

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ $\quad A = (a_{ij}) \quad 3 \times 3$

i, j minor (2×2) M_{ij}:

delete i^{th} row

" j^{th} col.

Cofactor $\pm M_{ij}$

$\pm = (-1)^{i+j}$

To expand along i^{th} row $(n \times n)$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} (\det M_{ij})$$

$$= \sum_{j=1}^{n} a_{ij} \underbrace{\left((-1)^{i+j} (\det M_{ij})\right)}_{\text{cofactor}}$$

Similarly: for j^{th} column.

Agrees with other method for 3×3.

Using minors: easier if have 0's

– expand along a row or col with 0's.

4×4: Expand along row or col

4 terms, each is a 3×3 det.

5×5: — — ——————

5 terms, - — —— 4×4 det.

"Generalized Diag's" to find det for $n \times n$:

Ex 4×4

$$\begin{array}{cccc} 3 & 1 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \leftarrow \text{permutation of } 1234$$

$$\begin{pmatrix} \cdot & \odot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \odot \\ \odot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \odot & \cdot \end{pmatrix}$$

$4!$ choices

One entry from each row & each col.

$n \times n$: $n!$ of these.

Take products of terms; add up, with $+$'s & $-$'s.

$(n > 1)$

which?

Sign of a permutation

Sign of a permutation:

Every perm. can be obtained by a sequence of transpositions.

Interchange two entries

1 2 3 4

2 1 3 4

3 1 2 4

3 1 4 2

3 steps: use $(-1)^3 = -1$

odd — odd perm.

If m steps: $\to$ m even: $(-1)^m = 1$ even perm

$\to$ m odd: $(-1)^m = -1$ odd perm

Use this sign.

This is well defined: # of steps always has the same parity

$$\det\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (-1)^3 = -1$$

3 1 4 2, perm: 3 steps.
odd. Get -1.

$\frac{1}{2}$ are odd $\qquad n \geq 2$

$\frac{1}{2}$ are even

$$\{\text{odd}\} \xleftrightarrow{\text{bijection}} \{\text{even}\}$$

$\longrightarrow$

follow by $1 \leftrightarrow 2$

$\longleftarrow$

$\frac{n!}{2} \qquad\qquad \frac{n!}{2}$

Key properties of det:

Get same result whether expand along a row, or a column, or use "gen'lized diag's".

A $n \times n$

$\det A = 0 \iff A$ singular (not inv)

$\iff \ker T_A \neq 0$ (T singular)

$\iff$ corresp homog sys ($AX = 0$) has non-trivial soln

$\iff \exists$ inhomog sys ($AX = B$) with >1 soln

$\iff \operatorname{im} T_A$ is not all of F^n ($\operatorname{rk} T_A < n$)

$\iff$ col's of A don't span F^n

$\iff$ col rk $(A) < n$ (col's are dependent)

$\iff$ row rk $(A) < n$ (rows dep)

$\iff$ rows don't span F^n

$\det(I) = 1$

If we fix all the rows except one, then the det is linear in that row:

$$\det\begin{pmatrix} - & \cdots & - \\ a_1 & \cdots & a_n \\ - & \cdots & - \end{pmatrix} + \det\begin{pmatrix} - & \cdots & - \\ b_1 & \cdots & b_n \\ - & \cdots & - \end{pmatrix}$$

$$= \det\begin{pmatrix} - & \cdots & - \\ a_1+b_1 & \cdots & a_n+b_n \\ - & \cdots & - \end{pmatrix}$$

$$c\det\begin{pmatrix} - & \cdots & - \\ a_1 & \cdots & a_n \\ - & \cdots & - \end{pmatrix} = \det\begin{pmatrix} - & \cdots & - \\ ca_1 & \cdots & ca_n \\ - & \cdots & - \end{pmatrix}$$

det is a multilinear function of the rows.

$$\det A = \det A^t$$

$\Rightarrow$ det is a multilinear function of the cols.

or: col's.

det is an <u>alternating</u> fn of the rows:

If two rows are $=$, $\det = 0$.

(b/c rows are dependent,
mx is singular)

Consequence:
If interchange two rows,
then det is replaced by $-$det.

