Math 104, Lecture 2 Problem Set 14 Spring 2011

Due Tues., April 26, 2011 (with automatic extension until Thurs., April 28).

Read sections 10.2-10.4 of Stewart. Optional: read section 10.5.

Do the first five core problems for sections 10.2, 10.3, and 10.4; but do not hand them in.

The problems below are to be handed in.

Part A: Do the following problems from Stewart:

Section 12.10: 71.
Chapter 12 review, Concept Check questions, p. 794: 1-4.
Chapter 12 review exercises, pages 795-796: 1-4, 11-14, 40-43.
Section 10.3: 1, 2, 5, 11, 13, 19.
Chapter 10 review exercises, pages 651-653: 6, 15.

Part B: Do the following problems:

1. In a power series, one can substitute complex numbers as well as real numbers for the variable. (Recall that a complex number has the form $a+bi$, where $i^2 = -1$.)

a) Find the power series for e^{ix}. [Hint: Substitute ix for x in the series for e^x.] Also find the series for $\cos(x) + i\sin(x)$. What do you notice?

b) Do the same for the series for e^{-ix} and for $\cos(x) - i\sin(x)$.

c) Use what you noticed to evaluate $e^{i\pi}$, $e^{i\pi/2}$, and $e^{i\pi/4}$ explicitly.

d) Find a complex number whose square is i. [Hint: Use part (c).]

2. The *hyperbolic functions* cosh and sinh are defined by

$$\cosh(x) = (e^x + e^{-x})/2 \quad \text{and} \quad \sinh(x) = (e^x - e^{-x})/2.$$

a) Express the derivatives of cosh and sinh in terms of hyperbolic functions, and simplify the expression $\cosh^2(x) - \sinh^2(x)$. (Here $\cosh^2(x)$ means $\big(\cosh(x)\big)^2$, and similarly for sinh.)

b) Show that the functions $\cosh, \sinh$ each satisfy the differential equation $f''(x) = f(x)$. Also find the values of $\cosh(0)$ and $\sinh(0)$.

c) Using your answers in parts (a) and (b), find the Maclaurin series for $\cosh(x)$ and $\sinh(x)$.

d) Describe the analogy between the pair of functions $\cosh, \sinh$ and the pair of functions $\cos, \sin$.

e) Using power series expansions, show that $\cos(x) = \cosh(ix)$ and $\sin(x) = \sinh(ix)/i$. [Note: This shows that one can avoid trigonometric functions and just use exponential functions, if complex numbers are allowed.]

3. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2(y^2+1)}{(x^2-1)}$.

a) Find all solutions to this differential equation.

b) A certain solution to this equation takes on the value $y = 0$ when $x = 2$. Find the value of y when $x = 3$ (both the exact value, and to three decimal places).

4. Consider these two initial value problems:

$$\frac{dP}{dt} = \frac{P}{20}, \quad P(0) = 100 \tag{1}$$

$$\frac{dQ}{dt} = \frac{Q}{20}\left(1 - \frac{Q}{1000}\right), \quad Q(0) = 100 \tag{2}$$

a) Describe situations in which these equations apply.

b) Find the solutions to these initial value problems, and sketch the graphs of these solutions.

c) For what value of t is $P(t) = 500$? For what value of t is $Q(t) = 500$? Which value of t is greater? How could you have predicted this in advance?

d) For what value of t (if any) is $P(t) = 2000$? For what value of t (if any) is $Q(t) = 2000$? How could you have predicted in advance how these two answers differ?

5. Let $p(Z) = AZ^2 + BZ + C$, where A, B, C are constants. Let r_1, r_2 be the two solutions to the quadratic equation $p(Z) = 0$, with $r_1 \neq r_2$.

a) Find all constants k such that e^{kx} is a solution to the differential equation $Ay'' + By' + Cy = 0$.

b) Find all the solutions you can to this differential equation. [Hint: If $y = f(x)$ and $y = g(x)$ are solutions, and if a is any number, then $y = f(x) + g(x)$ is also a solution and so is $y = af(x)$.]

c) Explain the relationship of part (b) to problem 5(c) on Problem Set 13.

d) Verify that $y = x^2 - 1$ is a solution to the differential equation

$$y'' + y' - 6y = 8 + 2x - 6x^2.$$

e) Find all the solutions you can to this differential equation. [Hint: Combine your answers in parts (c) and (d).]