Quantum Graphs and Quantum Cuntz-Krieger Algebras

Michael Brannan

Texas A&M University

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Today's Goals

- To describe a generalization of **graph theory** within the framework of operator algebras/non-commutative geometry: Quantum Graphs.
- To explain (briefly) why one might want to study quantum graphs.
- To introduce a new class of operator algebras that encode the "**symbolic dynamics**" associated to quantum graphs: Quantum Cuntz-Krieger algebras.

Joint work with:

- Kari Eifler (TAMU)
- Christian Voigt (Glasgow)
- Moritz Weber (Saarbrücken)

Graphs

A (finite, directed) graph is a tuple $\mathcal{G} = (V, E, s, t)$ where

- V, E are finite sets (vertices and edges, respectively)
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- Thus each edge $e \in E$ is thus associated to an ordered pair $(s(e), t(e)) \in V \times V$. $s(e)$ is the source of e and $t(e)$ is the target of e.
- Graphs $\mathcal{G}$ can multiple edges: The map $E \to V \times V; e \mapsto (s(e), t(e))$ need not be injective.

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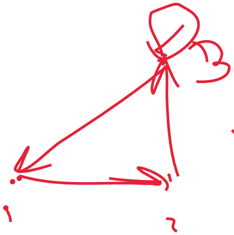
- Graphs $\mathcal{G}$ can multiple edges: The map $E \to V \times V; e \mapsto (s(e), t(e))$ need not be injective.
- For our purposes, we will ONLY consider graphs $\mathcal{G}$ with **no multiple edges**.
- In this case, we simply identify $E \subseteq V \times V$. In this case we can consider the adjacency matrix $A_G \in M_{V\times V}(\{0,1\})$ of $\mathcal{G}$:

$$A_{\mathcal{G}}(v, w) = 1 \iff (v, w) \in E.$$

Then $\mathcal{G}$ is encoded by the pair $(V, A_{\mathcal{G}})$.



$$A_G = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \\ 3 & & & \end{array}$$

How to "Quantize" a Graph: The NCG Perspective

Non-commutative Geometry (NCG) is all about finding **quantum** (i.e., non-commutative) generalizations of familiar topological/algebraic/geometric structures by generalizing **Gelfand duality**:

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Examples:

- **LC Hausdorff spaces** $X \longleftrightarrow$ abelian C*-algebras $A = C_0(X)$.
- **Quantum Topological Spaces** $\longleftrightarrow$ arbitrary C*-algebras $A \subset B(H)$.
- **Measure spaces** $(X, \mu) \longleftrightarrow$ abelian von Neumann algebras $M = L^\infty(X, \mu)$.
- **Quantum Measure spaces** $\longleftrightarrow$ arbitrary von Neumann algebras $M = M'' \subset B(H)$.
- **Quantum groups** $\longleftrightarrow$ nice Hopf algebras/tensor categories.
- **Quantum Riemannian manifolds** $\longleftrightarrow$ Connes' spectral triples.

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Our Problem: **What is the quantum analogue of a graph**?

Quantizing Graphs

Input: A finite directed graph $\mathcal{G} = (V, A_{\mathcal{G}})$ with $|V| = n$ vertices and no multiple edges.

Output: A C*-algebraic description of $\mathcal{G}$ that can be made non-commutative!

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- Let $B = C(V) = C^*\big((p_v)_{v\in V} \mid p_v^* = p_v^2 = p_v, \sum_v p_v = 1\big)$, the C*-algebra of functions on V.
- Equip B with the canonical trace functional $\psi : B \to \mathbb{C}$ given by counting measure on V. ψ is "**canonical**" because

  $$\psi = \mathrm{Tr}_{\mathrm{End}(B)}\Big|_B \quad \text{where } B \hookrightarrow \mathrm{End}(B) = \text{ left-regular representation.}$$

- Use (GNS) inner product $\langle f|g\rangle = \psi(f^*g)$ to turn B into a Hilbert space $B = L^2(B) = \ell^2(V)$.

