Recap:

Thm If V has a basis of m elts, then no lin. ind. set has $> m$ elts

Cor For a f.d.v.s. V, any two bases have the same # of elements

$\dim V$

Prop $S \subset V$ finite $\Rightarrow \exists T \subset S$ lin. ind. with same span.

Cor 1 Every finite spanning set contains a basis.

Cor 2 If V has a finite spanning set, the V has a finite basis. (f.d. v.s.)

Cor 3 In an n dim'l v.s, every spanning set has $\geq n$ elts.

Thm Every lin. ind. set in a f.d.v.s. is contained in a basis

Thm Let V is a f.d.v.s., & $W \subset V$ a subspace. then W has a finite basis, with $\leq \dim V$ elts.

Pf. If $W = 0 = \{0\}$, then ✓

If $W \neq 0$, then take a non-zero vector w_1 in W.

If w_1 spans W, then $\{w_1\}$ is a basis (b/c lin ind) ✓

If not, then $\operatorname{span}\{w_1\} \neq W$.

Take $w_2 \in W$, $w_2 \notin \operatorname{span}\{w_1\}$.

Then $\{w_1, w_2\}$ is lin. ind. (PS2 #4)

If this set spans, it's a basis, ✓.

If not, continue.

This must stop in $< n+1$ steps b/c $\dim V = n$. (no lin ind set in V with $> n$ elts)

Then: have a basis for W. ✓ $\leq n$ elts $= \dim V$.

Cor If $W \subset V$ is a subspace
$\uparrow$ fdvs
& $W \neq V$, then $\dim W < \dim V$.

Pf. By Thm, W has a finite basis $\{w_1, \dots, w_m\}$.
$n = \dim V$. Thm $\Rightarrow m \leq n$.

$W \subsetneq V \Rightarrow \exists v \in V$ st. $v \notin W$.

$W = \operatorname{span} \underbrace{\{w_1, \dots, w_m\}}$. basis of W, lin ind.

$\{w_1, \dots, w_m, v\}$ is lin ind. (PS 2 #4)
$\cap$ $\uparrow$ $m+1$ elts
$V \leftarrow \dim = n$.

$\therefore m+1 \leq n$. $\therefore m < n$. ✓

Prop Let V be an n dim'l v.s.
Let $S \subset V$ be a set of $\underline{\underline{n}}$ elts
TFAE:

i) S is a basis

ii) S is lin. ind.

iii) S span V.

Pf. (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (i) STS: S spans V.

Let $W = \operatorname{span} S$. $\quad W \underset{\text{subsp}}{\subset} V$

S is a basis of W.

S ← n elts.

$\dim W = n = \dim V$

Cor $\Rightarrow W = V$. $\therefore S$ spans V.

(iii) S spans $V \Rightarrow S$ contains $\underset{\text{basis}}{T} \subset V$

$\dim V = n \Rightarrow T$ has n elts.

$\underset{n}{T} \subset \underset{n}{S} \Rightarrow \underset{\text{basis}}{T} = S$. $\therefore$ basis.

$W_1, W_2 \subset V$ v.s.

subspaces

$W_1 \cap W_2$ subspace, cont. in W_1, W_2

← largest subspace cont. in W_1, W_2.

Smallest subsp containing W_1, W_2

$$\begin{array}{c} \text{Smallest subsp containing } W_1, W_2 \\ \| \\ W_1 + W_2 = \text{span}(W_1 \cup W_2) \\ \| \\ \{w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2\} \end{array}$$

$$\begin{array}{ccc} & V & \\ & \cup & \\ & W_1 + W_2 & \\ \cup & & \cup \\ W_1 & & W_2 \\ \cup & & \cup \\ & W_1 \cap W_2 & \\ & \cup & \\ & 0 & \end{array}$$

