Quantum Graphs
and
Quantum Cuntz-Krieger Algebras

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Today’s Goals

▶ To describe a generalization of graph theory within the framework of operator algebras/non-commutative geometry: Quantum Graphs.
▶ To explain (briefly) why one might want to study quantum graphs.
▶ To introduce a new class of operator algebras that encode the “symbolic dynamics” associated to quantum graphs: Quantum Cuntz-Krieger algebras.

Joint work with:
▶ Kari Eifler (TAMU)
▶ Christian Voigt (Glasgow)
▶ Moritz Weber (Saarbrücken)
Graphs

A (finite, directed) graph is a tuple $G = (V, E, s, t)$ where

- $V, E$ are finite sets (vertices and edges, respectively)
- $s, t : E \to V$ are functions (source and target maps).
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- Thus each edge $e \in E$ is thus associated to an ordered pair $(s(e), t(e)) \in V \times V$. $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$.
- Graphs $G$ can multiple edges: The map $E \to V \times V; e \mapsto (s(e), t(e))$ need not be injective.
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- Graphs $G$ can multiple edges: The map $E \rightarrow V \times V; e \mapsto (s(e), t(e))$ need not be injective.
- For our purposes, we will ONLY consider graphs $G$ with no multiple edges.
- In this case, we simply identify $E \subseteq V \times V$. In this case we can consider the adjacency matrix $A_G \in M_{V \times V} \{0, 1\}$ of $G$:

$$A_G(v, w) = 1 \iff (v, w) \in E.$$ 

Then $G$ is encoded by the pair $(V, A_G)$.
\[ A_G = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \]
How to “Quantize” a Graph: The NCG Perspective

Non-commutative Geometry (NCG) is all about finding quantum (i.e., non-commutative) generalizations of familiar topological/algebraic/geometric structures by generalizing Gelfand duality:

- **Examples**: ▶ LC Hausdorff spaces $X \leftrightarrow$ abelian $C^*$-algebras $A = C_0(X)$.
  ▶ Quantum Topological Spaces $\leftrightarrow$ arbitrary $C^*$-algebras $A \subset B(H)$.
  ▶ Measure spaces $(X, \mu) \leftrightarrow$ abelian von Neumann algebras $M = L_\infty(X, \mu)$.
  ▶ Quantum Measure spaces $\leftrightarrow$ arbitrary von Neumann algebras $M = M'' \subset B(H)$.
  ▶ Quantum groups $\leftrightarrow$ nice Hopf algebras/tensor categories.
  ▶ Quantum Riemannian manifolds $\leftrightarrow$ Connes' spectral triples.

Our Problem: What is the quantum analogue of a graph?
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Quantizing Graphs

**Input**: A finite directed graph $\mathcal{G} = (V, A_\mathcal{G})$ with $|V| = n$ vertices and no multiple edges.

**Output**: A C*-algebraic description of $\mathcal{G}$ that can be made non-commutative!
Quantizing Graphs

**Input:** A finite directed graph $G = (V, A_G)$ with $|V| = n$ vertices and no multiple edges.

**Output:** A $C^*$-algebraic description of $G$ that can be made non-commutative!

- Let $B = C(V) = C^*\left(\left(p_v\right)_{v \in V} \mid p_v^* = p_v^2 = p_v, \sum_v p_v = 1\right)$, the $C^*$-algebra of functions on $V$.
- Equip $B$ with the canonical trace functional $\psi : B \to \mathbb{C}$ given by counting measure on $V$. $\psi$ is “canonical” because
  \[ \psi = \text{Tr}_{\text{End}(B)} \bigg|_B \quad \text{where} \quad B \hookrightarrow \text{End}(B) = \text{left-regular representation}. \]
- Use (GNS) inner product $\langle f | g \rangle = \psi(f^* g)$ to turn $B$ into a Hilbert space $B = L^2(B) = \ell^2(V)$. 
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- Use (GNS) inner product $\langle f | g \rangle = \psi(f^*g)$ to turn $B$ into a Hilbert space $B = L^2(B) = \ell^2(V)$.
- View the adjacency matrix $A_G$ as a **linear map** $A_G \in \text{End}(B)$ in the obvious way:
  \[ A_G p_w = \sum_{w \in V} A_G(v, w)p_v. \]
Quantizing Adjacency Matrices

So far:

\[ G \leadsto \left( B = C(V), \psi = \text{Tr}_{\text{End}(B)}|_B, A_G \in \text{End}(B) \right). \]

**Problem**: How to intrinsically capture the fact that \( A_G \) is an **adjacency matrix** at the level of the function algebra \( B \)?
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**Problem:** How to intrinsically capture the fact that \( A_G \) is an adjacency matrix at the level of the function algebra \( B \)?