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- Use (GNS) inner product $\langle f|g\rangle = \psi(f^*g)$ to turn B into a Hilbert space $B = L^2(B) = \ell^2(V)$.
- View the adjacency matrix $A_{\mathcal{G}}$ as a **linear map** $A_{\mathcal{G}} \in \mathrm{End}(B)$ in the obvious way:

  $$A_{\mathcal{G}} p_w = \sum_{w\in V} A_{\mathcal{G}}(v,w) p_v.$$

Quantizing Adjacency Matrices

So far:

$$\mathcal{G} \rightsquigarrow \left(B = C(V), \psi = \mathsf{Tr}_{\mathsf{End}(B)}\Big|_B, A_{\mathcal{G}} \in \mathsf{End}(B)\right).$$

Problem: How to intrinsically capture the fact that $A_{\mathcal{G}}$ is an **adjacency matrix** at the level of the function algebra B?

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Problem: How to intrinsically capture the fact that $A_{\mathcal{G}}$ is an **adjacency matrix** at the level of the function algebra B?
Answer: $A_{\mathcal{G}} \in M_n(\{0,1\}) \iff A_{\mathcal{G}}$ is **idempotent** with respect to **Schur multiplication** in End(B)!

- Let $m : B \otimes B \to B$ be the algebra multiplication, $m^* : B \to B \otimes B$ its Hilbertian adjoint.

$$m(p_v \otimes p_w) = \delta_{v,w} p_v, \qquad m^*(p_v) = (p_v \otimes p_v).$$

- Simple calculation: Given $X, Y \in \mathrm{End}(B)$,

$$\text{Schur Multiplication} : X \star Y = m(X \otimes Y)m^*$$

Conclusion: $A_{\mathcal{G}} \in M_n(\{0,1\}) \iff m(A_{\mathcal{G}} \otimes A_{\mathcal{G}})m^* = A_{\mathcal{G}}$.

Quantum Graphs

We now go non-commutative! Let B be a finite-dimensional C*-algebra equipped with its canonical trace functional

$$\psi : B \to \mathbb{C}; \quad \psi = \mathsf{Tr}_{\mathsf{End}(B)}\Big|_B$$

(If $B = \bigoplus_k M_{n(k)}(\mathbb{C})$), $\psi = \sum_k n(k)\mathsf{Tr}_{n(k)}(\cdot) =$ Plancharel trace).

Definition

A linear map $A \in \mathsf{End}(B)$ is called a quantum adjacency matrix if

$$m(A \otimes A)m^* = A \quad (A \text{ is a "}\textbf{quantum Schur idempotent}\text{"})$$

We call the triple $\mathcal{G} = (B, \psi, A)$ a quantum graph.

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- Classical graphs $\iff$ quantum graphs with **abelian** B.
- For certain applications, we can consider quantum graphs $\mathcal{G} = (B, \psi, A)$ with **non-tracial states** ψ...but not today.

Some Basic Examples

Fix any pair (B, ψ).

1. The complete quantum graph over B is $\mathcal{K}(B) = (B, \psi, A)$,

$$A = \psi(\cdot)1_B \qquad \textbf{Classical case: } A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

2. The trivial quantum graph over B is $\mathcal{T}(B) = (B, \psi, A)$ where

$$A = \mathrm{id}_{B \to B} \qquad \textbf{Classical case: } A = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

3. If $B = \bigoplus_k M_{n(k)}(\mathbb{C})$ and $d = (d^{(k)})_k \in \bigoplus_k D_{n(k)} \subset B$ is **diagonal** with $\mathrm{Tr}_{n(k)}(d^{(k)}) = 1$ for all k, then

$$A_d(x) = dx \qquad \text{is a quantum adjacency matrix.}$$

We call $\mathcal{D}(B) = (B, \psi, A_d)$ a diagonal quantum graph over B.

