Explicit Salem Sets in Euclidean Space

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Outline

Hausdorff Dimension

Let $A \subseteq \mathbb{R}^d$ be Borel set. Let $\alpha \geq 0$.
$R =$ Rectangle $= \prod_{i=1}^d [a_i, b_i]$, $\mathsf{Vol}(R) = \prod_{i=1}^d (b_i - a_i)$.

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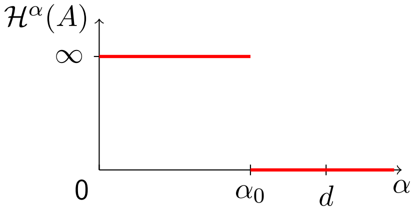

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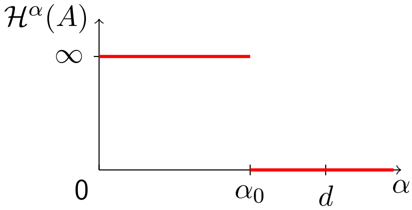

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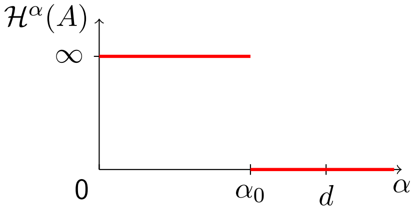


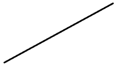
Hausdorff Dimension:

$\dim_H(A) = \alpha_0 =$ the number α where $\mathcal{H}^\alpha(A)$ jumps from 0 to ∞ $= \sup\{\alpha : \mathcal{H}^\alpha(A) > 0\}$

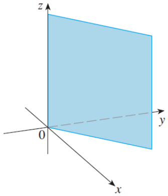
Hausdorff Dimension Agrees With Intuition

Point: Hausdorff Dimension = 0

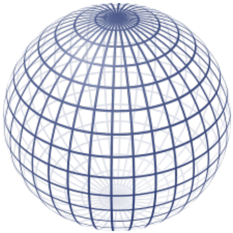
Line: Hausdorff Dimension = 1



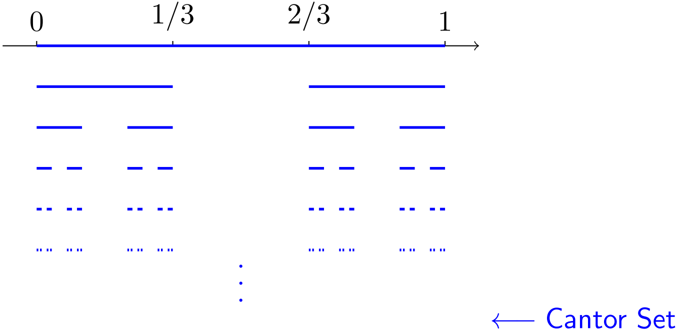
Plane: Hausdorff Dimension = 2



Sphere: Hausdorff Dimension = 2



Hausdorff Dimension of Fractals: Middle-$1/3$ Cantor Set



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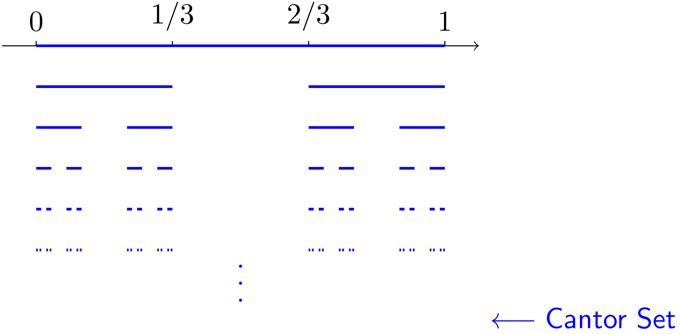

Lebesgue Measure = “Length” = 0

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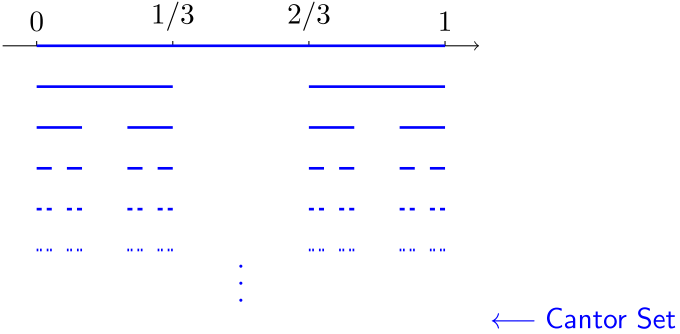