**Answer:** \( A_G \in M_n(\{0,1\}) \iff A_G \) is idempotent with respect to Schur multiplication in \( \text{End}(B) \)!

- Let \( m : B \otimes B \rightarrow B \) be the algebra multiplication, \( m^* : B \rightarrow B \otimes B \) its Hilbertian adjoint.

\[ m(p_v \otimes p_w) = \delta_{v,w} p_v, \quad m^*(p_v) = (p_v \otimes p_v). \]

- Simple calculation: Given \( X, Y \in \text{End}(B) \),

\[ \text{Schur Multiplication} : X \star Y = m(X \otimes Y)m^* \]

**Conclusion:** \( A_G \in M_n(\{0,1\}) \iff m(A_G \otimes A_G)m^* = A_G. \)
Quantum Graphs

We now go non-commutative! Let $B$ be a finite-dimensional $C^*$-algebra equipped with its canonical trace functional

$$
\psi : B \to \mathbb{C}; \quad \psi = \text{Tr}_{\text{End}(B)}|_B
$$

(If $B = \bigoplus_k M_n(k)(\mathbb{C})$, $\psi = \sum_k n(k) \text{Tr}_{n(k)}(\cdot) = \text{Plancharel trace}$).

Definition

A linear map $A \in \text{End}(B)$ is called a quantum adjacency matrix if

$$
m(A \otimes A)m^* = A \quad (A \text{ is a “quantum Schur idempotent”})
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We call the triple $\mathcal{G} = (B, \psi, A)$ a quantum graph.
Quantum Graphs

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- Classical graphs $\iff$ quantum graphs with abelian $B$.
- For certain applications, we can consider quantum graphs $\mathcal{G} = (B, \psi, A)$ with non-tracial states $\psi$...but not today.
Some Basic Examples

Fix any pair \((B, \psi)\).

1. The **complete quantum graph** over \(B\) is \(\mathcal{K}(B) = (B, \psi, A)\),

\[
A = \psi(\cdot)1_B \quad \textbf{Classical case: } A = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & 1 & \vdots \\
1 & \cdots & 1
\end{pmatrix}
\]

2. The **trivial quantum graph** over \(B\) is \(\mathcal{T}(B) = (B, \psi, A)\) where

\[
A = \text{id}_{B \to B} \quad \textbf{Classical case: } A = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & 1 & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\]

3. If \(B = \bigoplus_k M_{n(k)}(\mathbb{C})\) and \(d = (d^{(k)})_k \in \bigoplus_k D_{n(k)} \subset B\) is **diagonal** with \(\text{Tr}_{n(k)}(d^{(k)}) = 1\) for all \(k\), then

\[
A_d(x) = dx \quad \text{is a quantum adjacency matrix.}
\]

We call \(\mathcal{D}(B) = (B, \psi, A_d)\) a **diagonal quantum graph** over \(B\).
More Examples: Quantum Edge Space Picture

Writing down interesting examples of quantum adjacency matrices might at first seem not so easy.
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Alternate Approach [Nik Weaver]: Recall that a graph $\mathcal{G}$ is completely described by its edge relation $E \subseteq V \times V$.

- Consider the linear subspace

$$S_\mathcal{G} = \text{span}\{e_{x,y} : (x,y) \in E\} \subseteq B(\ell^2(V))$$

- Then $S_\mathcal{G}$ is a bimodule over $C(V) = C(V)' \subseteq B(\ell^2(V))$.

- (Weaver) Every $C(V)' - C(V)'$-bimodule $S \subseteq B(\ell^2(V))$ is of the form $S_\mathcal{G}$ for some unique $\mathcal{G} = (V, E)$. 
More Examples: Quantum Edge Space Picture

Writing down interesting examples of quantum adjacency matrices might at first seem not so easy.