Ex. $V = \mathbb{R}^3$. $W_1 = x, y$ plane
$W_2 = y, z$ plane

$W_1 + W_2 = \mathbb{R}^3$
$W_1 \cap W_2 = y$-axis

<u>Thm</u> If W_1, W_2 are fin. dim subspaces of a v.s. V, then so are $W_1 + W_2$ and $W_1 \cap W_2$, and
$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

In Ex: $2 + 2 = 3 + 1$

Pf W_1 fdvs

$\therefore W_1 \cap W_2$ is fdvs, of dim $\leq \dim W_1$

has basis: $B_0 = \{\alpha_1, \dots, \alpha_k\}$. of $W_1 \cap W_2$

lin ind set in V. (in W_1)

So $\exists$ basis $B_1 = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m\}$. of W_1

Similarly, $\exists$ basis $B_2 = \{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_n\}$ of W_2.

$$W_1 + W_2 = \text{Span}(W_1 \cup W_2)$$

spanned by

$B_1 \cup B_2 = B = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n\}$. finite set

fin sp. set. $\therefore$ fdvs: $\exists$ fin. basis

Claim: B is a basis of $W_1 + W_2$.

Pf of Claim:

$\mathcal{B}$ spans. STS $\mathcal{B}$ lin ind.

Say $(*)$ $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_\ell \gamma_\ell = 0$

WTS $x_i, y_j, z_\ell \in F$. are 0.

$$-\underbrace{\sum z_\ell \gamma_\ell}_{\in W_2} = \underbrace{\sum x_i \alpha_i + \sum y_j \beta_j}_{\in W_1}$$

in $W_1 \cap W_2$, has a basis $\mathcal{B}_0$ $\{\alpha_1, \ldots, \alpha_k\}$.

$-\sum z_\ell \gamma_\ell = \sum c_i \alpha_i$

$\sum c_i \alpha_i + \sum z_\ell \gamma_\ell = 0.$

$\{\alpha \ldots, \gamma \ldots\}$ basis, lin ind

all $c_i = z_\ell = 0.$ Similarly, all $y_j = 0.$

So $(*) \Rightarrow \sum x_i \alpha_i = 0.$ lin ind.

$\Rightarrow$ all $x_i = 0.$ α_i's : basis of $\mathcal{B}_1 \cap \mathcal{B}_2$

Proves the claim.

$\dim W_1 \cap W_2 = k$ α

$\dim W_1 = k + m$ α, β

$\dim W_2 = k + n$ α, γ

$\dim W_1 + W_2 = k + m + n.$ α, β, γ

$k + (k+m+n) = (k+m) + (k+n)$

Note: Office hours this week
Wed $1^{30} - 2^{30}$ instead of
Fri.

Back to matrices.

A $m \times n$ matrix

Each row is in F^n.

The rows span a subspace of F^n
— row space of A.

dim of row space: row rank of A.

Ex 1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ over $\mathbb{R}$.

Rows lin. ind. Rows span $\mathbb{R}^2$

row rank = 2 row space.

Ex 2. $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ over $\mathbb{R}$.

$R2 = 2 \cdot R1$; lin. dep.

Row space = mults of $(1, 2)$

line. row rk = 1

If do a row ops
row space doesn't change.

So: Row eq. mx's have
the same row space.

In partic: $A \xrightarrow[\text{row ops}]{} R$ red. row. ech form.

A, R have same row space.

Ex 1 above: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ row sp $= \mathbb{R}^2$ row rk $= 2$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ — — — —

Ex 2 above $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ row sp: line
mults of $(1, 2)$
row rk $= 1$

$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ "

Useful for large mx's.

For an $n \times n$ (square) matrix A:

A invertible $\iff$ A is row equiv to I
$\iff$ row space is F^n
$\iff$ the n rows span F^n
$\iff$ — — — — are a basis of F^n
$\iff$ — — — — — lin. ind.

A $\quad m \times n$ (not nec. square)

$\rightsquigarrow$

R red. row ech. form.