More Examples: Quantum Edge Space Picture

Writing down interesting examples of quantum adjacency matrices might at first seem not so easy.

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Alternate Approach [Nik Weaver]: Recall that a graph $\mathcal{G}$ is completely described by its **edge relation** $E \subset V \times V$.

- Consider the linear subspace

$$S_{\mathcal{G}} = \mathrm{span}\{e_{x,y} : (x,y) \in E\}\} \subset B(\ell^2(V)).$$

- Then $S_{\mathcal{G}}$ is a bimodule over $C(V) = C(V)' \subset B(\ell^2(V))$.
- (Weaver) Every $C(V)' - C(V)'$-bimodule $S \subset B(\ell^2(V))$ is of the form $S_{\mathcal{G}}$ for some unique $\mathcal{G} = (V, E)$.

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Definition (Weaver 2010)

Let $B \subseteq B(H)$ be a finite dimensional C*-algebra. A quantum graph on B is B'-B'-bimodule

$$_{B'}S_{B'} \subseteq B(H).$$

Think of $_{B'}S_{B'}$ as the collection of all "edge operators" that "connect quantum vertices" in the graph.

Quantum Edge Space vs Quantum Adjacency Matrices

Both approaches to quantum graphs are essentially the same:

- **Fact 1**: Given $B \subseteq B(H)$, a bimodule ${}_{B'}S_{B'} \subseteq B(H)$ is uniquely determined by a self-adjoint projection

$$p = \sum_i a_i \otimes b_i^{op} \in B \otimes B^{op}; S = p \cdot B(H) = \left\{ \sum_i a_i X b_i : X \in B(H) \right\}$$

- **Fact 2**: Any such $p = p^* = p^2 \in B \otimes B^{op}$ arises as the "**Choi-Jamiołkowski**" matrix of a **completely positive** quantum adjacency matrix $A \in \mathrm{End}(B)$ given by

$$p = (A \otimes 1)m^*(1_B).$$

Conclusion: Quantum graphs $\mathcal{G} = (B, \psi, A)$ (with completely positive A) $\longleftrightarrow$ operator spaces $S \subseteq B(H)$ that are B'-B'-bimodules.

Quantum Graphs in "Nature"

Quantum graphs arise naturally in problems in quantum information theory and representation theory.

- **Zero-error channel capacity**: Given finite sets V, W and a **noisy channel** with transition probabilities $(p(w|v))_{v\in V, w\in W}$, can form the confusability graph $\mathcal{G}_\Phi = (V, E)$ where

$$(v_1, v_2) \in E \iff \exists w \in W \text{ s.t. } p(w|v_1)p(w|v_2) \neq 0.$$

  The zero-error capacity of Φ: Is the Shannon capacity of $\mathbb{G}_\Phi$.
- In **Quantum Information Theory**, one considers **quantum channels**

$$\Phi : L^1(B, \psi) \to M_k(\mathbb{C}).$$

  The **Zero-error capacity of Φ**: $\rightsquigarrow$ study Shannon capacity of the non-commutative confusability graph S_Φ of Φ: $S_\Phi \subset B(L^2(B))$ is a B'-B'-bimodule.

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- Qgraphs $\mathcal{G} = (B, \psi, A)$ appear in the theory of non-local games (graph isomorphism games), quantum teleportation and superdense coding schemes in QIT, representation theory of quantum symmetry groups of graphs, ...

Cuntz-Krieger Algebras of graphs

Let $\mathcal{G} = (V, A)$ be a finite graph without multiple edges.

The Cuntz-Krieger algebra $\mathcal{O}(\mathcal{G}) = \mathcal{O}_A$ is the **universal C*-algebra** generated by partial isometries $(s_v)_{v \in V}$ satisfying the relations

(CK1) $\sum_{v \in V} s_v s_v^* = 1.$

(CK2) $s_v^* s_v = \sum_{w \in V} A(v, w) s_w s_w^*.$

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Why study these operator algebras?

