Name: Group: MATH 104

Moments and Centers of Mass

Snapshot

Major Concept: The center of mass of a thin wire or flat plate can be computed with one-dimensional integrals.

Before You Begin: Discuss in your group the various physical interpretations of the center of mass and how to find the center of mass of a finite collection of masses arranged on a line.

Standards for Practice and Evaluation: Be able to quickly identify when a question relates to center of mass. Know and use related terminology correctly, and be able to set up the necessary integrals, bearing in mind that not all slicing directions are always possible or useful.

Worksheet Objective

Explain the physical interpretation of centroids and centers of mass.

Set up calculations for centroids and centers of mass in various contexts, including planar regions with variable density and curves.

Calculate centroids and centers of mass using symmetries to simplify calculations when possible.

Use Physical Meanings to Calculate

Symmetry: If the shape (and the density if it's not uniform) have an axis with reflection symmetry, then the center of mass must lie somewhere on that axis.

Translation: If a shape is translated by a fixed amount in the x or y directions, the centroid and center of mass are translated by the same amount in the same direction.

Strips of Constant Density: If a thin strip has constant density along its entire length, then its center of mass is the literal center point of the strip. If possible, slice your region so that density is constant along the slice itself.

Integration as an Area: The integral of a function between to points is equal to the area under the graph (between the graph and the x-axis, that is) of that function between those two points. If the geometry makes the area easy to determine, skip the integral.

Remember | Understand | Apply | Analyze | Evaluate | Create

What's the difference between a center of mass and a centroid?

Remember | **Understand** | Apply | Analyze | Evaluate | Create

The book uses the integral notation

$$M_x = \int \widetilde{y}\ dm \text{ and } M_y = \int \widetilde{x}\ dm$$

to define the x- and y-moments of an object like a thin, two-dimensional plate.

- What is the physical meaning of dm? How does one compute it with or without a density function δ?

- When the density function δ only depends on one variable, how do you decide which way to slice the plate for integration?

Remember | Understand | Apply | Analyze | Evaluate | Create

- When slices are vertical lines, what are the physical meanings of $\widetilde{x}$ and $\widetilde{y}$? How would you compute the correct $\widetilde{x}$ and $\widetilde{y}$ for the integrals?

- When slices are horizontal lines, what are the correct formulas for $\widetilde{x}$ and $\widetilde{y}$?

Remember | Understand | Apply | Analyze | Evaluate | Create

Draw a picture of the region in the plane bounded below by $y = x^2$ and above by $y = \frac{1}{2}(x^2 + 1)$. Find the center of mass. Clearly identify the direction of slicing and use the same direction of slicing for all the integrals you compute.

Remember | Understand | **Apply** | Analyze | Evaluate | Create

Find the center of mass of a thin plate bounded by the curves $y = 0$, $x = 1$, and $y = x$ assuming that the density of the plate is given by $\delta = 1 - x$.

Remember | Understand | Apply | Analyze | Evaluate | Create

What is the formula for the center of mass of a curve? Give the formula assuming constant density and then determine how the formula would change for a curve of variable density.

Remember | Understand | **Apply** | Analyze | Evaluate | Create

Find the x-coordinate of centroid of the curve $y = \frac{x^3}{6} + \frac{x^{-1}}{2}$ between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Important! You are always allowed to use symmetry to simplify calculations. Often such considerations can make it possible to far less work than you might otherwise need to.

Remember | Understand | Apply | Analyze | **Evaluate** | Create

Find the centroid of the region bounded by the curves $y = 1 + \sin x$, $y = -1 - \sin x$, $x = 0$ and $x = \pi$.

Remember | Understand | Apply | Analyze | **Evaluate** | Create

Find the centroid of the region bounded below by $y = x^2$ and above by $x = y^2$.

Review Guidance

Compare the answers to each of the exercises you worked above, making note of the differences between the various exercises and how those differences influenced your approach. For example, when the density is a function of one variable, it is often best to take that variable to be the slicing variable (i.e., to imagine that the plate, etc., is composed of small slabs stacked along that variable's axis).

Practice all the applications of integration we have covered. Use questions which mix topics together and give minimal instructions (e.g., if it says "Use the shell method to..." then it's not good practice for exams because we expect you to make your own choices for how to best solve problems).

Prepare for the next topic in the course: techniques of integration. We will see many ways in which this new material interacts with ideas we have already encountered.