Lebesgue Measure = “Length” = 0

$$\text{Hausdorff Dimension} = \frac{\log 2}{\log 3} = 0.6309\ldots$$

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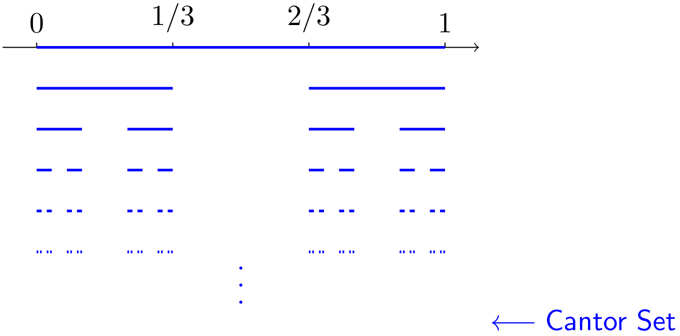

Lebesgue Measure = "Length" = 0

Hausdorff Dimension = $\frac{\log 2}{\log 3} = 0.6309\ldots$

$$C_{1/3} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right)$$

More Fractals

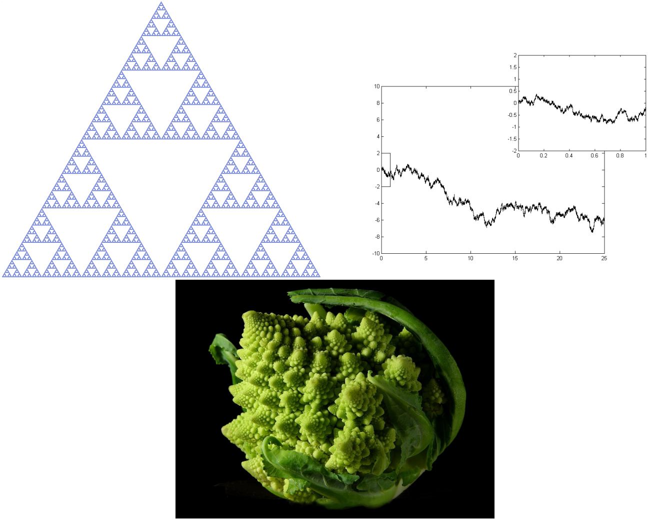

Figure: Sierpinski Triangle ($\dim_H = \frac{\log 3}{\log 2}$), graph of Brownian motion ($\dim_H = \frac{3}{2}$), and surface of Romanesco broccoli ("$\dim_H$" ≈ 1.26)

Hausdorff Dimension in Terms of Energy Integral

Theorem (Frostman)

$$\mathit{dim}_H(A) = \sup\{\alpha : \exists \mu \in \mathcal{M}(A) \ \mathit{s.t.}\ I_\alpha(\mu) < \infty\}$$

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Definition

$\mathcal{M}(A)$ is the set of all non-zero finite Borel measures on $\mathbb{R}^d$ with supp$(\mu) \subseteq A$.

Definition

supp(μ) is the smallest closed set C with $\mu(\mathbb{R}^d \setminus C) = 0$.

Fourier Transform of a Measure

Definition

If $f : \mathbb{R}^d \to \mathbb{R}$, the Fourier transform of f is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx \quad \text{for } \xi \in \mathbb{R}^d.$$

Definition

If μ is a measure on $\mathbb{R}^d$, the Fourier transform of μ is

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Proof of Second Equality.

By Parseval and the convolution theorem for Fourier transforms,

$$I_\alpha(\mu) = \int (|\cdot|^{-\alpha} * \mu)(y)\, d\mu(y) = \int (\widehat{|\cdot|^{-\alpha} *} \mu)(\xi)\ \overline{\widehat{\mu}}(\xi) d\xi$$
$$= \int \widehat{|\cdot|^{-\alpha}}(\xi)\ \widehat{\mu}(\xi)\ \overline{\widehat{\mu}}(\xi) d\xi = C \int |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi$$

□

Hausdorff Dimension in Terms of Fourier Transform

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$I_\alpha(\mu) < \infty$ is about the decay of $\widehat{\mu}(\xi)$ at ∞.