- These algebras turn out to encode the **symbolic dynamics** associated to a graph $\mathcal{G}$: They provide new conjugacy invariants for the topological Markov chain (X_A, σ_A) associated to $\mathcal{G}$.

$$X_A = \{(v_i)_{i \in \mathbb{N}} : A(v_i, v_{i+1}) = 1\}\} \text{ space of } \textbf{infinite paths},$$
$$\sigma_A : X_A \to X_A; \quad \sigma_A(v_1, v_2, \dots) = (v_2, v_3, \dots).$$

- They provide a rich class of C*-algebras that are amenable to classification.

Examples

1. $\mathcal{G} = K_n$ the complete graph on n vertices. Then $\mathcal{O}(K_n)$ has generators $(s_i)_{i=1}^n$ satisfying

$$(CK1)\ \sum_i s_i s_i^* = 1 \quad \& \quad (CK2)\ s_i^* s_i \left(= \sum_j s_j s_j^* \right) = 1.$$

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2. $\mathcal{G} = T_n$ (trivial graph on n vertices, with self-loops). Then $\mathcal{O}(T_n)$ has generators $(s_i)_{i=1}^n$ satisfying

$$\sum_i s_i s_i^* = 1, \quad s_i^* s_i = s_i s_i^* \implies \mathcal{O}(T_n) = C(\mathbb{T})^{\oplus n}.$$

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$$\sum_i s_i s_i^* = 1, \quad s_i^* s_i = s_i s_i^* \implies \mathcal{O}(T_n) = C(\mathbb{T})^{\oplus n}.$$

3. Let $\mathcal{G} \leftrightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathcal{O}(\mathcal{G}) = C^*\Big(s_1, s_2 \mid {}^{\ \ s_1 s_1^* + s_2 s_2^* = 1}_{s_1^* s_1 = 1,\ s_2^* s_2 = s_1 s_1^*} \Big).$$

General properties of Cuntz-Krieger algebras

- $\mathcal{O}(\mathcal{G}) \neq 0$ as long as $A_{\mathcal{G}} \neq 0$.
- $\mathcal{O}(\mathcal{G})$ is always nuclear.
- $\mathcal{O}(\mathcal{G})$ is always unital.
- $\mathcal{O}(\mathcal{G})$ are classified, up to isomorphism.
- Include many naturally occuring examples of C*-algebras.
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Goal: Generalize Cuntz-Krieger Algebras to Quantum Graphs!

Quantum Cuntz-Krieger Algebras

Now let $\mathcal{G} = (B, \psi, A)$ be a quantum graph. We want to build an operator algebra $\mathbb{F}\mathcal{O}(\mathcal{G})$ out of $\mathcal{G}$ as above. Here's our attempt.

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Definition (B-Eifler-Voigt-Weber)

The quantum Cuntz-Krieger Algebra associated to $\mathcal{G}$ is the universal C*-algebra $\mathbb{F}\mathcal{O}(\mathcal{G})$ generated by the range of a linear map $S : B \to \mathbb{F}\mathcal{O}(\mathcal{G})$ satisfying the relations

1. $\mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S$
2. $\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$

where $\mu : \mathbb{F}\mathcal{O}(\mathcal{G}) \otimes \mathbb{F}\mathcal{O}(\mathcal{G}) \to \mathbb{F}\mathcal{O}(\mathcal{G})$ is the multiplication map and $S^*(b) = S(b^*)^*$.

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Why is this a reasonable definition?

Theorem

If $\mathcal{G}$ is classical (i.e., $B = C(V)$ is abelian), then $\mathbb{F}\mathcal{O}(\mathcal{G})$ is the usual Cuntz-Krieger algebra $\mathcal{O}(\mathcal{G})$ associated to $\mathcal{G}$ (up to a KK-equivalence).