Ex. $R = \begin{array}{ccccc} k_1=1 & & k_2=3 & k_3=4 & \\ \end{array}$

$$R = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underset{m}{4} \times \underset{n}{5}$ $\quad d = 3$

Pivot entries $= 1$, other entries in those col's are 0

Rows: $\rho_1, \ldots, \rho_m$

Say $\rho_1, \ldots, \rho_d$ are non-0. Rest: 0

Let k_i = col # of i^{th} pivot

The rows $\rho_1, \ldots, \rho_d$ are lin. ind.

- b/c k_i entry of ρ_i is 1, others 0.

$\therefore$ $\rho_1, \ldots, \rho_d$ are a basis of row space (of A, or of R)

So row rk $= d$ for A + for R.

row rk(A) = dim of row sp. of A

= — — — — — — — R

= # of non-0 rows of R

$= d$.

A $m \times n$ $\rho_1, \ldots, \rho_d$ non 0 rows of R

In ρ_i, 1st non-0 entry is in k_i-col.

Say $v \in W$ = row space of A
= — — — — R.

$$v = \sum_{i=1}^{d} b_i \rho_i \in W \subset F^n$$

k_j-entry of ρ_i is $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

→ ∴ k_j-entry of v is b_j. (for all j)

The 1st non-0 entry in ρ_i is 1, in k_i-slot.

What is — — — — — — — v? + where?

Ans: b_i in the k_i-slot,

→ if b_i is the 1st non-0 coeff in $\sum b_i \rho_i$

Prop The row space W of A determines the pivot variables. ($k_1, \ldots, k_d$)

Proof By the above, the pivot vbls are in the same col's $k_1, \ldots, k_d$ as arise as 1st non-0 slots of vectors in W. ∎

<u>Prop</u> The row span W determines the row vectors $\rho_i \in W$ of R.

Pf. $W \rightsquigarrow \dim W = d$. $\rho_{d+1} = \cdots = \rho_m = 0$.

Prev. prop: have k_i's.

In ρ_i: k_i entry is 1, k_j entry is 0 for $j \neq i$. ($\star$)

<u>Claim</u> ρ_i is the <u>only</u> vector in W with property ($\star$)

<u>Pf of claim</u> For any $v \in W$,

$v = \sum b_j \rho_j$, & $b_j = k_j$-entry of v.

So if v satisfies ($\star$), then

$b_i = 1$, $b_j = 0$ for $j \neq i$

$v = \sum b_j \rho_j = \rho_i$. ✓

So: W determines pivot vbls, k_i.

& #'s k_i determine ρ_i's.

So W determines ρ_i's: rows of <u>R</u>.

R is determined by W = row sp

rref of A $\nearrow$ A determines W.

So A determines R.

<u>Th</u> Every mx A has a unique r.r.e.f. R.

Cor Let A, B be $m \times n$ mx's over F. TFAE

i) A, B are row equiv.

ii) A, B have same row space

iii) A, B have same red. row. ech. form mx.

Pf. (i) ⇒ (ii): Since row op's preserve row space.

(ii) ⇒ (iii): by Thm: row sp determines rref.

(iii) ⇒ (i): A, B have r.r.e.f. R.

Then A is row eq to R.

B - - - ——— R.

∴ A - -- - - B. ✓

Coordinates:

In F^n, std basis $e_1, \_, e_n$

$\underset{\in}{v} = (a_1, \dots, a_n) = \sum_{i=1}^{n} a_i e_i$

coord's $\qquad$ coeff's

In a more gen'l f.d.v.s. V over F,

basis $\alpha_1, \dots, \alpha_n$ $\leftarrow$ ordered basis $\mathcal{A}$

$\underset{\in V}{v} = \sum_{i=1}^{n} a_i \alpha_i$, uniquely

coeff's

Call a_i's the <u>coords</u> of v w.r.t. this basis $\mathcal{A}$.

