Oscillatory multilinear Radon-like transforms

Philip T. Gressman,
University of Pennsylvania

Joint work with
Ellen Urheim

In honor of the 90th birthday of Guido Weiss

Problem Statement

Let ρ and Φ be real-valued functions on some neighborhood U of the origin in $\mathbb{R}^{2d}$ and suppose that $\nabla\rho$ is nonvanishing there. Let $M := \{x \in U \mid \rho(x) = 0\}$ and let σ be Lebesgue measure on M. The main object:

$$I_\lambda(f_1, \ldots, f_{2d}) := \int_M e^{i\lambda\Phi(x)} \prod_{j=1}^{2d} f_j(x_j) d\sigma(x).$$

We wish to understand the asymptotic behavior of the norm as $\lambda \to \infty$ when $f_j \in L^{p_j}(\mathbb{R})$. Of particular concern:

- Obtaining sharp decay in excess of $|\lambda|^{-1/2}$
- Stability results

This falls within the framework of Christ, Li, Tao, and Thiele when ρ is affine linear.

Similar objects arise in recent work of Christ on "best of the best" decay rates. Here we sacrifice some decay for stability.

Our methods are in many ways classical but there are a few interesting deviations.

Theorem. *Suppose that there is a positive constant* c *such that for every* $x \in U$ *and every* $(\tilde{\tau}, \tau) \in \mathbb{R}^2$ *such that* $\tilde{\tau}^2 + \tau^2 = 1$*, the indices* $\{1, \ldots, 2d\}$ *may be partitioned into two sets* $\{i_1, \ldots, i_d\}$ *and* $\{j_1, \ldots, j_d\}$ *such that*

$$\left| \det \begin{bmatrix} \partial_{i_1}\rho(x) & \partial^2_{i_1 j_1}(\tilde{\tau}\Phi(x) + \tau\rho(x)) & \cdots & \partial^2_{i_1 j_d}(\tilde{\tau}\Phi(x) + \tau\rho(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{i_d}\rho(x) & \partial^2_{i_d j_1}(\tilde{\tau}\Phi(x) + \tau\rho(x)) & \cdots & \partial^2_{i_d j_d}(\tilde{\tau}\Phi(x) + \tau\rho(x)) \\ 0 & \partial_{j_1}\rho(x) & \cdots & \partial_{j_d}\rho(x) \end{bmatrix} \right| \geq c$$

Then for all $f_1, \ldots, f_{2d} \in L^2(\mathbb{R})$*,*

$$|I_\lambda(f_1, \ldots, f_{2d})| \lesssim |\lambda|^{-\frac{d-1}{2}} \prod_{j=1}^{2d} ||f_j||_2.$$

Proof: well-chosen wave packets. Hörmander at low freq and Radon at high freq.

The condition is a hybrid of the typical Hessian condition for Φ and rotational curvature condition for ρ. Think of it as an inhomogeneous FIO.

No faster decay in λ is possible for these exponents. In some cases we know faster decay is impossible for any exponents.

The switching around of indices $\{i_1, \ldots, i_d\}$, $\{j_1, \ldots, j_d\}$ comes from multilinearity and yields nontrivial results when linear objects must degenerate.

Sharpness and Stability of

$$|I_\lambda(f_1,\ldots,f_{2d})| \lesssim |\lambda|^{-\frac{d-1}{2}} \prod_{j=1}^{2d} ||f_j||_2 :$$

- It's relatively easy to show that no better decay is faster on products of $L^2(\mathbb{R})$ by using standard Knapp-type arguments.
- The hypotheses are stable, so remain true for sufficiently small smooth perturbations of Φ and ρ. This is quite different than overdetermined CLTT case:

$$\int_{x_1+\cdots+x_{2d}=\epsilon\Phi} e^{i\lambda\Phi(x)} \prod_{j=1}^{2d} f_j(x_j) d\sigma(x)$$

