Nonconcentration, L^p-Improving Estimates, and Multilinear Kakeya

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0. The Problem of Geometry in Fourier Analysis

- There are a number of deeply geometric operators, integrals, etc., in harmonic analysis. The structure is usually defined by smooth spaces with measures and maps between those spaces.
- With minimal structure, it's often not even clear what key quantities are or how to proceed.
- Introducing artificial structures (e.g., coordinates) reduces to more familiar settings, but doing so breaks fundamental invariances of the problem and risks missing important features.
- There is a third way: introduce artificial auxiliary structures and study the group action induced by transformations of these structures.

An Example: An n-dimensional vector space V is called a **Peano space** when it is equipped with a nontrivial alternating n-linear form $[v_1, \ldots, v_n]$. It is natural to use bases with $[v_1, \ldots, v_n] = 1$.

Suppose $U : V \times V \to \mathbb{R}$ is a symmetric bilinear functional on a real Peano space V such that $U(v, v) \geq 0$ for all $v \in V$. Is there an easy way to detect degeneracy or nondegeneracy of U?

- Option 1: Do calculations invariant under natural choices:

$$\det U := \det[U(v_i, v_j)]_{i,j=1,\ldots,n}.$$

- Option 2: Do anything, then optimize over all choices. E.g.:

$$\inf_{[v_1,\ldots,v_n]=1} \left[\sum_{j=1}^{n} |U(v_i, v_i)|^p \right]^{1/p}, \quad p \in (0, \infty].$$

Theorem: $\det U = \inf_{[v_1,\ldots,v_n]=1} \left[\frac{1}{n} \sum_{j=1}^{n} |U(v_i, v_i)|^p \right]^{\frac{n}{p}}.$

1. Brascamp-Lieb Constant

H, H_j : Hilbert(?) spaces of dimension d, d_j, $j = 1, \ldots, m$;
π_j : Surjective linear maps $H \to H_j$; θ_j: constants in $[0, 1]$.

Brascamp-Lieb Inequality

$$\mathrm{RBL}(\pi, \theta) \int_H \prod_{j=1}^{m} (f_j \circ \pi_j)^{\theta_j} \leq \prod_{j=1}^{m} \left(\int_{H_j} f_j \right)^{\theta_j}$$

Bennett, Carbery, Christ, and Tao (2005)

$\mathrm{RBL}(\pi, \theta) > 0$ if and only if

$$\dim V \leq \sum_{j=1}^{m} \theta_j \dim \pi_j(V) \text{ for all } V \subset H$$

with equality when $V = H$.

Bennett, Bez, Cowling, Flock (2016)

Fixing dimensions and θ, $\mathrm{RBL}(\pi, \theta)$ is continuous in π.

$$\mathrm{RBL}(\pi,\theta)\int_H \prod_{j=1}^m (f_j\circ\pi_j)^{\theta_j} \leq \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{\theta_j}$$

Lieb (1990): Gaussians extremize the inequality

$$\mathrm{RBL}(\pi,\theta) = \inf_{\substack{A_j\in \mathrm{GL}(H_j)\\ j=1,\ldots,m}} \frac{\left(\det \sum_{j=1}^m \theta_j \pi_j^* A_j^* A_j \pi_j\right)^{\frac{1}{2}}}{\prod_{j=1}^m |\det_{H_j} A_j|^{\theta_j}}$$

Change determinant to infimum of trace:

$$[\mathrm{RBL}(\pi,\theta)]^{\frac{2}{d}} = \inf_{\substack{A_j\in \mathrm{GL}(H_j)\\ A\in \mathrm{SL}(H)}} \frac{d^{-1}\mathrm{tr}\sum_{j=1}^m \theta_j A^*\pi_j^* A_j^* A_j \pi_j A}{\prod_{j=1}^m |\det_{H_j} A_j|^{2\theta_j/d}}$$

$$= \inf_{\substack{A_j\in \mathrm{SL}(H_j)\\ A\in \mathrm{SL}(H)\\ t_j\in(0,\infty)}} d^{-1} t^{-\frac{2\theta}{d}} \sum_{j=1}^m \theta_j t_j^{\frac{2}{d_j}} |||A_j\pi_j A|||^2$$

where $|||\cdot|||$ is the Hilbert-Schmidt (sum of squares) matrix norm.