**Alternate Approach [Nik Weaver]:** Recall that a graph $\mathcal{G}$ is completely described by its **edge relation** $E \subset V \times V$.

- Consider the linear subspace $S_{\mathcal{G}} = \text{span}\{e_{x,y} : (x, y) \in E\} \subset B(\ell^2(V))$.
- Then $S_{\mathcal{G}}$ is a **bimodule** over $C(V) = C(V)^{\prime} \subset B(\ell^2(V))$.
- (Weaver) Every $C(V)^{\prime} - C(V)^{\prime}$-bimodule $S \subset B(\ell^2(V))$ is of the form $S_{\mathcal{G}}$ for some unique $\mathcal{G} = (V, E)$.

**Definition (Weaver 2010)**

Let $B \subseteq B(H)$ be a finite dimensional C*-algebra. A **quantum graph** on $B$ is $B^{\prime}$-$B^{\prime}$-bimodule

$$B^{\prime}S_{B^{\prime}} \subseteq B(H).$$

Think of $B^{\prime}S_{B^{\prime}}$ as the collection of all “edge operators” that “connect quantum vertices” in the graph.
Quantum Edge Space vs Quantum Adjacency Matrices

Both approaches to quantum graphs are essentially the same:

▶ **Fact 1**: Given \( B \subseteq B(H) \), a bimodule \( B'S' \subseteq B(H) \) is uniquely determined by a self-adjoint projection

\[
p = \sum_i a_i \otimes b_i^{op} \in B \otimes B^{op}; \quad S = p \cdot B(H) = \left\{ \sum_i a_i Xb_i : X \in B(H) \right\}
\]

▶ **Fact 2**: Any such \( p = p^* = p^2 \in B \otimes B^{op} \) arises as the “Choi-Jamiołkowski” matrix of a completely positive quantum adjacency matrix \( A \in \text{End}(B) \) given by

\[
p = (A \otimes 1)m^*(1_B).
\]

**Conclusion**: Quantum graphs \( G = (B, \psi, A) \) (with completely positive \( A \)) \( \iff \) operator spaces \( S \subseteq B(H) \) that are \( B'-B' \)-bimodules.
Quantum Graphs in “Nature”

Quantum graphs arise naturally in problems in quantum information theory and representation theory.

▶ **Zero-error channel capacity:** Given finite sets $V, W$ and a noisy channel with transition probabilities $(p(w|v))_{v \in V, w \in W}$, can form the confusability graph $G_\Phi = (V, E)$ where

$$(v_1, v_2) \in E \iff \exists w \in W \text{ s.t. } p(w|v_1)p(w|v_2) \neq 0.$$ 

The zero-error capacity of $\Phi$: Is the Shannon capacity of $G_\Phi$.

▶ In Quantum Information Theory, one considers quantum channels

$$\Phi : L^1(B, \psi) \rightarrow M_k(\mathbb{C}).$$

The **Zero-error capacity of $\Phi$:** study Shannon capacity of the non-commutative confusability graph $S_\Phi$ of $\Phi$: $S_\Phi \subset B(L^2(B))$ is a $B'-B'$-bimodule.
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- **In Quantum Information Theory**, one considers quantum channels

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$$ S_\Phi \subset B(L^2(B)) \text{ is a } B'-B'-\text{bimodule}. $$

- **Qgraphs** $G = (B, \psi, A)$ appear in the theory of non-local games (graph isomorphism games), quantum teleportation and superdense coding schemes in QIT, representation theory of quantum symmetry groups of graphs, ...
Cuntz-Krieger Algebras of graphs

Let \( G = (V, A) \) be a finite graph without multiple edges.

The Cuntz-Krieger algebra \( \mathcal{O}(G) = \mathcal{O}_A \) is the universal C*-algebra generated by partial isometries \((s_v)_{v \in V}\) satisfying the relations

(CK1) \( \sum_{v \in V} s_v s^*_v = 1 \).

(CK2) \( s^*_v s_v = \sum_{w \in V} A(v, w) s_w s^*_w \).