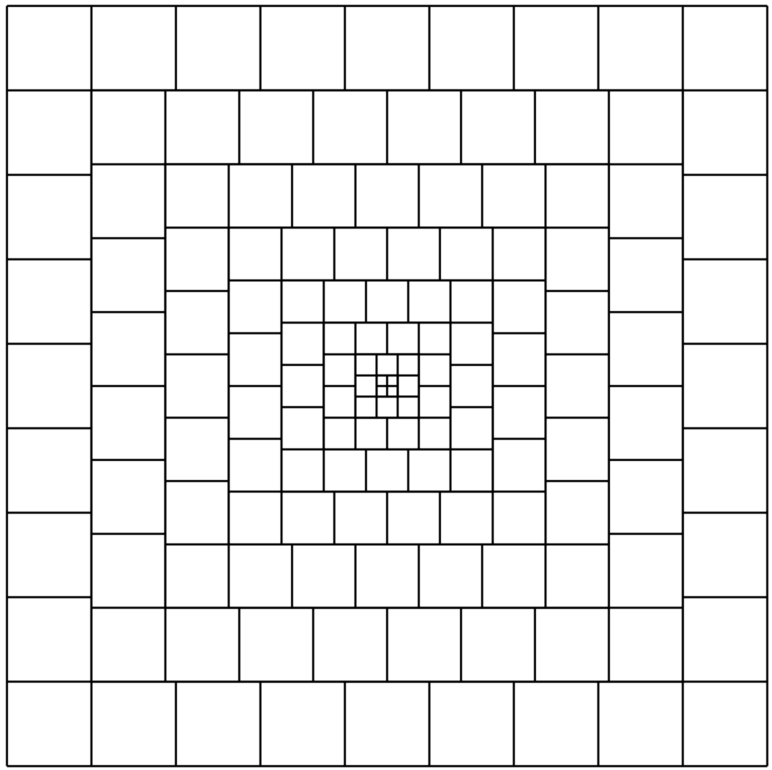
has no decay in λ for any $\epsilon \neq 0$.

For multilinearity of order $2d$, the analogous best Christ operator has decay like $|\lambda|^{-d+1}$. It's essentially an iterated Fourier transform.

Related to Christ's "Best of the Best": it seems quite hard to tell what the best possible stable decay rate is, but no better than $|\lambda|^{-d+\sqrt{d}-O(1)}$ (still a huge gap).

Algebra is hard and incomplete: How low does the rank go generically for real linear combinations of two symmetric matrices and diagonal matrices?

Main Decomposition



- Divide frequency space into tiles such that tile containing ξ has diameter $\sim |\xi|^{1/2}$.
- Resize innermost boxes so that none is smaller than $|\lambda|^{1/2}$.

Oscillatory integral operators are well-adapted to packet bases with uniform frequency resolution $\sim |\lambda|^{1/2}$, i.e., Gabor-like decompositions.

Radon-like transforms are often studied in Littlewood-Paley-type decompositions, where frequency resolution at frequency ξ is like $|\xi|$.

This basis falls halfway between extremes and almost diagonalizes the operator. Kernel decay is so good there's no need for TT^*.

Technical Details

There exists a map $f(x) \mapsto Vf(x, \xi)$ such that

$$f(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} Vf(y, \xi) \varphi_\xi(x - y) \, dy d\xi,$$

$$||Vf||_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \approx ||f||_{L^2(\mathbb{R}^d)},$$

$$\left| \partial_x^\alpha (e^{-2\pi i x \cdot \xi} \varphi_\xi(x)) \right| \lesssim (\max\{|\xi|, |\lambda|\})^{\frac{|\alpha|}{2} + \frac{1}{4}},$$

$$\varphi_\xi(x) = 0 \text{ when } |x| \gtrsim (\max\{|\xi|, |\lambda|\})^{-\frac{1}{2}}.$$

For our purposes, this continuous decomposition is easiest to work with.

Can the formula can be discretized? Work of Hernández, Labate, Weiss, and Wilson (ACHA 2004) suggest no tight frames will exist.

Even without λ correction, such a decomposition nearly diagonalizes Radon-like averages over hypersurfaces with nonvanishing rotational curvature.

Substitute in the expansion:

$$I_\lambda(f_1, \ldots, f_{2d}) := \int_M e^{i\lambda\Phi(x)} \prod_{j=1}^{2d} f_j(x_j) d\sigma(x)$$

$$= \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \mathcal{I}(y, \xi) \prod_{j=1}^{2d} V f_j(y_j, \xi_j) dy d\xi$$

For convenience, let $r_j := (\max\{|\lambda|, |\xi_j|\})^{-1/2}$. $\mathcal{I}(y, \xi)$ is supported on the region $|\rho(y)| \lesssim \max_j r_j$. Stationary phase gives:

$$|\mathcal{I}(y, \xi)| \lesssim \left(1 + \frac{(\min_j r_j)^2}{\max_j r_j} \left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y|^2\right)^{\frac{1}{2}}\right)^{-N} r_{j_0}^{-1} \prod_{j=1}^{2d} r_j^{\frac{1}{2}}$$

for every $j_0 \in \{1, \ldots, 2d\}$.

The vector fields X_i are orthonormal and span the tangent space of $M = \{y \ : \ \rho(y) = 0\}$ at every point.

We use a minor variation on the usual integration by parts technique for stationary phase to make things a little cleaner.

If the $r_j \not\approx r_{j'}$ then $|\xi| + |\xi'| \gtrsim |\lambda|$. As long as M is transverse to all coordinate directions, the phase is highly nonstationary and $|\mathcal{I}(y, \xi)|$ is very small.