Keep Going: Use AM-GM Inequality again to eliminate t_j:

$$[\mathrm{RBL}(\pi,\theta)]^{\frac{2}{d}} = \inf_{\substack{A_j \in \mathrm{SL}(H_j) \\ A \in \mathrm{SL}(H) \\ t_j \in (0,\infty)}} d^{-1} t^{-\frac{2\theta}{d}} \sum_{j=1}^{m} \frac{\theta_j d_j}{d} t_j^{\frac{2}{d_j}} \left(\frac{d}{d_j} |||A_j \pi_j A|||^2 \right)$$

$$= \left(\prod_{j=1}^{m} d_j^{-\frac{\theta_j d_j}{d}} \right) \inf_{\substack{A_j \in \mathrm{SL}(H_j) \\ A \in \mathrm{SL}(H)}} \prod_{j=1}^{m} |||A_j \pi_j A|||^{\frac{2\theta_j d_j}{d}}$$

Assuming rational θ_j, there exist integers N, N_j such that

$$\frac{\theta_j d_j}{d} = \frac{N_j}{N}, \qquad j = 1, \ldots, m,$$

$$[\mathrm{RBL}(\pi,\theta)]^{\frac{N}{d}} = \left(\prod_{j=1}^{m} d_j^{-\frac{N_j}{2}} \right) \inf_{\substack{A_j \in \mathrm{SL}(H_j) \\ A \in \mathrm{SL}(H)}} \prod_{j=1}^{m} |||A_j \pi_j A|||^{N_j}.$$

For integers $N = N_1 + \cdots + N_m$,

$$[\mathrm{RBL}(\pi, N)]^{\frac{N}{d}} \left\| \prod_{j=1}^m f_j \circ \pi_j \right\|_{L^{d/N}(H)} \leq \prod_{j=1}^m ||f_j||_{L^{d_j/N_j}(H_j)}.$$

Define $\Pi_N : H^N \times H_1^{N_1} \times \cdots \times H_m^{N_m} \to \mathbb{R}$ by the formula

$$\Pi_N(x^{(1)}, \ldots, x^{(N)}, x_1^{(1)}, \ldots, x_1^{(N_1)}, \ldots, x_m^{(N_m)}) \\ := \left\langle \pi_1 x^{(1)}, x_1^{(1)} \right\rangle_{H_1} \cdots \left\langle \pi_1 x^{(N_1)}, x_1^{(N_1)} \right\rangle_{H_1} \cdots \left\langle \pi_m x^{(N)}, x_m^{(N_m)} \right\rangle_{H_m}$$

and let $\mathcal{G} := \mathrm{SL}(H) \times \mathrm{SL}(H_1) \times \cdots \times \mathrm{SL}(H_m)$. Then

$$[\mathrm{RBL}(\pi, N)]^{\frac{N}{d}} = \left(\prod_{j=1}^m d_j^{-\frac{N_j}{2}} \right) \inf_{G \in \mathcal{G}} |||\rho_G \Pi_N|||,$$

ρ_G is the action of G on $H^N \times \cdots \times H_m^{N_m}$, $||| \cdot |||$ is Hilbert-Schmidt.

A Good Question: Why did we do this lovely calculation?

2. Geometric Nonconcentration Inequalities

Suppose Φ is some polynomial function from $(\mathbb{R}^n)^k$ into $\mathbb{R}^m$. $|\Phi(x_1, \ldots, x_k)|$ measures nondegeneracy of k-point configurations.
Example: if $\varphi(x) := (x^\alpha)_{|\alpha| \leq d}$, then

$$\Phi(x_1, \ldots, x_N) := \det(\varphi(x_1), \ldots, \varphi(x_N)) = 0$$

iff $x_1, \ldots, x_N$ lie on some real algebraic variety of deg. $\leq d$.

