Direct sum of subspaces

V v.s. / field F

$W_1, W_2 < V$ subspaces.

Can form

$$W = W_1 + W_2 \quad (\text{sum})$$

$$= \{ w_1 + w_2 \mid w_i \in W_i \}.$$

$$= \operatorname{span}(W_1 \cup W_2)$$

Say $W = W_1 \oplus W_2$

(direct sum) if

$$W_1 \cap W_2 = 0.$$

(Equiv: every $w \in W$ is uniquely $w_1 + w_2$, $w_i \in W_i$

$$\dim W_1 + \dim W_2 = \dim(W_1 \oplus W_2)$$

Can generalize to $\oplus$ of

more than two subspaces:

Say $W_1, \ldots, W_k \subset V$ subspaces.

Take $W = W_1 + \cdots + W_k$

Say $W = W_1 \oplus \cdots \oplus W_k$ <u>if</u> every elt of W is <u>uniquely</u> of the form $w_1 + \cdots + w_k$ with $w_i \in W_i$.

Equiv: $W_1 \cap W_2 = 0$

$$(W_1 + W_2) \cap W_3 = 0$$

$$\vdots$$

$$(W_1 + \cdots + W_{k-1}) \cap W_k = 0.$$

Equiv: If we pick a basis $\mathcal{B}_i$ of W_i for each i, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is a basis of W.

(See H+K, Lemma on p209 – pf is easy)

Ex. $V = \mathbb{R}^3$

Let W_1, W_2, W_3 be the x-, y-, z-axes respectively.

$V = W_1 \oplus W_2 \oplus W_3$

$v \in V \qquad v = (a, b, c)$

Can project onto each axis (each W_i)

1st projection

$P_1 : V \longrightarrow W_1 \subset V$

$(a, b, c) \longmapsto (a, 0, 0)$

2nd proj P_2, onto W_2 3rd proj P_3 onto W_3

Key property of P_i:

$$P_i \circ P_i = P_i$$

$$P_i\big|_{W_i} = id_{W_i}.$$

restriction $\leftarrow W_i = \mathrm{im}(P_i) = P_i(V)$

Motivated by this, we say that $T : V \longrightarrow V$ (lin transf) is a <u>projection</u> if $T \circ T = T$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{T^2}$

If $W = \mathrm{im}\, T$, T is a proj onto W; $T|_W = id_W$.

Here, let $N = \ker T$

Then: $V = W \oplus N$ (easy)

In Ex above: $V = \mathbb{R}^3$

$W = W_1 = x$-axis

$T = P_1 : (a,b,c) \mapsto (a,0,0)$

$N = yz$-plane $= W_2 \oplus W_3$

$V \stackrel{?}{=} W \oplus N = W_1 \oplus W_2 \oplus W_3$ ✓

Matrix of a projection

Take basis of W, $w_1 \ldots w_r$

— — — — — N, $z_1 \ldots z_s$

So $\dim V = \dim W + \dim N = r+s$

Union of these two bases is a basis of V. Use this.

Mx of T:

$$\begin{array}{c|c} & \begin{array}{cccc|ccc} w_1 & \cdots & & w_r & z_1 & \cdots & z_s \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_r \\ z_1 \\ \vdots \\ z_s \end{array} & \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 1 & & & \\ \hline \vdots & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right) \end{array} = \begin{array}{c} \\ w\text{'s} \\ z\text{'s} \end{array}\begin{array}{c} \begin{array}{cc} w\text{'s} & z\text{'s} \end{array} \\ \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

diagonal.

Eigenvalues 1, 0
$\uparrow$ $\uparrow$
w's z's.

