Math 314, Spring 2019
Linear Algebra
Prof. David Harbater

Study of vector spaces
& related concepts.
Assumes familiarity with basics
from Math 240 (vectors in n-space, matrices)
— will review.
Both theory & computations.
Alternative: Math 312
(312 has less theory, more computations)
(312 doesn't give Math major credit)

314 TA's: labs, 2hrs/wk
Ben Foster 402, 403 Tu Th 6:30-8:30
Juan Lanfranco 404, 405
Labs are <u>active</u>. (start Th this wk)
Weekly problem sets due in lab.
Three exams (not cumulative), in class
No final exam.

See web page for more info:
www.math.upenn.edu/~harbater
→ Courses → Math 314 Spring '19
Text: Hoffman & Kunze, 2nd ed.
MPA Library, bookstore web.

314 is pre-requisite for
Math 370-1. Helpful for 360-1.
more theoretical
than 314;
also assume 240.

See web pg for problem sets, exam & grading info.
(Grades: exams, homework, participation)

Grades: posted on Canvas.
Study groups, for homework.
Sign-up page:
indicate by lab #.

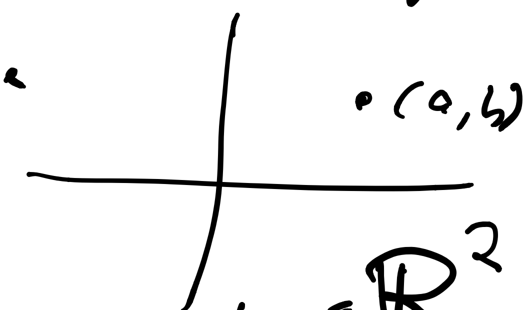
Today: Vectors, scalars;
Vector spaces, fields.

Vector spaces

Vectors in plane

— — — — 3-space

plane

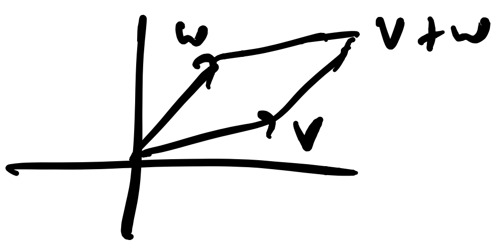


pair of real #s

plane = $\mathbb{R}^2$

3-space (a,b,c) $\mathbb{R}^3$

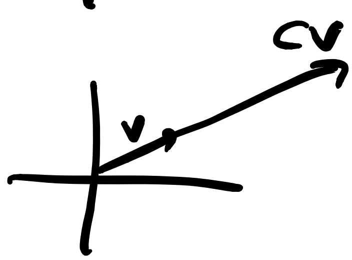
n-space $\mathbb{R}^n$



$v = (a_1, a_2)$

$w = (b_1, b_2)$

$v+w = (a_1+b_1, a_2+b_2)$



$c\,(a_1, a_2) = (ca_1, ca_2)$

OK in $\mathbb{R}^n$

Vectors & scalars obey alg. laws

Scalars $\mathbb{R}$

$+$, $\cdot$

both satisfy comm. law

& assoc. law

Identities $+$ 0

$a+0=a$

$\cdot$ 1

$a \cdot 1 = a$

Inverses $+$ a $-a$

$a+(-a)=0$

exc. for 0 $\longrightarrow$ $\cdot$ a $a^{-1} = 1/a$

$a \cdot a^{-1} = 1$

distrib law of $\cdot$ over $+$.

Summarize: $\mathbb{R}$ form a <u>field</u>

Other fields?

$\mathbb{C}$ complex #s

$a+bi$, $a, b \in \mathbb{R}$

$\mathbb{Q}$ rational #s

$\frac{a}{b}$ a, b integers, $b \neq 0$

$a, b \in \mathbb{Z}$

$\mathbb{F}_2$ field of 2 elements

$\mathbb{F}_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

3, 4, 5, 6, 7, 8, 9

Vector spaces

$\mathbb{R}^n$ addition
scalar mult

Alg. properties?

$+$ assoc, comm, identity, 0

inverse v $-v$

Scalar mult

$c \in \mathbb{R}$ $v \in \mathbb{R}^n$

$cv \in \mathbb{R}^n$

scalar · vector = vector

"assoc" $(c_1 c_2)v = c_1(c_2 v)$

"id" $1v = v$

"distrib laws"

$c(v+w) = cv + cw$

$(c_1 + c_2)v = c_1 v + c_2 v$

Can check by coords.