Quantum CK-Algebras: Unpacking the Definition

$$\mathcal{G} = (B, \psi, A) \rightsquigarrow \mathbb{F}\mathcal{O}(\mathcal{G}) = C^*\left((S(b))_{b\in B} \;\middle|\; \substack{S:B\to\mathcal{O}_{\mathcal{G}} \text{ linear} \\ \mu(1\otimes\mu)(S\otimes S^*\otimes S)(1\otimes m^*)m^*=S \\ \mu(S^*\otimes S)m^*=\mu(S\otimes S^*)m^*A} \right)$$

Write $B = \bigoplus_{k=1}^m M_{n(k)}(\mathbb{C})$ with standard matrix unit basis $\{e_{ij}^{(k)}\}$.
For $1 \leq k \leq m$, put $f_{ij}^{(k)} = n(k)^{-1} e_{ij}^{(k)}$ and

$$S^{(k)} = [S(f_{ij}^{(k)})] \in M_{n(k)}(\mathbb{F}\mathcal{O}(\mathcal{G})).$$

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Then $\mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S$

$\iff S^{(k)} S^{(k)*} S^{(k)} = S^{(k)}, \quad k = 1, \ldots, m$

$\iff$ each $S^{(k)}$ is a **partial isometry** in $M_{n(k)}(\mathbb{F}\mathcal{O}(\mathcal{G}))$.

$$\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$$

$$\iff S^{(k)*} S^{(k)} = (S^{(1)} S^{(1)*}, \ldots, S^{(m)} S^{(m)*}) \tilde{A}$$

where $\tilde{A}_{ijk}^{xyz} = \frac{n(z)}{n(k)} A_{ijk}^{xyz}$ (reweighted adjacency matrix).

Quantum CK-Algebras: Unpacking the Definition

Summary: Given $\mathcal{G} = (B, \psi, A)$, $B = \bigoplus_{k=1}^{m} M_{n(k)}$, the quantum Cuntz-Krieger Algebra $\mathcal{O}_{\mathcal{G}}$ is:

1. Generated by the coefficients of m **matrix partial isometries** $S^{(k)} \in M_{n(k)}(\mathbb{F}\mathcal{O}(\mathcal{G}))$,
2. Subject to the **quantum Cuntz-Krieger relations**

$$S^{(k)^*} S^{(k)} = (S^{(1)} S^{(1)^*}, \ldots, S^{(m)} S^{(m)^*}) \tilde{A}.$$

3. **BUT:** There seems to be no natural analogue of the relation

$$\text{“} \sum_k S^{(k)} (S^{(k)})^* = 1'' \qquad \text{(incompatible matrix sizes)}$$

$\Longrightarrow$ $\mathcal{O}(\mathcal{G})$ might not be unital in general.

Basic Examples, Basic Properties

Example 1: Trivial Quantum Graph $\mathcal{G} = \mathcal{T}(M_n)$.

$$\mathbb{F}\mathcal{O}(\mathcal{T}(M_n)) = C^*\left(S_{ij},\ 1 \le i,j \le n \,\middle|\, {}^{S=[S_{ij}] \text{ is a partial isometry}}_{\sum_l S_{li}^* S_{lj} = \sum_l S_{il} S_{jl}^* \iff S^*S = SS^*}\right)$$

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- $\mathbb{F}\mathcal{O}(\mathcal{T}(M_n))$ is **non-unital**, **non-nuclear**.
- We have quotient maps

$$\begin{aligned} \mathcal{O}_{\mathcal{T}(M_n)} &\to C(\mathbb{T})^{\star n} \quad \text{(non-unital free product)} \\ S_{ij} &\mapsto \delta_{ij} z_i. \\ \mathcal{O}_{\mathcal{T}(M_n)} &\to U^{nc}(n) \\ S_{ij} &\mapsto u_{ij}. \end{aligned}$$

  where $U^{(nc)}(n) = C^*\left(u_{ij},\ 1 \leq i,j \leq n \;\middle|\; U = [u_{ij}] \text{ unitary}\right)$ is Brown's Universal noncommutative unitary algebra.
- In fact $M_n(\mathbb{F}O(\mathcal{T}(M_n))^+) \cong M_n \star (C(S^1) \oplus \mathbb{C})$.