Nonconcentration Inequalities

For a given Φ and s, find the "best possible" measure μ such that

$$\mathcal{S}(E) := \underset{(x_1,\ldots,x_k) \in E^k}{\text{ess.sup}} |\Phi(x_1, \ldots, x_k)| \gtrsim (\mu(E))^{\frac{1}{s}},$$

$$\mathcal{I}(E) := \int_{E^k} |\Phi(x_1, \ldots, x_k)| d\mu(x_1) \cdots d\mu(x_k) \gtrsim (\mu(E))^{k + \frac{1}{s}}.$$

We call these inequalities "Nonconcentration Inequalities" because they dictate that product sets E^k cannot be degenerate (as measured by Φ) when $\mu(E) > 0$.

Simple Observations

- The inequality

$$\mathcal{S}(E) := \operatorname*{ess.sup}_{x_1,\ldots,x_k \in E} |\Phi(x_1,\ldots,x_k)| \gtrsim \mu(E)^{\frac{1}{s}}$$

is strictly easier to prove than

$$\mathcal{I}(E) := \int_{E^k} |\Phi(x_1,\ldots,x_k)| d\mu(x_1)\cdots d\mu(x_k) \gtrsim [\mu(E)]^{k+\frac{1}{s}}$$

- Looking at small sets suggests that the diagonal $x_1 = \cdots = x_k = x$ is the important part; presumably Φ and its derivatives through order $Q-1$ vanish there for some $Q \geq 1$.
- By a simple scaling argument, there is a particularly important exponent s,

$$\frac{1}{s} = \frac{Q}{n}, \text{ i.e., } s = \frac{n}{Q}.$$

One doesn't expect either inequality for $\mathcal{S}(E)$ or $\mathcal{I}(E)$ to hold for "nice" μ with larger values of s.

Basic Properties of Nonconcentration Functionals

Theorem 1 (G. 2018)

$$\forall F \; \mathcal{S}(F) \geq c[\mu(F)]^{\frac{1}{s}} \Leftrightarrow \forall F \; \mathcal{I}(F) \geq c'[\mu(F)]^{k+\frac{1}{s}}$$

Theorem 2

- For $s > \frac{n}{Q}$, only the zero measure satisfies the inequality.
- For $s = \frac{n}{Q}$, there is a "best possible" choice of μ which comes from a generalization of Hausdorff measure. It is possible to estimate the density (Think of relating Hausdorff and Lebesgue measures on a curve.)

Basic Properties of Nonconcentration Functionals

Theorem 3: Frostman's Lemma

Let weighted Φ-Hausdorff measure of dim. s, $\widetilde{\mathcal{H}}^s_\Phi(E)$, be given by

$$\liminf_{\delta \to 0^+} \left\{ \sum_i c_i \left[\mathcal{S}(E_i)\right]^s \;\middle|\; \chi_E \leq \sum_i c_i \chi_{E_i}, c_i \geq 0, \operatorname{diam} E_i \leq \delta \right\}.$$

Suppose E is compact. Then $\widetilde{H}^s_\Phi(E) > 0$ if and only if there exists a nonzero, nonnegative Borel measure μ supported on E such that

$$\mathcal{I}(F) \gtrsim [\mu(F)]^{k+\frac{1}{s}} \text{ and } \mathcal{S}(F) \gtrsim [\mu(F)]^{\frac{1}{s}}$$

for all Borel sets F.

- **A Good Question:** What does this measure "measure"?
- Note: For fixed n there are many possible interesting values of s because one can restrict to polynomial graphs in $\mathbb{R}^n$.

