Matrices $\longleftrightarrow$ Lin. transf.

$A \in M_{m,n}(F) \rightsquigarrow T_A : F^n \to F^m$

$$[T_A(v)] = A[v]$$

$$M_{m,n}(F) \xrightarrow[\text{iso}]{\text{bijection}} L(F^n, F^m)$$

$$A \longmapsto T_A$$

$$\exists A \longmapsto T_{\text{any}}$$

i.e. $T = T_A$

A: j^{th} col is $[T(e_j)]$

Generalize to $L(V, W)$

$\dim = n$ $\quad$ $\dim = m$

Pick basis of V $\mathcal{A} : v_1, \dots, v_n$

- - — — W $\mathcal{B} : w_1, \dots, w_m$

For $v \in V$, $v = \sum_{j=1}^{n} a_j v_j$, uniquely

coords of v wrt $\mathcal{A}$

$$[v]_{\mathcal{A}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

For $w \in W$, $\quad w = \sum_{i=1}^{m} b_i w_i$, uniquely

coords of w wrt $\mathcal{B}$

Using this, can generalize the above.

$$[w]_{\mathcal{B}} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{array}{ccc} M_{m,n}(F) & \xrightarrow{\text{iso}} & L(V,W) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ A & \longmapsto & T_A \end{array}$$

def similarly to before, but using $\mathcal{A}, \mathcal{B}$
V, W

$$\big[\underbrace{T_A(v)}_{\in W}\big]_{\mathcal{B}} = A[\underset{\underset{V}{\cap}}{v}]_{\mathcal{A}}$$

$m \times 1$ col. vector $\quad m \times n \quad n \times 1$

Given $T \in L(V,W)$

$\exists A$ st $T = T_A$ wrt $\mathcal{A}, \mathcal{B}$

$=$ j^{th} col is $[T(v_j)]_{\mathcal{B}}$

Ex 17, p 93.

$V = W = \mathcal{P}_3$

$D : V \xrightarrow{\text{lin tr}} V = W$ differentiation

Basis of V : $\underbrace{\{1, x, x^2, x^3\}}_{\mathcal{A} = \text{“}\mathcal{B}\text{”}}$

$D = T_A$; what is A?

$D(1) = 0,\ D(x) = 1,\ D(x^2) = 2x,\ D(x^3) = 3x^2$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

V vs, $\mathcal{A}$ basis

$T : V \longrightarrow V$ lin. op.

$T = T_A$ $\qquad A = [T]_{\mathcal{A}}$

---

mx mult $\longleftrightarrow$ composition

$$\begin{array}{llll} \text{dim} & n & m & p \\ \text{vs} & V & \xrightarrow{S} W & \xrightarrow{T} Z \\ \text{bases} & \mathcal{B}_V & \mathcal{B}_W & \mathcal{B}_Z \\ \text{mx} & & \underset{m \times n}{A} & \underset{p \times n}{B} \end{array}$$

$T_A = S$ wrt $\mathcal{B}_V, \mathcal{B}_W$

$T_B = T$ wrt $\mathcal{B}_W, \mathcal{B}_Z$

$\rightsquigarrow \underbrace{\underset{p\times m}{B}\ \underset{m\times n}{A}}_{p\times n} \longleftrightarrow T \circ S$

i.e. $T_{\underline{BA}} = T_B \circ T_A$

Th 13, p 90, H+K (as before)

Change of basis

V $\quad \mathcal{A} : \alpha_1, \ldots, \alpha_n$

$\quad \mathcal{B} = \beta_1, \ldots, \beta_n$ $\qquad n = \dim V$

Saw: $v \in V$

$\longrightarrow [v]_{\mathcal{A}} = \underset{n\times n}{C} [v]_{\mathcal{B}}$

Chg. of basis for coords

$C = (c_{ij})$

where $\beta_j = \sum_{i=1}^{n} c_{ij} \alpha_i \qquad (*)$

i.e. j^{th} col of $C \leftrightarrow$ coords of β_j wrt $\mathcal{A}$

$[v]_{\mathcal{B}} = C^{-1} [v]_{\mathcal{A}}$. (C invertible)

Change of basis for lin trans:

$$T : V \longrightarrow V$$

$$\mathcal{A}, \mathcal{B} \qquad \mathcal{A}, \mathcal{B}$$

$T = T_A$ for some mx A wrt $\mathcal{A}$

$$A = [T]_{\mathcal{A}}$$

$T = T_B$ - - - - - - B wrt $\mathcal{B}$

$$B = [T]_{\mathcal{B}}$$

$$A[v]_{\mathcal{A}} = [T(v)]_{\mathcal{A}}$$

$$B[v]_{\mathcal{B}} = [T(v)]_{\mathcal{B}}$$

Take the same mx $C = (c_{ij})$ as above (Ex 16, p92)

Th 14, p 92, H+K.