Basic Examples, Basic Properties

Example 2: Diagonal Quantum Graph $\mathcal{G} = \mathcal{D}(M_2)$
$A_d(e_{ij}) = \delta_{i,1} 2e_{ij}$.

$$\mathbb{F}\mathcal{O}(\mathcal{D}(M_2)) = C^*\left(S_{ij},\ 1 \leq i,j \leq 2 \mid {}^{S=[S_{ij}] \text{ is a partial isometry}}_{\sum_l S_{li}^* S_{lj} = \delta_{\{i=j=1\}} \sum_l 2S_{1l}S_{1l}^*}\right)$$

- Get $S_{l2}^* S_{l2} = 0 \implies S_{l2} = 0$.
- Thus, the canonical map $S : B \to \mathcal{O}_{\mathcal{G}}$ need not be injective!

Basic Examples, Basic Properties

Example 3: Complete Quantum Graph $\mathcal{G} = \mathcal{K}(M_n)$.

$$\mathbb{F}\mathcal{O}(\mathcal{K}(M_n)) = C^*\left(S_{ij},\ 1 \le i, j \le n \mid \begin{array}{c} S=[S_{ij}] \text{ is a partial isometry} \\ \sum_l S_{li}^* S_{lj} = \delta_{i,j}\left(n \sum_{x,l} S_{xl} S_{xl}^*\right) \end{array}\right)$$

Basic Examples, Basic Properties

Example 3: Complete Quantum Graph $\mathcal{G} = \mathcal{K}(M_n)$.

$$\mathbb{F}\mathcal{O}(\mathcal{K}(M_n)) = C^*\Big(S_{ij},\ 1 \le i,j \le n \mid \underset{\sum_l S_{li}^* S_{lj} = \delta_{i,j}\left(n \sum_{x,l} S_{xl} S_{xl}^*\right)}{S = [S_{ij}] \text{ is a partial isometry}}\Big)$$

- What is this algebra? Is it unital? Nuclear? ...
- We have a quotient map

$$\mathbb{F}\mathcal{O}(\mathcal{K}(M_n)) \to \mathcal{O}_{n^2} = \mathcal{O}(K_{n^2}) \quad \text{(the Cuntz algebra)}$$
$$S_{ij} \mapsto n^{-1/2} \hat{S}_{ij} \quad (\hat{S}_{ij} = \text{standard Cuntz isometry}).$$

Basic Examples, Basic Properties

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$$\mathbb{F}\mathcal{O}(\mathcal{K}(M_n)) = C^*\left(S_{ij},\ 1 \leq i,j \leq n \mid \substack{S=[S_{ij}] \text{ is a partial isometry} \\ \sum_l S_{li}^* S_{lj} = \delta_{i,j}\left(n \sum_{x,l} S_{xl} S_{xl}^*\right)}\right)$$

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- **Very surprising/mysterious theorem**: The above homomorphism is actually an isomorphism $\mathbb{F}\mathcal{O}(\mathcal{K}(M_n)) \cong \mathcal{O}_{n^2}$.
- So $\mathbb{F}\mathcal{O}(\mathcal{K}(M_n))$ is indeed unital, nuclear, etc.
- **The proof**: Uses the notion of a **quantum isomorphism** between quantum graphs (aka **superdense coding** in QIT): $\mathcal{K}(M_n)$ and K_{n^2} turn out to be quantum isomorphic.

Conclusion

Quantum graphs and Quantum Cuntz-Krieger Algebras seem to have some new and interesting properties! Lots of work to be done!

THANKS FOR LISTENING.