Math 104, Lecture 2 — Problem Set 8 — Spring 2011

Due the week of March 14, 2011, in recitation.

Read Stewart, sections 9.3-9.5.

Work through the following core problems, but do not hand them in:
Section 9.3: core problems beginning with #21.
Section 9.4: first four core problems.
Section 9.5: first two core problems.

The problems below are to be handed in.

Part A: Do the following problems from Stewart:

Chapter 9 review, pages 598-599: concept review problems 1, 2, 4, 8; exercises 13, 14.

Chapter 9 problems plus: exercises 1, 2.

Part B: Do the following problems:

1. a) Explain how one can define the arclength of a curve that is not necessarily given as the graph of $y = f(x)$ (because vertical lines can meet the graph more than once). This should be expressed as a limit of a sum, just as in the definition of arclength for the graph of a function.

b) Draw the portion of the graph of the curve $y^2 = x^3$ that lies to the left of the line $x = 1$. Explain why this is not the graph of a function. (Hint: consider all possible values of y.) Find the arclength of what you drew.

c) Draw the graph of the curve $y^2 = (1 - |x|)^3$ and find its arclength. (You may wish to consider separately the part where $x \geq 0$ and the part where $x \leq 0$.)

2. The Koch snowflake curve is obtained in the following way: First draw an equilateral triangle. Then modify it by adding on a smaller equilateral triangle protruding from the middle third of each edge; the modified object is the boundary of what you obtain. Then modify it again in the same way. This is repeated infinitely many times, and the Koch snowflake curve is the limit. (You can see an animation of the first seven steps of the construction at `http://upload.wikimedia.org/wikipedia/commons/f/fd/Von_Koch_curve.gif`.)

a) Find the area enclosed by the curve at the n-th step. Also find the arclength at the n-th step.

b) What happens to the area and the arclength as $n \to \infty$? Explain and resolve the apparent contradiction.

3. a) Suppose that a region in the plane is symmetric about the y-axis, in the sense that it is unchanged if reflected in that axis. Find the x-coordinate of the centroid of this region. Explain.

b) Find the centroid of the region lying above the graph of $y = x^n$ and below the graph of $y = 1$, where n is an even positive integer. (Hint: Use part (a) to simplify your computation.) What happens to the centroid as $n \to \infty$? Explain why this is plausible geometrically. (Hint: What does the graph of $y = x^n$ look like, for $-1 \leq x \leq 1$, when n is a very large even integer?)

c) Find the centroid of the region enclosed by the "fat circle" $x^n + y^n = 1$, where n is an even positive integer. (Hint: By using the idea of part (a) twice, you can avoid doing any computations.)

4. Consider the result of tossing a fair coin n times. The outcome can be expressed as a sequence like $HHTHTT\ldots TH$, to indicate the order in which heads and tails come up.

a) How many possible outcomes are there, for a given value of n? (Each sequence represents a different outcome.) What is the probability that any one given outcome happens?

b) If $n = 4$, what is the probability that there are no heads? that there is exactly one head? that there are exactly two heads? that there are exactly three heads? that there are all heads? Letting $f(i)$ denote the probability of getting exactly i heads, what is $\sum_{i=0}^{4} f(i)$? Why is this expected?

c) Illustrate the probability outcomes in part (b) by putting a point at $(i, f(i))$ for $i = 0, 1, 2, 3, 4$, where $f(i)$ is the probability of getting exactly i heads. What do you notice about the picture?

c) Redo parts (b) and (c) with $n = 5$. What do you think will happen to the shape of the probability graph as $n \to \infty$?