Also $T^2 = T$ $\quad T^2 - T = 0$

T satisfies $x^2 - x = x(x-1)$

$p_T(x) \mid x(x-1)$ (roots 0, 1)

prod of distinct lin factors

$\therefore$ diag'ble. (eig. vals: 0, 1)

Thm (Th 9, H+K, p 212)

a) Say $V = W_1 \oplus \cdots \oplus W_k$

Then $\exists$ proj's $E_1, \ldots, E_k$

of V onto $W_1, \ldots, W_k$ resp.,

st $E_1 + \cdots + E_k = I$ and $E_i E_j = 0 \; \forall i \neq j$.

b) Conversely, if we have

$W_1, \ldots, W_k \subset V$ + proj's

$E_1, \ldots, E_k$ with these properties

then $V = W_1 \oplus \cdots \oplus W_k$

Ex $V = \mathbb{R}^3$, $\quad W_1, W_2, W_3$ axes.

$E_i = P_i$, proj onto W_i

(a, b, c)

$E_1 : \mapsto (a, 0, 0)$, $E_2 : \mapsto (0, b, 0)$, $E_3 : \mapsto (0, 0, c)$

$E_1 + E_2 + E_3 : \mapsto (a, b, c)$. $\quad E_1 + E_2 + E_3 = I$.

$i \neq j$ $\quad E_i E_j = 0$. ✓ $\quad V = W_1 \oplus W_2 \oplus W_3$

Key special case:

$$V = W_1 \oplus \cdots \oplus W_k$$

$T : V \longrightarrow V$ lin. tr.

Suppose each W_i is invariant under T:

$$T(W_i) \subset W_i \quad \forall i.$$

Then: $T_i := T|_{W_i} : W_i \longrightarrow W_i$

Say "T is the direct sum of $T_1, \ldots, T_k$"

Pick a basis $\mathcal{B}_i$ of W_i for each i.

Then: $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$
is a basis of V. (since $\oplus$)

Mx for T wrt $\mathcal{B}$?

Let A_i = mx for $T_i : W_i \to W_i$ wrt $\mathcal{B}_i$

$$\begin{array}{c|c} & \begin{matrix} \mathcal{B}_1 & \mathcal{B}_2 & \cdots & \mathcal{B}_k \end{matrix} \\ \hline \begin{matrix} \mathcal{B}_1 \\ \vdots \\ \vdots \\ \mathcal{B}_k \end{matrix} & \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} \end{array}$$

block-diagonal

$V = W_1 \oplus \cdots \oplus W_k$

$\leadsto$ $E_1, \ldots, E_k$: corresp projs.

Ex. $V = \mathbb{R}^3 = \underset{x-}{W_1} \oplus \underset{y-}{W_2} \oplus \underset{z-\text{axes}}{W_3}$

$T: V \to V$

$T(a,b,c) = (2a, 3b, 4c)$

Each W_i is T-invariant.

$E_1 \circ T(a,b,c) = E_1(2a, 3b, 4c) = (2a, 0, 0)$

$T \circ E_1(a,b,c) = T(a,0,0) = (2a, 0, 0)$

$E_1 \circ T = T \circ E_1$. $\quad E_i \circ T = T \circ E_i$

Thm (Th 10, H+K, p214)

Say $T: V \to V$ lin tr.

$V = W_1 \oplus \cdots \oplus W_k$, with corresp

projs $E_1, \ldots, E_k$ onto W_i's.

Then:

Each W_i is T-inv $\iff TE_i = E_iT$ for all i.

Again Ex. $V = \mathbb{R}^3$

$\oplus$ of 3 axes.: W_1, W_2, W_3

$T(a,b,c) = (2a, 3b, 4c)$

axes are invariant

T is diag'ble: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

W_i's are the eigenspaces

(invariant wrt T)

This generalizes:

<u>Thm</u> (Th 11, H+K, p215)

V fdvs $/F$. $T: V \to V$ linear.

$\lambda_1, \ldots, \lambda_k \in F$.

with multiplicities disregarded

T diag'ble w eigenvalues $\lambda_1 \ldots \lambda_k$

$\Longleftrightarrow$ $\exists$ maps $E_1, \ldots, E_k : V \to V$

st $E_1 + \cdots + E_k = I$,

$E_i E_j = 0$ if $i \neq j$,

& $T = \lambda_1 E_1 + \cdots + \lambda_k E_k$.

Here, E_i is the proj onto the eigenspace W_i for eigenvalue λ_i.