$v + w = w + v$

$v = (a_1, \ldots, a_n)$, $w = (b_1, \ldots, b_n)$

$v + w = (a_1 + b_1, \ldots, a_n + b_n)$

$= (b_1 + a_1, \ldots, b_n + a_n)$

$= w + v$

Summarize: $\mathbb{R}^n$ is a <u>vector space</u>, with scalars $\mathbb{R}$.

Other ex's of V.S.:
(over $\mathbb{R}$

1) 2×3 real matrices

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2) Sequences of real #s

$a_1, a_2, a_3, \ldots$

$0 = 0, 0, 0, \ldots$

3) Real-valued functions on $[0,1]$

$\sin x + e^x$

0 : identically 0

4) Polys with real coeffs

$3x^2 + 2x - 5$

0 : 0-poly

VS's over other fields?

$\mathbb{C}^n$ n-tuples of cx #s $/\mathbb{C}$

$\mathbb{Q}^n$ — — — rat'l #s $/\mathbb{Q}$

F^n ----- elts of F $/F$

field

2×3 cx matrices

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad /\mathbb{C}$$

field

Seq's of elements in F $/F$

From basic properties of a field (or a V.s.), can deduce other properties.

Ex. F field. $a \in F$

$\Rightarrow 0 \cdot a = 0$

Proof $0 \cdot a \overset{\text{add. id.}}{=} (0 + 0) \cdot a$

$\overset{\text{dist.}}{=} 0 \cdot a + 0 \cdot a$

$0 \cdot a + (-(0 \cdot a))$

$= (0 \cdot a + 0 \cdot a) + (-(0 \cdot a))$

LHS RHS

$LHS \overset{\text{add. inv.}}{=} 0$

assoc.

$RHS = 0 \cdot a + (0 \cdot a + (-(0 \cdot a)))$

$= 0 \cdot a + 0$ (add inv)

$= 0 \cdot a$ (add id)

$\therefore 0 \cdot a = 0.$ ✓

V. S.: $0 \cdot v = 0$

Scalar 0 Vector 0

Pf is same, if replace $a \in F$ by $v \in V$

Another property of fields:

$a \in F \qquad (-1)a = -a$

Pf. WTS

$$(-1)a + a = 0$$

LHS

$$\begin{aligned} \text{LHS} &\overset{\text{comm}}{=} a + (-1)a \\ &\overset{\text{mult id}}{=} 1 \cdot a + (-1)a \\ &\overset{\text{dist}}{=} (1 + (-1))a \\ &= 0 \cdot a \quad \text{add inv} \\ &= 0 \quad \text{prev result} \end{aligned}$$

V. S. $\quad (-1)v = -v$

Pf is "same", replace a by v.

One vectors space inside another V. S.

– Subspace

Ex. (plane $\ni 0$) $\subset \mathbb{R}^3$

(solns to $f'' + 3f' + 2f = 0$) $\subset$ (cont fns)

(conv. seq's) $\subset$ (sequences)

(upper Δr 3x3 mxs) $\subset$ (3 x 3 mxs)

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix}$$

Subspace $\quad W \subset V$

V. S. with same $+$, sc. mult, 0

V. S. $+$, sc. mult, 0

How to check if W is a subspace?

Some properties are inherited from V to W

(assoc, "dist", etc.)

Still need:

$W \neq \emptyset$

$\rightsquigarrow 0 \in W$

closed under $+$, sc. mult

$\rightsquigarrow$ add. inv's.

Suff. to check:

$\rightarrow W \neq \emptyset$, closed under $+, \cdot$

Why? $W \neq \emptyset \Rightarrow \exists w \in W$

$0 \cdot w \in W$ cl. under $\cdot$

$\overset{\shortparallel}{0}$ ✓

$w \in W \Rightarrow -w \in W$

cl. under. $\quad (-1)w$

A way to get a subspace of v.s. V

Take a subset $S \subset V$.

$$a_1 s_1 + a_2 s_2 + \cdots + a_r s_r$$

linear comb of finitely many elements of S.

Let W = the set of all these.

W is a subspace of V; the subspace spanned by S.

$V = \mathbb{R}^3$

$S = \{v, w\}$

Get: $W =$ Plane, in genl.

Or: $W =$ Line, if

v or w is mult of the other.

Or: $W = 0$ if $v = w = 0$.

In general, for any

set $S \subset V$,
vs

get $W \subset V$, a subspace

subspace
spanned by S

Note: W is non-empty, &
closed under $+, \cdot$.