Quick Proof of Theorem 3

- The proof of the Frostman Lemma (in terms of weighted Φ-Hausdorff measure) follows essentially identically the proof (due to Howroyd) found in Mattila's book which uses Hahn-Banach.
  - Start with a homogeneous subadditive functional

$$p_\delta(f) := \inf \left\{ \sum_i c_i \left(\mathcal{S}(E_i)\right)^s \;\middle|\; f \leq \sum_i c_i \chi_{E_i}, c_i \geq 0, \operatorname{diam}(E_i) \leq \delta \right\}$$

  - Extend the functional which equals $p_\delta(\chi_E) > 0$ on χ_E; it's got to be a positive linear functional on continuous functions
  - Riesz Representation gives a measure which you (fix and) check works out.
- What about the non-weighted generalization of Hausdorff? In the classical case they are comparable in every dimension (see Federer's book), but those arguments break down.
- In this specific case, comparability for dimension $\frac{n}{Q}$ follows manually.

Upper Bounds on μ

$$\mathcal{S}(E) := \operatorname*{ess.sup}_{x_1,\ldots,x_k \in E} |\Phi(x_1,\ldots,x_k)| \geq (\mu(E))^{\frac{Q}{n}}$$

puts obvious constraints on the size of μ. Assume μ is absolutely continuous with respect to Lebesgue measure.

Pick a point $x \in \mathbb{R}^n$, and let $E = B_r(x)$ as $r \to 0^+$. Recall we assume derivatives of order $< Q$ of Φ vanish on the diagonal. Let P be the degree Q Taylor polynomial of Φ at $(x,\ldots,x)$. Then

$$\left[\frac{d\mu}{dx}\right]^{Q/n} |B_1(0)|^{Q/n} \leq \sup_{||x_1||,\ldots,||x_k|| \leq 1} |P(x_1,\ldots,x_k)|.$$

We could use any coordinates with the same volume element.

$$\left[\frac{d\mu}{dx}\right]^{Q/n} |B_1(0)|^{Q/n} \leq \inf_{G \in \mathrm{SL}(n,\mathbb{R})} |||\rho_G P|||$$

where ρ_G is natural action of $\mathrm{SL}(n,\mathbb{R})$ on polynomials and $||| \cdot |||$ is sup norm on $(B_1(0))^k$.

3. Geometric Invariant Theory

General Algebraic Setup

- V : Finite-dimensional real vector space
- $\mathcal{G}$: Real reductive algebraic group
- ρ : polynomial $\mathcal{G}$-Representation on V
- $|||\cdot|||$: Norm on V, ρ-invariant on maximal compact $K < \mathcal{G}$.

Main Question

Given $v \in V$, how does one understand, compute,

$$\inf_{G \in \mathcal{G}} |||\rho_G v|||?$$

Key Idea

Study ρ-invariant polynomials on V.

SL($d, \mathbb{R}$) Invariant Polynomials

Suppose $M \in \mathbb{R}^{n\times n}$. The Caley Ω operator is defined as

$$\Omega_M := \det \begin{bmatrix} \frac{\partial}{\partial M_{11}} & \cdots & \frac{\partial}{\partial M_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial M_{n1}} & \cdots & \frac{\partial}{\partial M_{nn}} \end{bmatrix}.$$

Basic Features:

- $\Omega_M [f(AM)] = (\det A) [(\Omega f)(AM)]$.
- $\Omega_M (\det M)^k = c_{n,k} (\det M)^{k-1}$, $c_{n,k} > 0$ when $k > 0$.

Renyolds Operator

When ρ is a polynomial representation, we can explicitly write a projection operator from polynomials on V to SL($n, \mathbb{R}$)-invariant polynomials. For homogeneous f of fixed degree:

$$f \mapsto c\, \Omega_M^k [f \circ \rho_M]$$

Thm (Hilbert): The algebra of $\mathcal{G}$-invariant polys is fin. gen'd.

Pf: Let I be the ideal generated by $\mathcal{G}$-invariant homogeneous polys (nonconstant); there must be homogeneous $\mathcal{G}$-invariant $f_1, \ldots, f_N$ generating I (unbdd N violates Noetherianity). If f is invariant,

$$f = \varphi_1 f_1 + \cdots + \varphi_N f_N.$$

Apply Renyolds operator:

$$f = (R\varphi_1) f_1 + \cdots + (R\varphi_N) f_N$$

where the $R\varphi_j$ are invariant and have lower degrees than f.