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$I_\alpha(\mu) < \infty$ is about the decay of $\widehat{\mu}(\xi)$ at ∞.

Remark

If $\mu \in \mathcal{M}(A)$ decays like $|\widehat{\mu}(\xi)|^2 \lesssim |\xi|^{-\beta}$, then $\beta \leq \dim_H(A)$.

Hausdorff Dimension and Fourier Dimension

Theorem (Hausdorff Dimension)

$$dim_H(A) = \sup \left\{ \alpha \in [0,d] : \exists \mu \in \mathcal{M}(A) \textit{ s.t. } \int |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi < \infty \right\}$$

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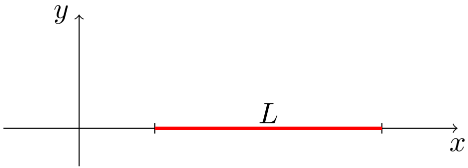
Theorem

$$dim_F A \leq dim_H A.$$

Hausdorff Dimension vs Fourier Dimension

Fourier dimension depends on the ambient space, while Hausdorff dimension does not.

Example



- If we view L as an interval in $\mathbb{R}$, then

$$\dim_F L = \dim_H L = 1.$$

- If we view L as a line segment in $\mathbb{R}^2$, then

$$\dim_F L = 0 \quad \text{and} \quad \dim_H L = 1.$$

Hausdorff Dimension vs Fourier Dimension

Examples

- If A is a k-dimensional plane in $\mathbb{R}^d$ with $k < d$, then

$$\dim_F A = 0 \quad \text{and} \quad \dim_H A = k.$$

- If $A \subseteq (d-1)$-dimensional plane in $\mathbb{R}^d$, then

$$\dim_F A = 0 \quad \text{and} \quad \dim_H A \in [0, d-1]$$

Hausdorff Dimension vs Fourier Dimension

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Proof.

If $A \subseteq \{x \in \mathbb{R}^d : x \cdot \xi_0 = c\}$, and $\mu \in \mathcal{M}(A)$, then
$$\widehat{\mu}(n\xi_0) = \int_A e^{-2\pi i n \xi_0 \cdot x} d\mu(x) = e^{-2\pi i n c} \mu(A) \neq 0,$$
which does not go to zero as $\xi = n\xi_0 \to \infty$. □

Hausdorff Dimension vs Fourier Dimension

Examples

- If $C_{1/3}$ = middle-$1/3$ Cantor set in $\mathbb{R}$, then

$$\dim_F C_{1/3} = 0 \quad \text{and} \quad \dim_H C_{1/3} = \frac{\log 2}{\log 3}$$

- If C_δ = middle-δ Cantor set in $\mathbb{R}$, then

$$\dim_F C_\delta < \dim_H C_\delta \quad \text{for all } \delta \in (0,1)$$

and

$$0 < \dim_F C_\delta \quad \text{for almost every } \delta \in (0,1)$$

- If $\dim_F C_\delta > 0$, then $2/(1-\delta)$ is not a Pisot number (i.e., an algebraic integer whose conjugates are strictly less than 1 in absolute value).
- If $\dim_F A > 0$, then A generates $\mathbb{R}^d$ as an additive group.

Salem Sets

Theorem

$$dim_F A \leq dim_H A.$$

Definition

A set Borel set $A \subseteq \mathbb{R}^d$ is called a Salem set if

$$\dim_F A = \dim_H A.$$

Examples

- For some non-Salem sets, see the previous slide.
- Point = Salem set of dimension 0
- Sphere = Salem set of dimension $d-1$
- Ball = Salem set of dimension d
- Salem sets of dimensions $\alpha \neq 0, d-1, d$ are harder to find.

Salem Sets of Every Dimension

Theorem (Salem (1951))

For every $\alpha \in (0, 1)$, there exists a Salem set $A \subseteq \mathbb{R}$ with dimension α.