Pf is similar to above example
— H+K, pp 215–216 for pf
Also: $d_i = \dim W_i$.

In particular: $T : V \longrightarrow V$ (n dim'l/F)

with n distinct eigenvalues

$c_1, \ldots, c_n$. $\Leftrightarrow$ non-0 eigenvectors $v_1, \ldots, v_n$.

$$P_T(x) = (x - c_1) \cdots (x - c_n) = p_T(x)$$

Eigenspace W_i for c_i

$$W_i = \operatorname{Span}(v_i), \text{ 1-dim'l.}$$

$$V = W_1 \oplus \cdots \oplus W_n$$

W_i is invariant under T

$$T_i := T|_{W_i} : W_i \longrightarrow W_i$$

min poly $p_i(x) = x - c_i$

$$W_i = \ker(T - c_i I)$$
$$= \ker(p_i(T))$$

More generally:

V n dim'l vs $/F$

$T : V \longrightarrow V$, with min-poly $p_T(x)$

$(*)$ $p_T(x) = p_1(x)^{r_1} \cdots p_k(x)^{r_k}$

factorization into powers of distinct monic irred. poly's

(In above ex, each $p_i(x)$ had deg $=1$, & each $r_i = 1$)

Turns out: get same conclusion:

$V = W_1 \oplus \cdots \oplus W_k$,

Each $W_i = \ker\left(p_i(T)^{r_i}\right)$

Each W_i is invariant under T

& min'l poly of $T_i = T|_{W_i}$

is $p_i(x)^{r_i}$.

H+K, p 220, Th 12.

"Primary decomposition thm"

Pf uses projections — see H+K.

Pf in H+K shows more:

Cor, p 221 (of the pf):

Let $E_1, \dots, E_k$ be the projections assoc to $V = \underbrace{W_1 \oplus \cdots \oplus W_k}_{T\text{-inv subspaces}}$

Then each E_i is a poly in T

(i.e. $E = f_i(T)$ for some $f_i(x) \in F[x]$).

As a consequence:

Every lin op $U: V \to V$

that commutes with T

also " " each E_i;

& so W_i is then invariant under U.

At one extreme

ex. with distinct eigenvalues,

& n eigenspaces W_j of dim=1.

At other extreme:

$p_T(x)$ is irred, & just one factor (itself)

$$V = W_1$$

Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, rotation $/\mathbb{R}$, $p_A(x) = x^2+1$, irred.

Ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longleftrightarrow T$

one repeated factor

min. poly $p_A(x) = p_T(x) = (\underline{x-1})^2$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{I \\ \text{diagible}}} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B}$$

$B^2 = 0$

$\underbrace{\text{nil}}_{0}\underbrace{\text{potent}}_{\text{power}}$

$B^m = 0$.

A = diagible + nilpotent

these commute.

More generally, when can we write a mx A = diagible + nilpotent

commute?

Ans: OK if A is Δble

Equiv: $\underline{p_A(x) = \text{prod of lin. factors}}$

Recall:

1^{st} case: just one eigenvalue, c.

$$p_A(x) = (x-c)^r$$

$$A \sim \begin{pmatrix} c & & * \\ & \ddots & \\ 0 & & c \end{pmatrix} = \underbrace{\begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix}}_{cI} + \underbrace{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}}_{\text{nilp.}}$$

(rth power = 0)

commute

More gen'l case of Δ'ble mx's

$$p_A(x) = \prod_{i=1}^{k} (x - c_i)^{r_i}$$

We use primary decomp. to break V into a $\bigoplus$ of W_i's, $i = 1, \dots, k$.

On W_i, $p_{T_i}(x) = (x - c_i)^{r_i}$ & prev. case.

$$\begin{array}{c} W_1 \quad \cdots \quad W_k \\ \begin{pmatrix} \boxed{\;} & & 0 \\ & D & \\ & & \ddots \\ 0 & & D \end{pmatrix} \end{array}$$

block diag

as in special case

$D_i + N_i$; do this on each block.

diag nilp, commute Get $A = D + N$

More is true: D & N are unique & each is a poly in T (or A)

For details: H + K, Th 13, p 222.