Write: $[v]_{\mathcal{A}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$\mathcal{B}$: $\beta_1, \dots \beta_n$ another ordered basis of V

$v = \sum_{i=1}^{n} b_i \beta_i$

coords of v w.r.t. $\mathcal{B}$

$[v]_{\mathcal{B}} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

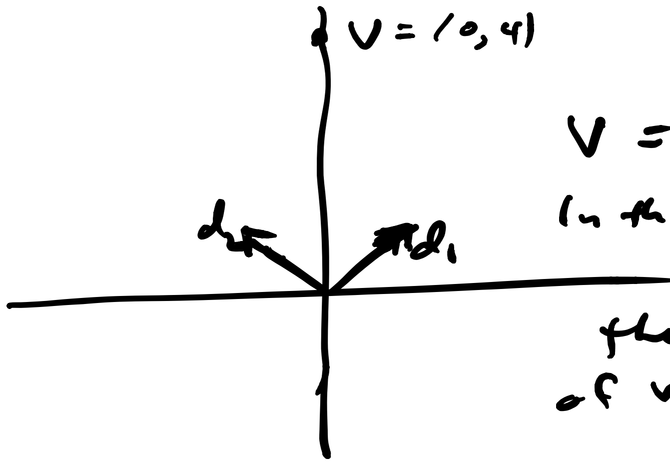
Ex. In $\mathbb{R}^2$ $\quad v = (0,4)$

Std. basis: e_1, e_2

$v = 0e_1 + 4e_2$ $\quad$ coords: $0,4$

<u>Another basis</u>: $d_1 = (1,1)$

$d_2 = (-1,1)$



$v = 2d_1 + 2d_2$

In this basis, the coords of v are $2,2$.

For $v \in V$, & two bases $\mathcal{A}, \mathcal{B}$, want to pass from coords of v in one basis to coords in other basis.

<u>Change of basis</u>.

$\mathcal{A}: \alpha_1, \ldots, \alpha_n \qquad \mathcal{B}: \beta_1, \ldots, \beta_n$

$$v = \sum a_i \alpha_i = \sum b_i \beta_i$$

Say we know b_i's (coords w.r.t. $\mathcal{B}$)

Want a_i's (coords w.r.t. $\mathcal{A}$)

First: do for $v = \beta_j$.
Write β_j in terms of α's

$$\rightarrow \beta_j = \sum_{i=1}^{n} c_{ij}\alpha_i \qquad c_{ij} \in F$$

$$C = (c_{ij}) \qquad n \times n \quad \text{mx}$$

j^{th} col of C : coords of β_j in terms of α's.

For a genl $v \in V$,

$$v = \sum_j b_j \beta_j = \sum_j b_j \Big(\sum_i c_{ij}\alpha_i\Big)$$

b_j know

$\| \; \sum_i a_i \alpha_i$ a_i want

$$= \sum_i \Big(\sum_j c_{ij} b_j\Big)\alpha_i$$

$$\therefore \; a_i = \sum_j c_{ij} b_j$$

Expresses coords of v wrt $\mathcal{A}$
in terms of . . . . . $\mathcal{B}$.

$$\underset{n\times 1}{[v]_{\mathcal{A}}} = \underset{n\times n}{C}\,\underset{n\times 1}{[v]_{\mathcal{B}}}$$

Ex. $\mathbb{R}^2$ $\quad a = (d_1, d_2)$

$(1,1)$ $\quad (-1,1)$

$B = (e_1, e_2)$

1st express e_1, e_2 in terms of d_1, d_2

$$e_1 = \tfrac{1}{2} d_1 - \tfrac{1}{2} d_2$$

$$e_2 = \tfrac{1}{2} d_1 + \tfrac{1}{2} d_2$$

$$C = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$[v]_a = C\,[v]_B$$

$v = (0,4)$ $\quad [v]_B = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$$[v]_a = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Linear transformations
(Chap 3)

Before: A $m \times n$ matrix

Sys of eqns $AX = B$

A: $m \times n$, X: $n \times 1$, B: $m \times 1$

Given B, want X

B in F^m, X in F^n

Can turn around:

Given X, get B. $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

X in F^n, B in F^m

Using A $m \times n$

to each vector $v \in F^n$,

associate a vector $T(v) \in F^m$

by: if $v = (x_1, \_ x_n)$,

then $T(v) = (b_1, \_ b_m)$ where $\underline{AX = B}$

T: transformation
function
map

The above transf. T satisfies:

i) $T(v+w) = T(v) + T(w)$

ii) $T(cv) = cT(v)$

Reason:

For i) $A(X+Y) = AX + AY$

ii) $A(cX) = c(AX)$

A transformation satisfying (i) & (ii): called <u>linear</u>.

linear transf., linear map, homomorphism of v.s.'s

V, W v.s.'s over F.

$$\begin{array}{ccc} T: V & \longrightarrow & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & T(v) = w \end{array}$$

Ex. 1. $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(v) = 3v$ ✓ (linear)

2. $T : \mathbb{R}^3 \to \mathbb{R}^4$

$T(a,b,c) = (a,b,c,0)$ ✓

3. $T : \mathbb{R}^3 \to \mathbb{R}^2$

$T(a,b,c) = (a,b)$ ✓

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(a,b) = (a+1, b+1)$

$v = 0, \quad c = 0$

$T(cv) = T(0) = T(0,0) = (1,1)$

$cT(v) = 0\,T(v) = 0^{*}$

X Not linear

this shows If T is lin, then $T(0) = 0$.

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$

rotate ccw by $30°$ ✓

6. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(a,b) = (a^2, b^2)$

$T(0) = T(0,0) = (0,0) = 0$

$v = (1,0), \quad w = (1,0)$

$T(v+w) \stackrel{?}{=} T(v) + T(w)$

$T(v) = (1,0), \quad T(w) = (1,0)$

$T(v) + T(w) = (2,0)$

$T(v+w) = T(2,0) = (4,0) \neq$

$\times$ Not linear

(quadratic transformation)

$T : V \longrightarrow W$ lin. transf.

V: domain (source)

W: codomain (target)

range of T

(image)

$\{ w \in W \mid w = T(v) \text{ for some } v \in V \}$

Ex 1. image $(T) = \mathbb{R}^2$

Ex 2. image $= \{(x, y, z, 0) \mid x, y, z \in \mathbb{R}\}$

Ex 3. image $= \mathbb{R}^2$

Ex 5. image $= \mathbb{R}^2$

Here, image $\subset W$
is a subspace.

<u>Prop</u> The image of a lin. transf.
$T: V \longrightarrow W$ is a subspace of W.

<u>Proof</u> STS: image $\neq \phi$,
closed under $+$, & under sc. mult.

$\neq \phi$: $0 \in V$ $\quad \underset{\substack{\parallel \\ 0}}{T(0)} \in$ image $\neq \phi$

$+$: $T(v_1) + T(v_2)$
$= T(v_1 + v_2)$ ✓

$\cdot$: $cT(v) = T(cv)$ ✓

$T : V \to W$ lin transf

$\text{image}(T) \subset W$ subsp

$\underbrace{\dim(\text{image}(T))}$

rank of T

Ex 1 rk = 2

Ex 2 rk = 3

Ex 3 rk = 2

Ex 5 rk = 2

$T: V \to W$ lin tr.

$\text{image}(T) \subset W$ subsp.

$\text{rk}(T) = \dim(\text{im}(T)) \leq \dim W$

$\dim V = n$

basis of V: $v_1, \dots, v_n$

Claim: $\text{im}(T)$ is spanned by $T(v_1), \dots, T(v_n)$.

Reason: $w \in \mathrm{im}(T)$

$$\begin{aligned} w &= T(v) = T\left(\sum a_i v_i\right) \\ &\qquad\qquad\;\; = \sum a_i \underbrace{T(v_i)}_{} \end{aligned}$$

$\sum a_i v_i$ ✓

Claim $\Rightarrow$ $\dim(\mathrm{im}\, T) \leq n$

$$\mathrm{rk}\, T \leq \dim V$$

(as well as $\mathrm{rk}\, T \leq \dim W$)

$T: V \to W$ lin. tr.