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Remarks

- Salem's Construction: Random Cantor sets
- Kahane's Construction: Images of Brownian motion
- There are many other **random** constructions (e.g., by Kahane, Shapiro, Bluhm, Łaba and Pramanik, Chen and Seeger).

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Problem (Kahane (1966))

Can we find explicit (i.e., non-random) Salem sets in $\mathbb{R}^d$ of every dimension?

Explicit Salem Sets in $\mathbb{R}$

Definition (Set of τ-Well-Approximable Numbers)

$$E(\tau) = \left\{ x \in \mathbb{R} : \left| x - \frac{r}{q} \right| \leq |q|^{-\tau} \text{ for } \infty\text{-many } (q, r) \in \mathbb{Z} \times \mathbb{Z} \right\}$$

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Example

Given $x = \pi$, find $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ such that

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Taking $\frac{r}{q} = \frac{314159}{10^5} = 3.14159$ doesn't work:

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But the simpler rational $\frac{r}{q} = \frac{22}{7} = 3.\overline{142857}$ does:

$$\left| x - \frac{r}{q} \right| = \left| \pi - \frac{22}{7} \right| = 0.0012644\ldots < 7^{-2} = |q|^{-2}$$

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Theorem (Dirichlet (1834))

$E(\tau) = \mathbb{R}$ *when* $\tau \leq 2$.

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$E(\tau)$ *has Hausdorff dimension* $2/\tau$ *when* $\tau > 2$.

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$E(\tau)$ *has Hausdorff dimension* $2/\tau$ *when* $\tau > 2$.

Theorem (Kaufman (1981))

$E(\tau)$ *is a Salem set of dimension* $2/\tau$ *when* $\tau > 2$.

Explicit Salem Sets in $\mathbb{R}^d$: $d > 1$?

Definition

$$E_{\mathsf{rot}}(\tau) = \left\{ x \in \mathbb{R}^d : |x| \in E(\tau) \right\}$$

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$E_{rot}(\tau)$ is a Salem set with dimension $d - 1 + \frac{2}{\tau}$ for every $\tau > 2$.

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Remarks

- Gives explicit Salem sets in $\mathbb{R}^d$ of every dimension $\alpha \in (d-1, d)$.
- Leaves $\alpha \in (0, d-1)$.

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In number theory, the natural multi-dimensional version of $E(\tau)$ is:

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Definition

$E(m, n, \tau) = \{x \in \mathbb{R}^{mn} : |xq - r| \le |q|^{-\tau+1}$ for ∞-many $(q, r) \in \mathbb{Z}^n \times \mathbb{Z}^m\}$

Here $x \in \mathbb{R}^{mn}$ is viewed as an $m \times n$ matrix and $|\cdot|$ is the max norm.

Explicit Salem Sets in $\mathbb{R}^d$: $d > 1$?

In number theory, the natural multi-dimensional version of $E(\tau)$ is:

Definition

$E(m, n, \tau) = \{x \in \mathbb{R}^{mn} \colon |xq - r| \le |q|^{-\tau+1}$ for ∞-many $(q, r) \in \mathbb{Z}^n \times \mathbb{Z}^m\}$

Here $x \in \mathbb{R}^{mn}$ is viewed as an $m \times n$ matrix and $|\cdot|$ is the max norm.

This is about simultaneous Diophantine approximation of linear forms, i.e., having good approximate integer solutions of several linear forms at once:

$$|x_{11}q_1 + x_{12}q_2 + \cdots + x_{1n}q_n - r_1| \le |q|^{-\tau+1}$$
$$|x_{21}q_1 + x_{22}q_2 + \cdots + x_{2n}q_n - r_2| \le |q|^{-\tau+1}$$
$$\vdots$$
$$|x_{m1}q_1 + x_{m2}q_2 + \cdots + x_{mn}q_n - r_m| \le |q|^{-\tau+1}$$

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$\dim_H E(m,n,\tau) = m(n-1) + \frac{m+n}{\tau}$ for every $\tau > 1 + \frac{n}{m}$

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$\dim_F E(m,n,\tau) \ge \frac{2n}{\tau}$ for every $\tau > 1 + \frac{n}{m}$

But we don't know whether $E(m,n,\tau)$ is Salem because

$$m(n-1) + \frac{m+n}{\tau} > \frac{2n}{\tau}$$

Explicit Salem Sets in $\mathbb{R}^d$: $d = 2$

Definition

$$E(\mathbb{C}, \tau) = \left\{ x \in \mathbb{R}^2 : \left| x - \frac{r}{q} \right| \leq |q|^{-\tau} \text{ for } \infty\text{-many } (q, r) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \right\}$$

Here r/q is interpreted via the identification $\mathbb{R}^2 \simeq \mathbb{C}$.