Computing the Infimum

If $f_1, \ldots, f_N$ are homogeneous and generate the algebra, then

$$\inf_{G \in \mathcal{G}} |||\rho_G v||| \approx \sum_{i=1}^{N} |f_i(v)|^{1/\deg f_i}.$$

Proof: $\gtrsim$ is trivial; $\lesssim$ is a compactness argument. Key point: if $f_i(v) = 0$ for all i iff $\inf = 0$ (aka Hilbert-Mumford Criterion).

A Good Question: Why do I care so much about the algebra?

The following are equivalent:

- $\mathrm{RBL}(\pi, N) > 0$
- **For all** $V \subset H$,

$$\dim V \leq \sum_{j=1}^{m} \frac{N_j d}{N d_j} \dim \pi_j(V)$$

- **There exists** $\mathrm{SL}(H) \times \mathrm{SL}(H_1) \times \cdots \times \mathrm{SL}(H_m)$-invariant polynomial f with $f(0) = 0$, $f(\Pi_N) \neq 0$.

- It is easy to prove that all $\mathrm{SL}(d, \mathbb{R})$-invariant polynomials of M-linear forms must be expressible as d-linear contractions:

$$A_{i_1 \cdots i_d} \mapsto \sum_{\sigma \in \mathfrak{S}_d} (-1)^\sigma A_{\sigma_1 \cdots \sigma_d}.$$

  In our case, these are known as "dotted bracket polynomials."
- Harder to know when two such polynomials are independent and when you can stop looking.
- To do analysis, it is sometimes hard to find easily-computable polynomials, sometimes easier to work with the infimum.

4. Multilinear Kakeya and L^p-Improving Estimates

Consider a geometric averaging operator which integrates functions on $\mathbb{R}^n$ over k-dimensional algebraic submanifolds: $Tf(x) := \int_{{}^x\Sigma} f d\sigma$. Let $\rho(x,y) = 0$ be the incidence relation.

Theorem

For any nonnegative continuous functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \left[\int_{{}^x\Sigma} \cdots \int_{{}^x\Sigma} [\mathrm{RBL}(D_x \rho)]^{\frac{m(n-k)}{n}} \prod_{j=1}^m f_j(y_j) d\sigma(y_1) \cdots d\sigma(y_m) \right]^{\frac{n}{m(n-k)}} dx \lesssim \prod_{j=1}^m ||f_j||_{L^1(\mathbb{R}^n)}^{\frac{n}{m(n-k)}}.$$

This is simply a continuous version of Zorin-Kranich's Kakeya-(Rogers)-Brascamp-Lieb inequality.

A machine to prove L^p-improving estimates I

1. Weighted Kakeya-Brascamp-Lieb is an inequality which relies **transversality** of cotangent spaces. The Brascamp-Lieb weight compensates for lack of transversality on the diagonal.
2. The stuff inside weighted Kakeya-Brascamp-Lieb is a nonconcentration quantity. Precisely, pick m points $y_1, \ldots, y_m$ on ${}^x\Sigma$ (submanifold associated to x).

$$\Phi(y_1, \ldots, y_m) := (\mathrm{RBL}(D_x \rho(x, y_1), \ldots, D_x \rho(x, y_m)))^{\frac{m(n-k)}{n}}.$$

3. **Curvature = Infinitesimal Transversality of Cotangent Spaces**
4. Extracting curvature effects reduces to proving nonconcentration inequality.
5. Exploit that BLW is effectively a polynomial in $D_x \rho(x, y_i)$.

A machine to prove L^p-improving estimates II

Benefits:

- "Good Transversality" and "Good Curvature" mean some polynomial is nonzero. Consequently valid proofs of this sort for example operators will automatically remain valid for the right kind of small algebraic perturbations.
- This way of packaging things avoids some seemingly very difficult challenges posed by the method of inflation. For example, there are no longer any arithmetic constraints on dimension and codimension.