Prop $B = C^{-1} A C$.

Pf. $[v]_{\mathcal{A}} = C[v]_{\mathcal{B}}$

$$[T(v)]_{\mathcal{A}} = C[T(v)]_{\mathcal{B}}$$

$$C\,[T(v)]_{\mathcal{B}} = [T(v)]_{\mathcal{A}}$$
$$= A\,[v]_{\mathcal{A}}$$
$$= A\,C\,[v]_{\mathcal{B}}$$

$$[T(v)]_{\mathcal{B}} = \underline{\underline{C^{-1}AC}}\,[v]_{\mathcal{B}}$$

$$\underline{\underline{B}}\,[v]_{\mathcal{B}}$$

True for all

get $B = C^{-1}AC$. ✓

In gen'l, if C is any $n\times n$ invertible mx,
& A, B are $n\times n$ mx's,
we say that A, B are <u>similar</u>
if $B = C^{-1}AC$.

So: If A, B express the same lin. tr. $V \longrightarrow V$ wrt diff. bases, then A, B are similar (& conversely)

j^{th} col of C
expresses j^{th} basis vector in $\mathcal{B}$ — B
in terms of basis vectors in $\mathcal{A}$. — A

$$V \longrightarrow V \qquad \mathcal{a}, \mathcal{B}$$

$$A, B$$

$$B = C^{-1} A C.$$

$$V \longrightarrow W$$

$$\begin{array}{ccc} \mathcal{a} & A & \mathcal{a}', \\ \mathcal{B} & B & \mathcal{B}' \end{array} \qquad C, \ C'$$

$$B = C'^{-1} A C.$$

$$\underset{\text{dim} = mn}{L}\,(\underset{n}{V}, \underset{m}{W}) = \mathrm{Hom}(V, W)$$

Ex. $V = W$. $L(V) = \mathrm{End}(V)$ for short

n $\dim = n^2$

Ex. V, any, dim n. $W = F = F'$

$L(V, F) = \mathrm{Hom}(V, F) = V^*$

$\dim V^* = n$ (iso to V; depends on basis)

V^*: dual space of V.

elements in V^*: linear functionals

Ex. $V = \mathbb{R}^3$

basis $\mathcal{B}$: e_1, e_2, e_3

$$\begin{array}{ccc} e_1 & e_2 & e_3 \\ \shortparallel & \shortparallel & \shortparallel \\ \vec{i} & \vec{j} & \vec{k} \end{array}$$

Functions on $\mathbb{R}^3$, ex. $x^2 + yz$ (not linear)

Value at $(1,2,3) = \vec{i} + 2\vec{j} + 3\vec{k}$

is $1^2 + 2 \cdot 3 = 7$ lin. functional

Ex of a linear fn on V:

$\mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad 3x - y + z$.

Value at $(1,2,3)$ is $3 \cdot 1 - 2 + 3 = 4$

Take <u>all</u> the lin. fcts on $\mathbb{R}^3$

(of the form $ax + by + cz$

$a, b, c \in \mathbb{R}$.

This is a v.s.: $V^* = \mathbb{R}^{3*}$

$x \in \mathbb{R}^{3*}$ $\quad x(\vec{i}) = 1,\ x(\vec{j}) = 0,\ x(\vec{k}) = 0$

$y, z \in \mathbb{R}^{3*}$

x, y, z: basis of $\mathbb{R}^{3*}$ dual basis

Ex. $V = \mathbb{R}^n$. Std basis $e_1, \ldots, e_n$

Elts of V^*:

Lin. fn'ls on V, $\sum_{i=1}^{n} a_i x_i$

$x_i(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}$

$x_1, \ldots, x_n$ basis for V^*

Dual basis.

More generally:

V fdvs, $\dim V = n$

Basis $\mathcal{B}$: $v_1, \ldots, v_n$

$V^* = \{\text{lin. fn'ls on } V\}$

Dual basis of V^*

$\mathcal{B}^*$: $f_1, \ldots, f_n$ $\quad f_i(v_j) = \delta_{ij}$

Can also take dual of inf. dim. v.s.

Ex. $V = \{\text{cont. fns } [0,1] \to \mathbb{R}\}$

v.s. over $\mathbb{R}$, inf. dim'l

n integer $\quad e^{nx}, \cos nx, \sin nx, \ldots$

Dual space V^*:

Lin. fnls on V.

Ex. $\quad \underset{V}{\overset{}{f}} \longmapsto \int_0^1 f(x)\,dx$

Ex. $\alpha \in [0,1]$. $\quad ev_\alpha : V \longrightarrow \mathbb{R}$

evaluation at $\alpha \qquad f \longmapsto f(\alpha)$

V v.s. $\qquad V^*$ dual space.