Why study these operator algebras?
▶ These algebras turn out to encode the symbolic dynamics associated to a graph \( G \): They provide new conjugacy invariants for the topological Markov chain \((X_A, \sigma_A)\) associated to \( G \).

\[ X_A = \{ (v_i)_{i \in \mathbb{N}} : A(v_i, v_{i+1}) = 1 \} \]
 space of infinite paths, \( \sigma_A : X_A \to X_A ; \sigma_A(v_1, v_2, ...) = (v_2, v_3, ...) \).

▶ They provide a rich class of C*-algebras that are amenable to classification.
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The Cuntz-Krieger algebra $O(G) = O_A$ is the universal $\mathbf{C}^*$-algebra generated by partial isometries $(s_v)_{v \in V}$ satisfying the relations

\begin{align*}
(CK1) \quad & \sum_{v \in V} s_v s_v^* = 1. \\
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- They provide a rich class of $\mathbf{C}^*$-algebras that are amenable to classification.
Examples

1. \( \mathcal{G} = K_n \) the complete graph on \( n \) vertices. Then \( \mathcal{O}(K_n) \) has generators \( (s_i)_{i=1}^n \) satisfying

\[
(CK1) \sum_i s_is_i^* = 1 \quad \& \quad (CK2) \quad s_i^*s_i = \sum_j s_js_j^* = 1.
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This is the famous Cuntz algebra \( \mathcal{O}_n \)!
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2. \( \mathcal{G} = T_n \) (trivial graph on \( n \) vertices, with self-loops). Then \( \mathcal{O}(T_n) \) has generators \( (s_i)_{i=1}^n \) satisfying

\[
\sum_i s_is_i^* = 1, \quad s_i^*s_i = s_is_i^* \quad \implies \quad \mathcal{O}(T_n) = C(\mathbb{T})^{\oplus n}.
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2. $G = T_n$ (trivial graph on $n$ vertices, with self-loops). Then $\mathcal{O}(T_n)$ has generators $(s_i)_{i=1}^n$ satisfying

$$\sum_i s_is_i^* = 1, \quad s_i^*s_i = s_is_i^* \implies \mathcal{O}(T_n) = C(\mathbb{T})^\oplus n.$$ 

3. Let $G \leftrightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathcal{O}(G) = C^* \left( s_1, s_2 \mid s_1 s_1^* + s_2 s_2^* = 1, \ s_1^* s_1 = 1, \ s_2^* s_2 = s_1 s_1^* \right).$$
General properties of Cuntz-Krieger algebras

- \( \mathcal{O}(\mathcal{G}) \neq 0 \) as long as \( A_\mathcal{G} \neq 0 \).
- \( \mathcal{O}(\mathcal{G}) \) is always nuclear.
- \( \mathcal{O}(\mathcal{G}) \) is always unital.
- \( \mathcal{O}(\mathcal{G}) \) are classified, up to isomorphism.
- Include many naturally occurring examples of \( C^* \)-algebras.
- Can be studied using graph theoretical-symbolic dynamical tools.

Goal: Generalize Cuntz-Krieger Algebras to Quantum Graphs!
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Quantum Cuntz-Krieger Algebras

Now let $\mathcal{G} = (B, \psi, A)$ be a quantum graph. We want to build an operator algebra $\mathcal{F}\mathcal{O}(\mathcal{G})$ out of $\mathcal{G}$ as above. Here’s our attempt.

Definition (B-Eifler-Voigt-Weber)

The quantum Cuntz-Krieger Algebra associated to $\mathcal{G}$ is the universal $C^*$-algebra $\mathcal{F}\mathcal{O}(\mathcal{G})$ generated by the range of a linear map $S: B \to \mathcal{F}\mathcal{O}(\mathcal{G})$ satisfying the relations:

1. $\mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*) = S$
2. $\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$

where $\mu: \mathcal{F}\mathcal{O}(\mathcal{G}) \otimes \mathcal{F}\mathcal{O}(\mathcal{G}) \to \mathcal{F}\mathcal{O}(\mathcal{G})$ is the multiplication map and $S^*(b) = S(b^*)^*$. Why is this a reasonable definition?