Challenges:

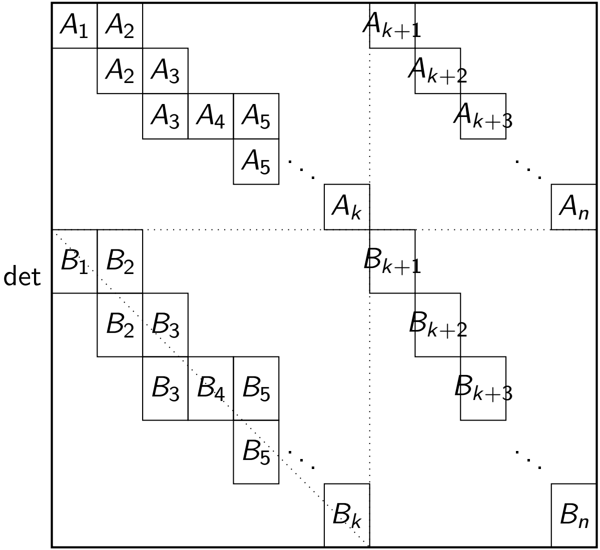
- There is potential to prove a very general weighted L^p-improving inequality with this machinery, but there are a number of additional obstacles to overcome.
- Comparing to Tao-Wright, it seems that the story is not yet finished for multilinear Kakeya?

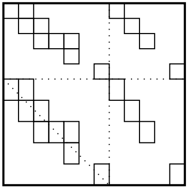
A Good Question: Can this be worked out in any concrete cases?

A machine to prove L^p-improving estimates III

- Example: convolution with measures on the surface
$$\left(t_1, \ldots, t_k, \left\{ \sum_{j=1}^{k} \lambda_{ij} t_j^2 \right\}_{i=1}^{n-k} \right)$$
where $k \geq n/2$ and in λ_{ij}, all $(n-k) \times (n-k)$ minors of cyclically adjacent columns are nondegenerate.
- Identify a workable degree of multilinearity and a workable transversality polynomial. In this case, $m = n$ copies of the submanifold works.
- Replace Brascamp-Lieb weight with this polynomial.
- To prove the nonconcentration inequality, you must bound transversality below in small ball limit in an effectively arbitrary coordinate system.
- The calculation still has high algebraic complexity despite the initial reduction.

A nice invariant polynomial



$\Phi :=$ det 

- Each $B_1, \ldots, B_k$ needs $n-k$ derivatives (let derivs act on cols; undifferentiated cols will be zero after column operations).
- In our example, the various t-coordinate functions all reside on their own rows. Differentiate with respect to the variables that cross the diagonal and argue that lower-priority derivatives must always be zero (so coordinate independent).
- **Conclusions:** (N.B. $N = m = n$; Φ is degree $n-k$ in Π_N)
  - $[\mathrm{RBL}(D_x \rho)]^{n-k} \approx \inf_G |||\rho_G \Pi_N|||^{n-k} \gtrsim |\Phi(t^{(1)}, \ldots, t^{(n)})|$
  - $\sup_{t^{(1)}, \ldots, t^{(n)} \in {}^x\Sigma \cap E} |\Phi(t^{(1)}, \ldots, t^{(n)})| \gtrsim |{}^x\Sigma \cap E|^{n-k}$.
  - Convolution satisfies: $||T\chi_E||_{\frac{2n-k}{n-k}} \lesssim |E|^{\frac{n}{2n-k}}$.

Where do things stand?

- Arbitrary dimension and codimension: Yes?
- Weighted inequalities: Some but likely not all
  - But even for curves, do we fully understand the implication relationships between weighted estimates?
- Kakeya-fication of Brascamp-Lieb is only the first step?
- Better understanding of algebra and geometry is needed
  - There exists a contraction-type formula using Ω to compute all invariants of fixed degree.
  - GIT people never tried to actually compute the infimum.
  - Is there a nice explicit formula for the Brascamp-Lieb constant?