Nullspace (kernel) of T is

$$\{v \in V \mid T(v) = 0\}$$

In Ex 1. 0

2. 0

3. z-axis

5. 0

In these ex's, nullspace is a subspace of V.

Prop If $T: V \to W$ is a lin. transf, then $\ker T$ is a subspace of V.

Pf. $\neq \emptyset$: $T(0) = 0$, so $0 \in \ker T$.

$+$: If $v, v' \in \ker T$ want $v + v' \in \ker T$.

$T(v+v') = T(v) + T(v') = 0 + 0 = 0$ ✓

$\cdot$: If $v \in \ker T$, $c \in F$,

$T(cv) = cT(v) = c \cdot 0 = 0$

$\therefore cv \in \ker T$.

$T: V \to W$

$\cup$

$\ker T$ = nullspace T

$\dim(\ker T)$ = nullity of T

	nullity	+ rank	= dim V
Ex 1.	0	2	2
2.	0	3	3
3.	1	2	3
5.	0	2	2

Prop If $T: V \to W$ is a lin. transf. & V is a f.d.v.s., then

$$\mathrm{rk}(T) + \mathrm{nullity}(T) = \dim V.$$

Pf Let $n = \dim V$.

Let $k = \text{nullity}(T) \leq n$

$= \dim(\underbrace{\ker T}_{\text{subsp of } V})$

n vectors

Let $\underbrace{v_1, \ldots, v_k}_{\text{lin ind in } \ker T \subset V}$ be a basis of $\ker T$.

n-k more

$\exists\, v_{k+1}, \ldots, v_n$ st

$\to v_1, \ldots, v_n$ is a basis of V

$$\begin{array}{ccccccc} v_1, & \ldots, & v_k, & v_{k+1}, & \ldots, & v_n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ 0 & & 0 & T(v_{k+1}) & & T(v_n) \end{array}$$

$\to w_i = T(v_i)$ $\quad w_1, \ldots, w_k = 0$

$w_{k+1}, \ldots, w_n$ $\quad (n-k)$

$v \in V$ $\quad v = \sum_{i=1}^{n} a_i v_i$

$T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$

in im(T)

$= \sum_{i=1}^{n} a_i \underbrace{T(v_i)}_{w_i} = \sum_{i=1}^{n} a_i w_i$

$= \sum_{i=k+1}^{n} a_i w_i$

Image(T) is spanned by $w_{k+1}, \ldots, w_n$

<u>Claim:</u> $w_{k+1}, \dots, w_n$ are lin ind.
(& so are a basis of $\mathrm{im}(T)$).

Once we show this,

$\mathrm{im}(T)$ has a basis of $n-k$ elts

→ $\mathrm{rk}\, T = \dim(\mathrm{im}\, T) = n-k$

→ $\mathrm{nullity}\, T = \dim(\ker T) = k$

→ $\dim V = n$

$(n-k) + k = n.$

It remains to prove the claim.

For this:

$$0 = \sum_{i=k+1}^{n} a_i w_i = \sum_{i=k+1}^{n} a_i T(v_i)$$

$$= T\Big(\sum_{i=k+1}^{n} a_i v_i\Big) \Rightarrow \sum_{i=k+1}^{n} a_i v_i \in \ker T$$

Basis of $\ker T$ is $v_1, \dots, v_k$.

$$\therefore \sum_{i=k+1}^{n} a_i v_i = \sum_{i=1}^{k} b_i v_i$$

$$\to \sum_{i=1}^{k} b_i v_i - \sum_{i=k+1}^{n} a_i v_i = 0$$

$v_1, \dots, v_n$: basis of V, so lin ind

All $a_i = 0$, all $b_i = 0$.