Cor (H+K, p 223):
If F is alg. closed (e.g. $F = \mathbb{C}$). & V fdvs $/F$. Then every lin trT: $V \to V$ is Δble & its T can be written in above form.

F field

V n-diml vs $/ F$

$T: V \longrightarrow V$

$v \neq 0$, $v \in V$

Look at

$v, T(v), T^2(v), \ldots$

Since $\dim V = n$, the 1st $n+1$

$v, T(v), \ldots, T^{n-1}(v), T^n(v)$

are lin dep. $\exists\, a_i \in F$, not all 0, :

$a_0 v + a_1 T(v) + \cdots + a_n T^n(v) = 0$

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

$= f(T)(v)$

$f(T) = a_0 I + a_1 T + \cdots + a_n T^n$

annihilates v.

$I = \{ \text{all } f(x) \in F[x] \mid f(T) \text{ annih. } v \}$

Let $f_0(x)$ be the monic poly in I of least (minimal) degree, d.

By div. alg., f_0 divides <u>every</u> $f \in I$.

(I is an ideal; so = multiples of f_0)

Write $p_v(x) = f_0(x)$, $\deg = d$

$$\overset{\shortparallel}{\longrightarrow}\ x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$$

$v, T(v), \ldots, T^{d-1}(v)$ are lin <u>ind</u>.

(by minimality of d)

$v, T(v), \ldots, T^{d-1}(v), T^d(v)$ lin <u>dep</u>.

$p_v(T)(v) = 0$

Under T,

$v \mapsto T(v) \mapsto T^2(v) \mapsto \cdots \mapsto T^{d-1}(v) \mapsto T^d(v)$

$p_v(T)(v) = 0 \longrightarrow$ $T^d(v) =$ a lin comb of $v, T(v), \ldots, T^{d-1}(v)$

$T^d(v) = -c_0 v - c_1 T(v) - \cdots - c_{d-1} T^{d-1}(v)$ $(*)$

$\rightarrow$ $v, T(v), T^2(v), \ldots, T^{d-1}(v)$

span a subspace $W \subset V$

They are a basis for W (b/c lin indep).

$\dim W = d$

W is T-invariant:

STS basis elts of W are sent to " " ".

$T^i(v) \mapsto T^{i+1}(v)$, so ok for $i = 0, 1, \ldots, d-2$.

$T^{d-1}(v) \mapsto T^d(v) \in W$, by $(*)$.

$\therefore T|_W : W \to W$

(In H+K, $W = Z(v, T)$)

v: cyclic vector for T on W

What is the mx for $T|_W$ on W w.r.t. the above basis of W?

$$\begin{array}{c|cccccc|}
 & v & Tv & T^2v & \cdots & T^{d-2}v & T^{d-1}v \\
\hline
v & 0 & 0 & 0 & & & -c_0 \\
Tv & 1 & 0 & 0 & & 0 & -c_1 \\
T^2v & 0 & 1 & 0 & & & -c_2 \\
 & 0 & 0 & 1 & & & \vdots \\
\vdots & \vdots & \vdots & 0 & \ddots & & \vdots \\
 & \vdots & \vdots & \vdots & 0 & & \vdots \\
T^{d-1}v & 0 & 0 & 0 & & 1 & -c_{d-1}
\end{array}$$

$d \times d$ mx.

<u>Companion matrix</u> of $p_v(x)$

Its min poly of the mx is $p_v(x)$ of $T|_W$
of deg d

& this is also the ch. poly of $T|_W$.

In the special case when $W = V$:

$v, T(v), T^2(v), \ldots, T^{d-1}(v)$ span V.

& are a basis of V

$$V = Z(v, T) \text{ in this case.}$$

The companion mx is the mx of T on V.
in this basis

– – –

More generally:

$$W = Z(v, T) \subset V$$

T-invariant subsp, arising from $v \neq 0$.

Can also pick other vectors...