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Remarks

- Gives Salem sets in $\mathbb{R}^2$ of every dimension $\alpha \in (0, 2)$.
- Resolves Kahane's problem when $d = 2$.

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Remarks on Proof

Kaufman's proof applies almost verbatim. The hard part was coming up with the set $E(\mathbb{C}, \tau)$ where Kaufman's proof would work.

Explicit Salem Sets in $\mathbb{R}^d$: $d = 4$?

Since $\mathbb{R}^2 \simeq \mathbb{C}$ worked, it is natural to try

$$\mathbb{R}^4 \simeq \mathbb{H} \, (= \text{the set of quaternions})$$
$$(a, b, c, d) = a + bi + cj + dk$$
$$i^2 = j^2 = k^2 = ijk = -1$$

Definition

$$E(\mathbb{H}, \tau) = \left\{ x \in \mathbb{R}^4 : \left| x - \frac{r}{q} \right| \leq |q|^{-\tau} \text{ for } \infty\text{-many } (q, r) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \right\}$$

Remarks

The proof that $E(\mathbb{H}, \tau)$ is Salem fails because there is no good divisor bound for the quaternions.

Explicit Salem Sets in $\mathbb{R}^d$: All d

Definition

$$E(K, B, \tau) = \left\{ x \in \mathbb{R}^d : \left| x - \frac{r}{q} \right| \leq |q|^{-\tau} \text{ for } \infty\text{-many } (q, r) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}$$

$K =$ degree d field extension of $\mathbb{Q}$ (i.e., a number field)
$\mathcal{O}(K) =$ ring of integers of K
$B = \{\omega_1, \ldots, \omega_d\} =$ integral basis for K

$$\mathbb{Q}^d \simeq K, \quad \mathbb{Z}^d \simeq \mathcal{O}(K), \quad \mathbb{R}^d \simeq \mathbb{R}\omega_1 + \cdots + \mathbb{R}\omega_d$$

$$(q_1, \ldots, q_d) = q_1\omega_1 + \cdots + q_d\omega_d$$

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Theorem (Fraser, Hambrook (2019))

$E(K,B,\tau)$ *is a Salem set with dimension* $2d/\tau$ *for every* $\tau > 2$.

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Remarks

- Gives Salem sets in $\mathbb{R}^d$ of every dimension $\alpha \in (0,d)$.
- Completely resolves Kahane's problem.

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Theorem (Fraser, Hambrook (2019))

$E(K, B, \tau)$ is a Salem set with dimension $2d/\tau$ for every $\tau > 2$.

Remarks on Proof

The proofs for $E(\tau)$ and $E(\mathbb{C}, \tau)$ rely on features of $\mathbb{R}$ and $\mathbb{C}$ that don't generalize easily to number fields K:

- Divisor bounds in $\mathbb{Z}$ and $\mathbb{Z}^2 \simeq \mathbb{Z}[i]$ (which come from unique factorization and finiteness of the unit group)
- Transpose of matrix for $x \in \mathbb{C}$ is matrix for $\overline{x}$.
- For $q = a + ib \in \mathbb{Z}[i]$, $N(\langle q \rangle) = a^2 + b^2 = |q|^2$.

Explicit Salem Sets in $\mathbb{R}^d$: All d

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$E(K, B, \tau)$ is a Salem set with dimension $2d/\tau$ for every $\tau > 2$.

Remarks on Proof

To overcome these obstacles, we:

- Use unique factorization of ideals in $\mathcal{O}(K)$ and Dirichlet's unit group theorem to obtain an appropriate divisor bound.
- Rediscover an algebra theorem: Transpose of matrix for $q \in K$ is matrix for q in a *different basis*.
- Use pigeonholing argument to eliminate dependence on comparability of algebraic norm $N(\langle q \rangle)$ and geometric norm $|q|$.