Reason: WTS

$$\det\begin{pmatrix} - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \\ b_1 & \cdots & b_n \\ - & - & - \end{pmatrix} + \det\begin{pmatrix} - & - & - \\ b_1 & \cdots & b_n \\ - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \end{pmatrix} \overset{\text{want}}{=} 0$$

$$\det\begin{pmatrix} - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \end{pmatrix} + \det\begin{pmatrix} - & - & - \\ b_1 & \cdots & b_n \\ - & - & - \\ b_1 & \cdots & b_n \\ - & - & - \end{pmatrix} = 0$$

Add this to LHS of 1st eq'n.

Get

$$\det\begin{pmatrix} - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \\ a_1+b_1 & \cdots & a_n+b_n \\ - & - & - \end{pmatrix} + \det\begin{pmatrix} - & - & - \\ b_1 & \cdots & b_n \\ - & - & - \\ a_1+b_1 & \cdots & a_n+b_n \\ - & - & - \end{pmatrix}$$

WTS this is 0.

$$= \det\begin{pmatrix} - & - & - \\ a_1+b_1 & \cdots & a_n+b_n \\ - & - & - \\ a_1+b_1 & \cdots & a_n+b_n \\ - & - & - \end{pmatrix} \overset{\text{2 rows are =}}{=} 0$$

$\det A = \det A^t$

$\Rightarrow$ If two col's are $=$, $\det = 0$

$\Rightarrow$ If interchange two cols, $\det \leadsto -\det$

Other properties:

$$\det(AB) = (\det A)(\det B)$$

$$\Rightarrow \det(AB) = \det(BA)$$

b/c each $= (\det A)(\det B)$

$\Rightarrow$ If A is invertible

then $\det(A^{-1}) = (\det A)^{-1}$

Reason:

$$(\det A)(\det A^{-1}) = \det(AA^{-1})$$
$$= \det I = 1$$

Another consequence:

If A is similar to B

(i.e. $\exists C$ invertible s.t. $A = C^{-1} B C$)

then $\det A = \det B$

Reason:

$$\det A = \det(C^{-1})(\det B)(\det C)$$

$$= (\det C)^{-1}(\det B)(\det C)$$

$$= \det B.$$

Recall: $T: V \to V$ lin tr.

Choose a basis: $T \leftrightarrow A$

" another basis: $T \leftrightarrow B$

A, B similar (via change of basis mx)

Can refer to $\det T$: $\det A$ for any $A \leftrightarrow T$

Some bases lead to mx's where det's are easier to compute.

$\det(A+B)$ cannot be expressed in terms of $\det A$ and $\det B$.

Ex

$$\begin{array}{ccccc} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & & 0 & & 0 \end{array}$$

$$\begin{array}{ccccc} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & & 0 & & 1 \end{array}$$

Opposite for traces:

$$\operatorname{tr}(a_{ij}) = a_{11} + \cdots + a_{nn}$$

$$\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$$

$$\det 0 = 0$$

det (diagonal mx) = product of diag entries.

$$\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

det (Δ'r mx) = product of diag entries.

upper Δ'r : $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

lower Δ'r : $\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}$

Relationship between $\det A$ and $\det A^{-1}$

&

Formula for A^{-1} in terms of det?

If $\det A = 0$: A^{-1} does not exist

If $\det A \neq 0$: formulas

$$A^{-1} = \frac{1}{\det A}\left(\quad \right)$$

$n \times n$ matrix whose (i,j) entry is the cofactor A_{ji}

$= (-1)^{i+j}\det(M_{j,i})$

named "cofactor matrix"

"classical adjoint"

"adjunct"

Ex. $n=2$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{as usual}$$

Can use determinants to find a formula for soln to non-singular system of n eqns in n vbls:

$\underset{n\times n}{A}\,\underset{n\times 1}{X} = \underset{n\times 1}{B}$ $\qquad \det A \neq 0$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Cramer's Rule:

$$x_j = \frac{\det B_j}{\det A}$$

B_j = mx obtained by replacing j^{th} col of A by col. vector B.

Ex $5x_1 + 3x_2 = 11$
$3x_1 + 2x_2 = 12$ $\Big\}$ $AX = B$

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 11 \\ 12 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 11 & 3 \\ 12 & 2 \end{pmatrix} \quad B_2 = \begin{pmatrix} 5 & 11 \\ 3 & 12 \end{pmatrix}$$

$\det A = 5 \cdot 2 - 3 \cdot 3 = 1$ $\qquad x_1 = \frac{-14}{1} = -14$

$\det B_1 = 22 - 36 = -14$ $\qquad x_2 = \frac{27}{1} = 27$

$\det B_2 = 60 - 33 = 27$

For a large system $AX = B$, row reduction is faster. $(A|B)$

For inv. of a large mx: row red is faster $(A|I) \rightsquigarrow (I|A^{-1})$

For finding det of large mx: row red. is faster.

