Math 104, Lecture 2 Problem Set 1 Spring 2011

Due the week of January 17, 2011, in recitation.

Review Stewart, chapters 2 and 3.

Do the diagnostic test to see if you have sufficient background for Math 104. (It is at ⟨`http://www.math.upenn.edu/~harbater/104diagnostic.pdf`⟩.) If you are unsure whether you did well on this test, you can hand it in to your TA to be graded, for diagnostic purposes only (not counted in your grade). Otherwise, you need not hand it in.

Problems to hand in, on the review material:

1. Determine whether the following functions are continuous. If they are, cite one or more theorems from Stewart, Chapter 2, to justify your conclusion. Also indicate any points at which the function is not continuous, including points at which it is not defined.

a) $f(x) = x^{300} + x - 1$ b) $f(x) = \dfrac{\sin(x) + \cos(x)}{x^2}$ c) $\sin(x^2 - 1)$

2. Determine whether the following limits exist. If they do, evaluate the limits. Explain your answers.

a) $\lim_{x\to 1} x^9 - x^5 - 1$

b) $\lim_{x\to 2}(x+2)/(x-2)$

c) $\lim_{x\to 0} \sin(x) + \cos(x) + |x|$

3. Define a function f as follows: $f(x) = 1/x^2$ if $x < 0$; $f(x) = 1/x$ if $0 < x < 1$; $f(x) = 0$ if $x = 1$; $f(x) = x$ if $1 < x \leq 2$; $f(x) = 2$ if $2 < x < 3$; $f(x) = 0$ if $x \geq 3$.

a) Draw the graph of $y = f(x)$.

b) For which real numbers a is $f(a)$ defined?

c) For which a does $\lim\limits_{x\to a} f(x)$ exist?

d) At which a is $f(x)$ continuous?

e) At which a is $f(x)$ differentiable?

4. For any real number x, define $f(x)$ to be the smallest integer that is greater than or equal to x. (For example, $f(1.3) = 2$ and $f(-1.3) = -1$.)

a) Evaluate $f(.01)$, $f(0)$, and $f(-.01)$. Does $\lim\limits_{x\to 0} f(x)$ exist? Is f continuous at $x = 0$? Explain your answer.

b) Let $g(x) = f(x^2)$. Evaluate $g(.01)$, $g(0)$, and $g(-.01)$.

c) Does $\lim\limits_{x\to 0} g(x)$ exist? Is g continuous at $x = 0$?

5. Consider the function $f(x) = 2^{-1/x^2}$.
   a) For which x is $f(x)$ defined?
   b) What can you say about the value of $f(x)$ if x is close to 0?
   c) Based on your answer in (b), what can you say about $\lim_{x\to 0} f(x)$?

6. a) Explain why there is a real number x between 1 and 2 such that $x^3+x = 7$. (Hint: Consider the function $f(x) = x^3+x$, and use an important theorem.)

   b) Is there a real number x between 1 and 2 such that $x^3 + x = 100$? Explain why or why not.

7. a) Draw the graph of $y = x^2$, and explain *from the graph* where the slope of the tangent line is negative, where it is zero, and where it is positive. Now compute the derivative of x^2 and explain how your answer gives another way to see this behavior of the slope, even if you hadn't drawn the graph.
   b) Do the same with the graph of $y = \sin(x)$ from $x = 0$ to $x = 2\pi$.

8. a) Evaluate the derivative of $x\sin(x) + \cos(x^2) - 6243$ at $x = 0$.
   b) Do the same for $(x^5 + 1)^{10}$. (Hint: Do *not* expand this expression.)

9. a) Can the derivative of $\cos^2(x)$ be equal to 3 for some value of x? Explain.
   b) Can the derivative of x^3+x be equal to 0 for some value of x? Explain.