Proof

Want:

$$\dim_F E(K,B,\tau) = \dim_H E(K,B,\tau) = 2d/\tau$$

• $\dim_F E(K,B,\tau) \le \dim_H E(K,B,\tau)$ by definition of Fourier dimension.

• $\dim_H E(K,B,\tau) \le 2d/\tau$ by standard covering argument, which comes from writing

$$\begin{aligned} E(K,B,\tau) &= \left\{ x \in \mathbb{R}^d : \left| x - \frac{r}{q} \right| \le |q|^{-\tau} \text{ for } \infty\text{-many } (q,r) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{|q|>N} \bigcup_{r \in \mathbb{Z}^d} \overline{B}(r/q, |q|^{-\tau}) \end{aligned}$$

• $2d/\tau \le \dim_F E(K,B,\tau)$ proved by constructing a measure ...

Proof

$$\mu = \operatorname*{w-lim}_{k\to\infty} F_{M_k} F_{M_{k-1}} \cdots F_{M_1}\, dx$$

$$M_1 \le M_2 \le \ldots \to \infty \text{ rapidly}$$

$$F_M(x) = \sum_{\substack{q\in\mathbb{Z}^d \\ \frac{M}{2}<|q|\le M}} \sum_{r\in\mathbb{Z}^d} \underbrace{\phi_\epsilon(x - r/q)}_{\text{normalized bump on } \overline{B}(r/q, M^{-\tau})}$$

Here $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$, $\epsilon = M^\tau$, and ϕ is positive, smooth, L^1-normalized, and supported in $\overline{B}(0,1)$. Then

$$\text{supp}(\mu) \subseteq \bigcap_{k=1}^{\infty} \text{supp}(F_{M_k}) \subseteq E(K, B, \tau)$$

and ...

Proof

$$\widehat{F_M}(s) = \widehat{\phi}(s/M^\tau) \sum_{\substack{q \in \mathbb{Z}^d \\ M/2 < |q| \le M}} \sum_{r \in R_q} e(s \cdot r/q) \quad \text{for } s \in \mathbb{Z}^d$$

where R_q = set of representatives of $\mathcal{O}(K)/\langle q \rangle$.
Matrix Games: There is a $L \in \mathbb{Z}$ depending on K and B such that

$$\left| \sum_{r \in R_q} e(s \cdot r/q) \right| \quad \begin{cases} \le N(\langle q \rangle) & \text{if } q \text{ divides } Ls \\ = 0 & \text{otherwise} \end{cases}$$

Problem: Need bound on number of divisors q of Ls such that $|q| \le M$.
Solution: Unique factorization of ideals in $\mathcal{O}(K)$, Dirichlet's unit theorem.

$$|\widehat{F_M}(\xi)|^2 \le C|\xi|^{-2d/\tau} \exp\left(\frac{\log|\xi|}{\log\log|\xi|}\right) (\log M)^C$$

An induction argument gives

$$|\widehat{\mu}(\xi)|^2 \le |\xi|^{-2d/\tau} \exp\left(\frac{C \log|\xi|}{\log\log|\xi|}\right)$$

What Else?

A Sample of Related Problems:

- Exact Fourier Dimension of $E(m, n, \tau)$
- Restricted Diophantine Approximation
- Fourier Restriction

Restricted Denominators

For infinite $Q \subseteq \mathbb{Z}$, define

$$E(\tau, Q) = \left\{ x \in \mathbb{R} : \left| x - \frac{r}{q} \right| \leq \frac{1}{|q|^\tau} \text{ for infinitely many } (q, r) \in Q \times \mathbb{Z} \right\}$$

and

$$\nu(Q) = \inf\{\nu \geq 0 : \sum_{q \in Q} |q|^{-\nu} < \infty\}$$

Theorem (Borosh-Fraenkel (1972))

If $\tau > 2$, then $\dim_H E(\tau, Q) = \dfrac{1 + \nu(Q)}{\tau}$.