U subset.

S

Define a subset of V^*

annihilator of S

$$\operatorname{ann} S = S^\circ \subset V^*$$

$$/\!/$$

$$\{ f \in V^* \mid f(v) = 0 \text{ for all } v \in S \}$$

Easy to check: $\operatorname{ann} S \subset V^*$ is a subspace.

Easy to check:

If $S \subset V$ any subset, & $W = \operatorname{span} S$, then

$\operatorname{ann} S = \operatorname{ann} W$

Ex. $V = \mathbb{R}^3$

$\mathcal{B} = \{ e_1, e_2, e_3 \}$

$\quad\quad\;\; \vec{\imath} \quad \vec{\jmath} \quad \vec{k}$

$S = \{ e_1 \} = \{\vec{\imath}\}$ $\qquad W = \operatorname{span} S$

$\qquad\qquad\qquad\qquad\qquad = x\text{-axis}$

$V^* = \mathbb{R}^{3*} = \{ \text{lin. fnls on } \mathbb{R}^3 \}$

$= \{ ax + by + cz \mid a, b, c \in \mathbb{R} \}$

$f \in V^* \qquad f = ax + by + cz$

$f(\vec{\imath}) = a$

$\operatorname{ann} S = \operatorname{ann} W = \{ \text{elts of } V^* \text{ s.t. } a = 0 \}$

$= \operatorname{span}\{ y, z \} = \{ \text{lin fnls } by + cz \}$

$\dim V = 3 = \dim V^*$
$\wr \mathbb{R}^3$

$\dim W = 1$

$\dim(\operatorname{ann} W) = 2$

Here: $\dim V = \dim W + \dim(\operatorname{ann} W)$
$3 = 1 + 2$

This holds in general

Thm (Th 16, p 101, H + K)
If V is a fdVS, & if W is a subspace of V, then

$$\dim V = \dim W + \dim(\operatorname{ann} W)$$

Proof. Take a basis of W
$\{v_1, \dots, v_k\}$
$k = \dim W \leq \dim V = n$

Extend to a basis of V:

$\mathcal{B}: \{v_1, \dots, v_k, \dots, v_n\}$

Dual basis of V^*

$\mathcal{B}^*: \{f_1, \dots, f_k, \dots f_n\}$ $\quad f_i(v_j) = \delta_{ij}$

Say $f \in V^*$. $f = \sum_{j=1}^{n} a_j f_j$ s
$\in F$

$\Rightarrow f(v_i) = a_i$

$f \in \text{ann}\, W \Leftrightarrow f(v_i) = 0$ for $i = 1, \ldots, k$.

$\Leftrightarrow a_1 = \cdots = a_k = 0$

$\Leftrightarrow f \in \text{span}\{f_{k+1}, \ldots f_n\}$.

$\therefore \text{ann}\, W = \underbrace{\text{span}\{f_{k+1}, \ldots f_n\}}_{n-k \text{ dim'l}}$

$\dim V = n = k + (n-k)$
$= \dim W + \dim(\text{ann}\, W)$. ✓

__Cor.__ (1st Cor on p. 102 of H+K)

If W is a k-dim'l subspace of a fdvs V of dim n, then W is the intersection of $n-k$ hyperplanes in V

"hyperspace"

Ex. $n = 3$. $k = 1$. W: line in $\mathbb{R}^3$.
$n-k=2$
$\cap$ of 2 planes ✓

Pf of Cor: W $\quad v_1, \dots, v_k$

$\mathcal{B}$ $\quad V$ $\quad v_1, \dots, v_k, \dots, v_n$

$\mathcal{B}^*$, V^*: $f_1, \dots, f_n$. $\quad f_i(v_j) = \delta_{ij}$

For $i = k+1, \dots, n$, let

$$W_i = \{v \in V \mid f_i(v) = 0\},$$

a hyperplane in V.

$n-k$ of these. Will show: $W = W_{k+1} \cap \dots \cap W_n$.

For this:

Say $v \in V$. $\quad v = \sum_{i=1}^{n} c_i v_i$.

$$f_i(v) = c_i$$

$$\begin{aligned} v \in W &\iff v \in \operatorname{span}\{v_1, \dots, v_k\} \\ &\iff c_{k+1} = \dots = c_n = 0 \\ &\iff f_{k+1}(v) = \dots = f_n(v) = 0 \\ &\iff v \in W_{k+1} \cap \dots \cap W_n \end{aligned}$$

$$W = W_{k+1} \cap \dots \cap W_n.$$

$W \subset V$ $\quad$ codimension of W is $n-k$.

$k \quad n$ $\quad$ hyp.pl: codim $= 1$. $\quad$ cod $= m \Rightarrow \cap$ of m hyp.pl.