$v_1, v_2, \dots, v_r \qquad W_i = Z(v_i, T) \subset V$

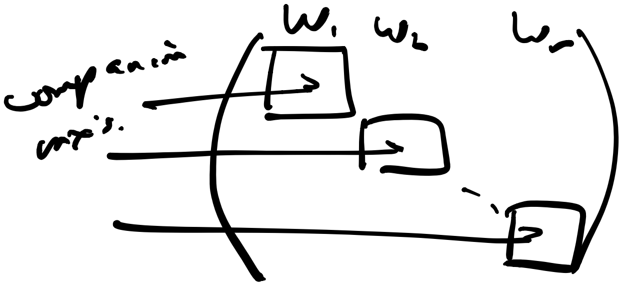
In fact: $V = \bigoplus_{i=1}^{r} W_i = W_1 \oplus \dots \oplus W_r$.

Each W_i has a cyclic basis

dim d_i $\qquad v_i, Tv_i, \dots, T^{d_i-1}v_i$

Get a companion mx for each

each is invariant under T



$p_i(x)$ = min'l poly for $T|_{W_i}$

coeff's go in last column of comp. mx for W_i.

Can be made unique:

e.g. if we require $p_r(x) \mid p_{r-1}(x) \mid p_{r-2}(x) \mid \dots \mid p_1(x)$

(H+K)

min'l poly of T

rat'l canonical form.

Min poly of T: $p_T(x) = \prod_{j=1}^{s} q_j(x)^{e_j}$ irred.

(To make unique: Cd also: min poly of each W_i

is $q_j(x)^{a_j}$ $\quad 0 \le a_j \le e_j$, for some j

H+K $\qquad \forall j \ \exists$ block W_i with $a_j = e_j$.)

Thm 5, p238.

At one extreme: $W = V$, one block.

rat'l can. form = one companion mx

Ex. $\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$

min. poly: $x^2 + 3x + 2$.

At other extreme: Each block is 1×1 (c).

$\dim W_i = 1$.

Ex. $\left(\begin{array}{c|c} \boxed{c} & 0 \\ \hline 0 & \boxed{c} \end{array}\right)$

$p_1(x) = x - c = P_1(x)$

$p_2(x) = x - c = P_2(x)$

$p_T(x) = x - c$

$P_T(x) = (x-c)^2$

This works over <u>any</u> field.

Another canonical form:

Jordan canonical form:

Works if A Δble. ($A \leftrightarrow T$)

Equiv: if $P_A(x)$ = prod. of lin factors / F.

" : - - $p_A(x)$ = - - - - -

In partic: always works if

F alg. closed (e.g. $F = \mathbb{C}$)

Again: Can put into blocks $\longleftrightarrow$ T-inv subspaces W_i.

$$T_i = T|_{W_i} : W_i \to W_i$$

$$\begin{array}{c} \quad W_1 \quad W_2 \; \cdots \\ \begin{pmatrix} \square & & & \\ & \square & & \\ & & \ddots & \\ & & & \square \end{pmatrix} \end{array}$$

Each block W_i is of the form:

$$\begin{pmatrix} c & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & c \end{pmatrix} \qquad d_i \times d_i$$

<u>**Jordan block**</u>. (1×1 block is just (c))

Ex. A is 2×2 s.t. Δ'ble.

$$A \sim \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad P_A(x) = (x-a)(x-c)$$

Eigenvalues: a, c.

<u>If</u> A diag'ble, $A \sim \begin{pmatrix} \boxed{a} & 0 \\ 0 & \boxed{c} \end{pmatrix}$

2 Jordan blocks, each 1×1.

min poly = char poly = $(x-a)(x-c)$

<u>If</u> A <u>not</u> diag'ble: must have $a=c$.

$$A \sim \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \qquad b \neq 0$$

since not diag'ble

$v_1 \quad v_2$

interchange

$$A \sim \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}.$$

Replace new v_1 by $\frac{1}{b} v_1$:

$$A \sim \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix},$$ one Jordan block.

In gen'l: $\begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix}$ if $\exists$ ble (if F alg cl).

§ 7.3 of H+K

Pf uses: cyclic subspaces + primary decomposition.