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Theorem (Hambrook (2015))

If $\tau > 2$, then $\dim_F E(\tau, Q) \geq \frac{2\nu(Q)}{\tau}$. In particular, if $\nu(Q) = 1$ (eg. $Q =$ primes), then $E(\tau, Q)$ is Salem.

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If $\tau > 2$*, then* $\dim_F E(\tau, Q) \geq \dfrac{2\nu(Q)}{\tau}$*. In particular, if* $\nu(Q) = 1$ *(eg.* $Q =$ *primes), then* $E(\tau, Q)$ *is Salem.*

Problem

Increase lower bound when

$$\nu(Q) = \inf\{\nu \geq 0 : \sum_{q \in Q} |q|^{-\nu} < \infty\} < 1?.$$

For example, when $Q =$ squares and $\nu(Q) = 1/2$.

Restricted Denominators and Numerators

For infinite $Q, R \subseteq \mathbb{Z}$, define

$$E(\tau, Q, R) = \left\{ x \in \mathbb{R} : \left| x - \frac{r}{q} \right| \leq \frac{1}{|q|^\tau} \text{ for infinitely many } (q, r) \in Q \times R \right\}$$

Theorem (Harman (1988))

If $\tau > 2$ and $Q = R =$ primes, then $\dim_H E(\tau, Q, R) = \dfrac{2}{\tau}$.

Problem

If $\tau > 2$ and $Q = R =$ primes, then $\dim_F E(\tau, Q, R) = \dfrac{2}{\tau}$?

Restricted Denominators and Numerators

Problem

If $\tau > 2$ and $Q = R =$ primes, then $\dim_F E(\tau, Q, R) = \frac{2}{\tau}$?

Reduces to...

Problem

Are there infinitely many integers M such that for every prime q and integer k satisfying $M/2 \leq q \leq M$ and $q \nmid k$ and for every $\epsilon > 0$, we have

$$\left| \sum_{\substack{0 \leq r < q \\ r \text{ prime}}} e^{2\pi i k r / q} \right| \leq C_\epsilon |k|^\epsilon M^\epsilon ?$$

Remark

For primes, this looks unlikely. But maybe there's another approach. Or maybe for another set R.

Fourier Restriction

Fourier Restriction Problem

Given a measure μ on $\mathbb{R}^d$, determine the exponents $1 \leq p \leq 2$ and $q \geq 1$ for which

$$\text{(R)} \qquad \left(\int |\widehat{f}(\xi)|^q d\mu(\xi) \right)^{1/q} \leq C \left(\int |f(x)|^p dx \right)^{1/p}$$

for all functions f in a dense subspace of $L^p(\lambda)$. In other words, determine when the Fourier transform $f \mapsto \widehat{f}$ is a continuous operator from $L^p(\lambda)$ to $L^q(\mu)$.

Applications

- Strichartz estimates in PDE
- Exponential sum estimates in number theory
- Kakeya problem in geometric measure theory

Sharpness

Mockenhaupt-Mitis-Bak-Seeger Restriction Theorem

If $\dim_H(\mu) \geq \alpha$ and $\dim_F(\mu) \geq \beta$, then the restriction inequality (R) holds whenever $1 \leq p \leq p_0$ and $q = 2$, where
$p_0 = (4d - 4\alpha + 2\beta)/(4d - 4\alpha + \beta)$.

The range of p is best possible on $\mathbb{R}^d$ (Knapp example) and $\mathbb{R}$:

Theorem (Hambrook-Łaba (2013))

There is a measure μ on $\mathbb{R}$ that satisfies $\dim_H(\mu) \geq \alpha$ and $\dim_F(\mu) \geq \beta$, but the restriction inequality (R) *fails whenever $p > p_0$ and $q = 2$.*

However, as shown by Chen and Seeger and by Łaba and Wang, there are measures μ that satisfy $\dim_H(\mu) \geq \alpha$ and $\dim_F(\mu) \geq \beta$ and (R) for some $p > p_0$. The constructions are **random**.

Problem

Are there explicit (i.e., non-random) measures μ that satisfy $\dim_H(\mu) \geq \alpha$ and $\dim_F(\mu) \geq \beta$ and (R) *for some $p > p_0$? In particular, is there such a measure on $E(\tau)$?*

The End

Thank You for Your Attention

Any Questions?