For convenience, let $r := (\max\{|\lambda|, |\xi|\})^{-1/2}$.

$$|\mathcal{I}(y,\xi)| \lesssim \left(1 + r\left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y|^2\right)^{\frac{1}{2}}\right)^{-N} r^{d-1}\chi_{|\rho(y)|\lesssim r}$$

Now manually add another derivative in the direction of $\nabla\rho(y)$ to make life easier:

$$|\mathcal{I}(y,\xi)| \lesssim \int (1 + r|\lambda\nabla\Phi(y) + \tau\nabla\rho(y) + 2\pi\xi)|)^{-N} r^d \chi_{|\rho(y)|\lesssim r}\, d\tau$$

Again, we may assume $|\tau|+|\lambda| \approx |\xi|+|\lambda| \approx r^{-2}$ as otherwise the phase is nonstationary.

$$\int_E \frac{r^d \chi_{|\rho(y)|\lesssim r} \prod_{j=1}^{2d} |Vf_j(y_j,\xi_j)|}{(1 + r|\nabla(\lambda\Phi(y) + \tau\rho(y)) + 2\pi\xi|)^N} dy\, d\xi\, d\tau$$

where $E := \{|\tau| + |\lambda| \approx |\xi| + |\lambda| \approx r^{-2}\}$. Minor nuisance that r depends on ξ.

Having good control on $|\tau| + |\lambda|$ is important because it will come up as a Jacobian determinant of some change of variables.

So far, there's nothing special about multilinearity; the analysis proceeds similarly for fewer functions of higher dimensions.

The quantity to the left is similar to things you'd get from TT^* but it has better localization and works better with decomposition after the fact.

$$\int_E \frac{r^d \chi_{|\rho(y)|\lesssim r} \prod_{j=1}^{2d} |Vf_j(y_j,\xi_j)|}{(1+r|\nabla(\lambda\Phi(y)+\tau\rho(y))+2\pi\xi|)^N} dy\,d\xi\,d\tau$$

Now interpolate: Put half of the $Vf_j \in L^\infty(\mathbb{R}\times\mathbb{R})$ and the others in $L^1(\mathbb{R}\times\mathbb{R})$. Reduces to estimating an integral

$$\int_E \frac{r^d \chi_{|\rho(y)|\lesssim r} dy_{j_1}\cdots dy_{j_d} d\tau}{(1+r|P_{i_1\cdots i_d}(\nabla(\lambda\Phi(y)+\tau\rho(y))+2\pi\xi)|)^N}$$

where $i_1,\ldots,i_d,j_1,\ldots,j_d$ are distinct. This integral can be understood via change of vars:

$$(y_{j_1},\ldots,y_{j_d},\tau)\mapsto (\partial_{i_1}(\lambda\Phi(y)+\tau\rho(y)), \ldots, \partial_{i_d}(\lambda\Phi(y)+\tau\rho(y)), \rho(y))$$

Good news: freezing a ξ_j essentially freezes r, so it behaves like a constant even though it isn't.

Bad News: For fixed choice of $i_1,\ldots,j_d$, there are rarely any instances where the change of variables is nonsingular.

Good News: Because our estimate is pointwise, we can chop up the domain as we wish and do the interpolation different ways on different pieces.

Jacobian determinant of

$$(\tau, y_{j_1}, \ldots, y_{j_d}) \mapsto (\partial_{i_1}(\lambda\Phi(y) + \tau\rho(y)), \ldots, \partial_{i_d}(\lambda\Phi(y) + \tau\rho(y)), \rho(y))$$

is

$$\det \begin{bmatrix} \partial_{i_1}\rho(y) & \partial^2_{i_1 j_1}(\lambda\Phi(y) + \tau\rho(y)) & \cdots & \partial^2_{i_1 j_d}(\lambda\Phi(y) + \tau\rho(y)) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{i_d}\rho(y) & \partial^2_{i_d j_1}(\lambda\Phi(y) + \tau\rho(y)) & \cdots & \partial^2_{i_d j_d}(\lambda\Phi(y) + \tau\rho(y)) \\ 0 & \partial_{j_1}\rho(y) & \cdots & \partial_{j_d}\rho(y) \end{bmatrix}.$$

Det is homogeneous of degree $(d-1)$ in the pair (λ, τ). For even d and any fixed y, it must vanish along some line in (λ, τ). BUT if $y = (s, t, u, v) \in (\mathbb{R}^{d/2})^4$, then

$$\Phi(s, t, u, v) = s \cdot t + u \cdot v$$

$$\rho(s, t, u, v) = \overrightarrow{1} \cdot (s + t + u + v) + s \cdot u + t \cdot v$$

has one partition with determinant $c\tau^{d-1}$ and one with $c'\lambda^{d-1}$, so it's all good.

No need for Φ, ρ to be polynomials: noncompactness of domain in τ can be handled by scaling properties of the system of equations.

Special cases: $\tau = 0$ implies infinitesimal CLTT-type nondegeneracy. $\lambda = 0$ is rotational curvature.

It should be possible to study the singular linear operator as well. The simplest singularities are like inhomogeneous folds and may not need new machinery.