Theorem

If $\mathcal{G}$ is classical (i.e., $B = C(V)$ is abelian), then $\mathcal{F}\mathcal{O}(\mathcal{G})$ is the usual Cuntz-Krieger algebra $\mathcal{O}(\mathcal{G})$ associated to $\mathcal{G}$ (up to a $KK$-equivalence).
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Quantum CK-Algebras: Unpacking the Definition

\[ \mathcal{G} = (B, \psi, A) \leadsto \mathcal{F} \mathcal{O}(\mathcal{G}) = C^* \left( (S(b))_{b \in B} \mid \begin{array}{c} S : B \rightarrow \mathcal{O}_G \text{ linear} \\ \mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S \\ \mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A \end{array} \right) \]

Write \( B = \bigoplus_{k=1}^m M_{n(k)}(\mathbb{C}) \) with standard matrix unit basis \( \{ e_{ij}^{(k)} \} \).

For \( 1 \leq k \leq m \), put \( f_{ij}^{(k)} = n(k)^{-1} e_{ij}^{(k)} \) and

\[ S^{(k)} = [S(f_{ij}^{(k)})] \in M_{n(k)}(\mathcal{F} \mathcal{O}(\mathcal{G})). \]
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Then \( \mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S \)

\[ \iff S^{(k)} S^{(k)*} S^{(k)} = S^{(k)}, \quad k = 1, \ldots, m \]

\[ \iff \text{each } S^{(k)} \text{ is a partial isometry in } M_{n(k)}(\mathbb{F}\mathcal{O}(G)). \]

\[ \mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A \]

\[ \iff S^{(k)*} S^{(k)} = (S^{(1)} S^{(1)*}, \ldots, S^{(m)} S^{(m)*}) \tilde{A} \]

where \( \tilde{A}^{xyz}_{ijk} = \frac{n(z)}{n(k)} A^{xyz}_{ijk} \) (rewighted adjacency matrix).
Quantum CK-Algebras: Unpacking the Definition

**Summary:** Given $\mathcal{G} = (B, \psi, A)$, $B = \bigoplus_{k=1}^{m} M_{n(k)}$, the quantum Cuntz-Krieger Algebra $\mathcal{O}_{\mathcal{G}}$ is:

1. Generated by the coefficients of $m$ matrix partial isometries $S^{(k)} \in M_{n(k)}(\mathbb{F} \mathcal{O}(\mathcal{G}))$,
2. Subject to the quantum Cuntz-Krieger relations

$$S^{(k)} S^{(k)*} = (S^{(1)} S^{(1)*}, \ldots, S^{(m)} S^{(m)*}) \tilde{A}.$$

3. **BUT:** There seems to be no natural analogue of the relation

$$\sum_{k} S^{(k)} (S^{(k)})^{*} = 1$$  \quad (incompatible matrix sizes)

$$\Rightarrow \mathcal{O}(\mathcal{G}) \text{ might not be unital in general.}$$
Example 1: Trivial Quantum Graph $G = T(M_n)$.

$\mathbb{F}O(T(M_n)) = C^*\left(\begin{array}{c} S_{ij}, 1 \leq i, j \leq n \end{array} | \begin{array}{c} S = [S_{ij}] \text{ is a partial isometry} \\ \sum_i S_{ii}^* S_{ij} = \sum_i S_{ii} S_{ij}^* \iff S^* S = SS^* \end{array} \right)$
Example 1: Trivial Quantum Graph $\mathcal{G} = \mathcal{T}(M_n)$.

$$\mathcal{FO}(\mathcal{T}(M_n)) = C^*\left(S_{ij}, 1 \leq i, j \leq n \mid S = [S_{ij}] \text{ is a partial isometry} \right.$$ 

$$\sum_l S^*_l S_{lj} = \sum_l S_{li} S^*_j \iff S^* S = S S^*$$

- $\mathcal{FO}(\mathcal{T}(M_n))$ is non-unital, non-nuclear.
- We have quotient maps

$$\mathcal{O}_\mathcal{T}(M_n) \to C(\mathbb{T})^n$$

(non-unital free product)

$$S_{ij} \mapsto \delta_{ij} z_i.$$ 

$$\mathcal{O}_\mathcal{T}(M_n) \to U^{nc}(n)$$

$$S_{ij} \mapsto u_{ij}.$$ 

where $U^{(nc)}(n) = C^*\left(u_{ij}, 1 \leq i, j \leq n \mid U = [u_{ij}] \text{ unitary} \right)$ is Brown's Universal noncommutative unitary algebra.