How to use row red. to find $\det A$?

$A \rightsquigarrow \cdots \rightsquigarrow \cdots \longrightarrow \cdots$

If get row of 0's: $\det = 0$.

Otherwise:

3 types of row ops:

① Interchange two rows: $\det \to -\det$

② Mult a row by $c \neq 0$: $\det$ is mult by c.

③ Subtract a mult of one row from another:

$\to$ $\det$ is unchanged.

Keep track, get $\det A$.

Enough to get into up. Δ'r form

$\det = \prod$ diag. entries

Reason for ③:

$$\begin{array}{cc} \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ a_1 \cdots & a_n \\ \cdots & \cdots \\ b_1 \cdots & b_n \\ \cdots & \cdots \end{pmatrix} & \begin{array}{c} \\ \\ R_i \\ \\ R_j \\ \\ \end{array} \end{array}$$

Subtract $c \cdot R_i$ from row j.

$$\det \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_1 & \cdots & a_n \\ \cdots & \cdots & \cdots \\ b_1 - ca_1 & \cdots & b_n - ca_n \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$= \det\begin{pmatrix} - & - & - \\ a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ - & - & - \end{pmatrix} - \det\begin{pmatrix} - & - & - \\ a_1 & \cdots & a_n \\ - & - & - \\ ca_1 & \cdots & ca_n \\ - & - & - \end{pmatrix}$$

original mx $\qquad\qquad$ $= 0$ ✓

Another reason for ① – ③:

Elem. row op $\longleftrightarrow$ mult. by elem mx.

Effect of row op: mult det by det of elem. mx.

✓

Ex. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 5 & 7 \end{pmatrix}$ $\qquad$ $\det = 2.$ ✓

$R_2 - 2R_1$, $R_3 - 3R_1$

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & -2 \end{pmatrix}$ $\qquad$ $\det = 2$

$R_2 \leftrightarrow R_3$

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ $\qquad$ Δr mx

$\det = 1(-1)(2) = \underline{\underline{-2}}$

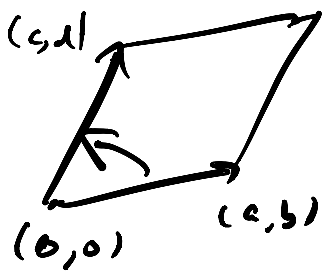
This works over any field F.

Case $F = \mathbb{R}$: geom interp of det

$\mathbb{R}^2$ $\quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\parallel$

signed area of $\square$



rows (or col's)

$\mathbb{R}^3$ = signed vol of parallelopiped formed by the rows. (or col's)

$\rightsquigarrow$ Chg of vbls formula for mult. integration

$x_1, \ldots, x_n \rightsquigarrow y_1, \ldots, y_n$ fns of x's.

$z = f(x_1, \ldots, x_n) = g(y_1, \ldots, y_n)$

$\int \cdots \int g(y_1, \ldots, y_n)\, \overline{dy_1 \wedge \cdots \wedge dy_n}$

$= \int \cdots \int f(x_1, \ldots, x_n) \det\left(\frac{\partial y_i}{\partial x_j}\right) \overline{dx_1 \wedge \cdots \wedge dx_n}$

Need 1-1 if omit abs.

Jacobian det (what vol is mult by, near any pt)

Why all methods of computing det give same answer:

Show all methods give fns

$M_n(F) \longrightarrow F$ with three properties:

$\uparrow$

View as $\underbrace{F^n \times \cdots \times F^n}_{n} \longrightarrow F$

(1) Multilinear

(2) Alternating

(3) $I \longmapsto 1$ $\quad (e_1, e_2, \ldots, e_n) \longmapsto 1$

Then show: there is only one fn

$M_n(F) \to F \quad (F^n \times \cdots \times F^n \to F)$

satisfying (1), (2), (3).