<u>Cor (of Thm):</u>

$W_1, W_2 \underset{\text{subsp}}{\subset} V$ f dims

$\text{ann}\, W_1 \subset \text{ann}\, W_2 \iff W_2 \subset W_1.$

<u>Pf.</u> ($\Leftarrow$) Easy:

If $f \in \text{ann}\, W_1$,

then $f(w) = 0$ for all $w \in W_1$.

In particular --------------------- $w \in W_2 \subset W_1$.

i.e. $f \in \text{ann}\, W_2$. ✓.

($\Rightarrow$)

$\text{ann}\, W_1 \subset \text{ann}\, W_2$

$$\Rightarrow \text{ann}\, W_1 = \text{ann}\, W_1 \cap \text{ann}\, W_2$$

$$= \text{ann}(W_1 \cup W_2)$$

$$= \text{ann}(\text{span}(W_1 \cup W_2))$$

$$= \text{ann}(W_1 + W_2)$$

$$\dim(\text{ann}\, W_1) \overset{\text{Thm}}{=} \dim V - \dim W_1$$

$$\|$$

$$\dim(\text{ann}(W_1 + W_2)) \overset{\text{Thm}}{=} \dim V - \dim(W_1 + W_2)$$

$\dim W_1 = \dim(W_1 + W_2)$. But $W_1 \subset W_1 + W_2$.

$\therefore W_1 = W_1 + W_2 \supset W_2 \qquad W_2 \subset W_1$ ✓

Cor: W_1, W_2 subsp of V, fdvs

Thm: $\operatorname{ann} W_1 = \operatorname{ann} W_2 \iff W_1 = W_2$.

Pf. $= \rightsquigarrow \subset \xrightarrow[\text{cor}]{\text{prev}} \supset$

$= \rightsquigarrow \supset \longrightarrow \subset$

$W_1 \subset W_2 \quad W_2 \subset W_1 \quad \therefore = .$ ✓

For computations with ann's:
Ex 23 p103, Ex 24 p104
H+K

$\dim V = n \rightsquigarrow \dim V^* = n$

$\therefore V, V^*$ are iso.

Take a basis $\mathcal{B}$ of V, $v_1, \dots, v_n$

Take dual " $\mathcal{B}^*$ of V^*, $f_1, \dots, f_n$

$f_i(v_j) = \delta_{ij}$

Iso $V \longrightarrow V^*$: $v_i \longmapsto f_i$

$\sum a_i v_i \longmapsto \sum a_i f_i$

Depends on choice of $\mathcal{B}$.

Iso "not natural"

Take vs	V	n	Pick $\mathcal{B}$
dual	V^*	n	$\mathcal{B}^*$
doubledual	V^{**}	n	$\mathcal{B}^{**}$

$$V \xrightarrow{\sim} V^* \xrightarrow{\sim} V^{**}$$

$$\mathcal{B} \qquad \mathcal{B}^* \qquad \mathcal{B}^{**}$$

$$V \xrightarrow{\sim} V^{**} \qquad \text{composition}$$

Amazing fact: This iso is natural; independent of choice of $\mathcal{B}$.

from $\mathcal{B}, \mathcal{B}^*, \mathcal{B}^{**}$: PS 7 #5

⇝ This nat. iso is given by

$$\begin{tikzcd}
V \arrow[r, "\text{lin}", "\sim"'] & V^{**} \\
v \arrow[r, mapsto] & ev_v
\end{tikzcd}$$

$ev_v \in V^{**}$ $\qquad$ $ev_v : V^* \xrightarrow{\text{lin.}} F$

$$f \longmapsto f(v)$$

For any vs V, $V \to V^{**}$, $v \mapsto ev_v$
lin tr, natural.

If V is fdvs, natural iso.

Th 17, p107, H + K: nat. iso. $ev_V = L_V$

This implies: if V is fdvs every elt of V^{**} is of form ev_v.

$$V \xrightarrow{\sim} V^* \xrightarrow{\sim} V^{**} \quad (V \text{ fdvs})$$

$V \xrightarrow{\sim} V^{**}$ natural.

regard as =

$v \leftarrow ev_v$

$W \subset \underset{\text{fdvs}}{V} \longrightarrow$ view W as in V^{**}

view W as in V or $W \subset V^{**}$

$W \subset V$

$\underbrace{\text{ann}(\text{ann}\underbrace{W}_{\text{in } V^*})}_{\text{in } V^{**}\text{, regard as } V}$ is regarded as subsp. of V.

Which? Ans W.

Th 18 p108 H+K

More generally: if $S \subset \underset{\text{fdvs}}{V}$ subset and identify V with V^{**} ann(ann S) is regarded as span(S) $\subset V$.

See H+K §3.8 for more.