- In fact $M_n(\mathcal{FO}(\mathcal{T}(M_n))^+) \cong M_n \ast (C(S^1) \oplus \mathbb{C})$. 
Example 2: Diagonal Quantum Graph $G = \mathcal{D}(M_2)$

$A_d(e_{ij}) = \delta_{i,1}2e_{ij}$.

$\mathcal{F}\mathcal{O}(\mathcal{D}(M_2)) = C^* \left( S_{ij}, 1 \leq i, j \leq 2 \mid S = [S_{ij}] \text{ is a partial isometry} \sum_l S^*_l S_{lj} = \delta_{i=j=1} \sum_l 2S_{1l}S^*_{1l} \right)$

- Get $S^*_{i2} S_{l2} = 0 \implies S_{l2} = 0$.
- Thus, the canonical map $S : B \to \mathcal{O}_G$ need not be injective!
Basic Examples, Basic Properties

Example 3: Complete Quantum Graph $\mathcal{G} = \mathcal{K}(M_n)$.

$$\mathcal{F}\mathcal{O}(\mathcal{K}(M_n)) = C^*(S_{ij}, 1 \leq i, j \leq n | S = [S_{ij}] \text{ is a partial isometry} \sum_l S^*_{li} S_{lj} = \delta_{i,j} \left( n \sum_x, l S_{xl} S^*_{xl} \right))$$
Basic Examples, Basic Properties

**Example 3:** Complete Quantum Graph $\mathcal{G} = \mathcal{K}(M_n)$.

$$\mathbb{F}O(\mathcal{K}(M_n)) = C^*\left( S_{ij}, \ 1 \leq i, j \leq n \mid \begin{array}{l}
S = [S_{ij}] \text{ is a partial isometry} \\
\sum_l S^*_liSlj = \delta_{i,j} \left( n \sum_x l S_{xl}S^*_{xl} \right)
\end{array} \right)$$

- What is this algebra? Is it unital? Nuclear? ...
- We have a quotient map

$$\mathbb{F}O(\mathcal{K}(M_n)) \to O_{n^2} = O(K_{n^2}) \text{ (the Cuntz algebra)}$$

$$S_{ij} \mapsto n^{-1/2} \hat{S}_{ij} \quad (\hat{S}_{ij} = \text{standard Cuntz isometry}).$$
Basic Examples, Basic Properties

**Example 3:** Complete Quantum Graph \( \mathcal{G} = \mathcal{K}(M_n) \).

\[
\mathcal{FO}(\mathcal{K}(M_n)) = C^* \left( S_{ij}, 1 \leq i, j \leq n \mid S = [S_{ij}] \text{ is a partial isometry} \right)
\]

\[
\sum_l S_{i*}^l S_{lj} = \delta_{i,j} \left( n \sum_x S_{x*}^l S_{xl} \right)
\]

- What is this algebra? Is it unital? Nuclear? ...
- We have a quotient map

\[
\mathcal{FO}(\mathcal{K}(M_n)) \to O_{n^2} = \mathcal{O}(K_{n^2}) \quad \text{(the Cuntz algebra)}
\]

\[
S_{ij} \mapsto n^{-1/2} \hat{S}_{ij} \quad (\hat{S}_{ij} = \text{standard Cuntz isometry}).
\]

- **Very surprising/mysterious theorem:** The above homomorphism is actually an isomorphism

\[
\mathcal{FO}(\mathcal{K}(M_n)) \simeq O_{n^2}.
\]

- So \( \mathcal{FO}(\mathcal{K}(M_n)) \) is indeed unital, nuclear, etc.

- **The proof:** Uses the notion of a quantum isomorphism between quantum graphs (aka superdense coding in QIT): \( \mathcal{K}(M_n) \) and \( K_{n^2} \) turn out to be quantum isomorphic.
Conclusion

Quantum graphs and Quantum Cuntz-Krieger Algebras seem to have some new and interesting properties! Lots of work to be done!

THANKS FOR LISTENING.