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(1) V v.s. over F.

fns on r vectors in V

$M^r(V) = \{ \underbrace{V \times \cdots \times V}_{r} \longrightarrow F, \text{ multilinear} \}$

v.s. over F

Ex. $r = 1$ $\quad \underset{M^1(V) =}{\{ V \longrightarrow F, \text{ linear} \}} = V^*$

$r=2.$ $\quad M^2(V)$

$\{V\times V \to F,\ \text{bilinear}\}$

(v_1, v_2) bilinear forms

r $\quad M^r(V)$ $\quad$ r-linear

$\underbrace{V\times\cdots\times V}_{r} \to F$

Ex. of bilinear fns on $V=F^3$.

$$V\times V \to F$$

V: std basis $\quad i, j, k$

V^*: dual basis $\quad x, y, z$

Some elts in $M^2(V)$

$(v_1, v_2)\in V\times V$

$v_1 = (a_1, b_1, c_1)$ $\quad v_2 = (a_2, b_2, c_2)$

$T_1: (v_1, v_2)\mapsto a_1a_2 + b_1b_2 + c_1c_2 = v_1\cdot v_2$ if $F=\mathbb{R}$

$T_2: (v_1, v_2)\mapsto a_1b_2 - c_1a_2$

$T_3: (v_1, v_2)\mapsto a_1b_2 = \underline{\underline{x}}(v_1)\,\underline{\underline{y}}(v_2)$

$\quad = \underline{x}\otimes\underline{y}$ $\quad = (x\otimes y)(v_1, v_2)$

$$T_1 = x\otimes x + y\otimes y + z\otimes z$$

$$T_2 = x\otimes y - z\otimes x$$

In $M^2(\underset{\dim 3}{V})$ here

$x\otimes x,\ x\otimes y,\ x\otimes z$
$y\otimes x,\ y\otimes y,\ y\otimes z$
$z\otimes x,\ z\otimes y,\ z\otimes z$

Basis of $M^2(V)$

$$\dim M^2(\underset{3}{V}) = 9 = 3^2$$

$M^r(\underset{\dim = n}{V})$ $\dim = n^r$

V basis $v_1, \dots, v_n$

V^* basis $\delta_1, \dots, \delta_n$

$$\underbrace{\delta_{i_1}\otimes\delta_{i_2}\otimes\cdots\otimes\delta_{i_r}}\qquad 1\le i_1,\dots,i_r\le n$$

These form a basis of $M^r(V)$.

$\det \in M^n(F^n)$, $\leftarrow \dim = n^n$

(2) Alternating (as well as multilinear)

$\Lambda^r V = \{$ alt. mult. r-forms on $\underset{\dim n}{V}\} \subset M^r(V)$ subsp

Ex. $r = 2$. 2 forms on F^n

$$\delta_i \otimes \delta_j - \delta_j \otimes \delta_i =: \delta_i \wedge \delta_j \qquad 1 \le i, j \le n$$

$$\delta_i \wedge \delta_j (v, v) = 0 \qquad \text{alt.}$$

$$\delta_i \wedge \delta_j (v, w) = - \delta_i \wedge \delta_j (w, v)$$

$$\delta_j \wedge \delta_i = - \delta_i \wedge \delta_j. \qquad \delta_i \wedge \delta_i = 0$$

$$\underline{\underline{\delta_i \wedge \delta_j}} \qquad 1 \le i, j \le n.$$

$$1 \le i < j \le n$$

basis

$$\dim \underset{\uparrow n}{\Lambda^2 V} = \binom{n}{2} = \frac{n(n-1)}{2}$$

In gen'l, $\Lambda^r V$ $\dim \binom{n}{r}$

$$\delta_{i_1} \wedge \cdots \wedge \delta_{i_r} \qquad \text{basis}$$

$$= \sum_{\substack{\text{perms} \\ \sigma \text{ of } 1 \ldots r}} (\pm 1)\, \delta_{\sigma(i_1)} \otimes \cdots \otimes \delta_{\sigma(i_r)}$$

use sign of perm.

$$\dim \underset{\uparrow n}{\Lambda^r V} = \binom{n}{r}$$

$\det \in M^n(F^n)$ $\quad$ dim $= n^n$

$\det \in \Lambda^n(F^n)$ $\quad$ dim $= \binom{n}{n} = 1$

$\therefore$ Any two alt mult n-forms on F^n are mults of each other.

So: If $\det(I) = 1$: they agree.

$(e_1, \ldots, e_n)$

$\therefore$ det is unique