The HRT Conjecture (for real-valued functions)

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MAAM 2020
Saturday October 17, 2020

Partially supported by NSF grant: DMS 1814253

Outline

The conjecture

$g : \mathbb{R} \to \mathbb{C}$, $a, b \in \mathbb{R}$: $M_b g(x) = e^{2\pi i b x} g(x)$, $T_a g(x) = g(x-a)$.

Conjecture (C. Heil, J. Ramanathan, and P. Topiwala (HRT) 96)

For $0 \neq g \in L^2(\mathbb{R})(S(\mathbb{R}))$ *and* $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$, *the set*

$$\mathcal{G}(g, \Lambda) = \{M_{b_k} T_{a_k} g = e^{2\pi i b_k \cdot} g(\cdot - a_k)\}_{k=1}^N$$

is a linearly independent set in $L^2(\mathbb{R})$.
That is, if $\{c_k\}_{k=1}^N \subset \mathbb{C}$ *is such that*

$$\sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k) = 0 \, a.e,$$

can we conclude that $c_k = 0$ *for all* $k = 1, \ldots, N$*?*

The case of pure translates

Example

Pure translates (Edgar and Rosenblatt (1980's)): $0 \neq g \in L^2(\mathbb{R})$, $\Lambda = \{(a_k, b)\}_{k=1}^{M}$.

$$\sum_{k=1}^{N} e^{2\pi i b x} c_k g(x - a_k) = 0 \iff p(\xi)\hat{g}(\xi) = 0$$

Some examples for fun

Question

Let $g \in L^2(\mathbb{R})$, or even $\mathcal{S}(\mathbb{R})$. Define

$$\mathcal{G}_1(g) = \{g(x), g(x-1), e^{2\pi i x} g(x), g(x-\sqrt{2})\}$$

$$\mathcal{G}_2(g) = \{g(x), g(x-1), e^{2\pi i x} g(x), e^{2\pi i \sqrt{2} x} g(x-\sqrt{2})\}$$

$$\mathcal{G}_3(g) = \{g(x), g(x-1), e^{2\pi i x} g(x), e^{2\pi^2 i x} g(x-\sqrt{2})\}.$$

Is $\mathcal{G}_i(g)$ linearly independent for $i = 1, 2, 3$?

Why the HRT conjecture?

Claim (J. Von Neumann (1932)/ D. Gabor (1944))

Let $g(t) = e^{-\pi t^2}$.

- $\mathcal{G}(g,1,1) = \{g_{n,k}(t) : k,n \in \mathbb{Z}\} = \{e^{2\pi i k t} g(t-n) : k,n \in \mathbb{Z}\}$ *spans a dense subspace of* $L^2(\mathbb{R})$.
- *For* $f \in L^2(\mathbb{R})$:

$$f = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{n,k} e^{2\pi i k t} g(t-n).$$

Theorem (V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder (1971); A. M. Perelomov (1971))

$\mathcal{G}(g,1,1) = \{g_{n,k}(t) : k,n \in \mathbb{Z}\} = \{e^{2\pi i k t} g(t-n) : k,n \in \mathbb{Z}\}$ *spans a dense subspace of* $L^2(\mathbb{R})$.

Why the HRT conjecture?

- Given $g \in L^2(\mathbb{R})$, and $\{(a_k, b_k)\}_{k\in I} \subset \mathbb{R}^2$,

$$\{M_{b_k} T_{a_k} g\}_{k\in I} = \{e^{2\pi b_k i \cdot} g(\cdot - a_k)\}_{k\in I}$$

is called a Weyl-Heisenberg (or Gabor) system.

- Gabor systems are examples of coherent systems.
- Other example: Wavelet systems.
- Let $g(x) = 1_{[0,1)}(x)$ be the Haar scaling function, then

$$g(x) = g(2x) + g(2x - 1).$$

- In fact, any wavelet arising from a MRA will work, e.g., the Daubechies scaling functions:

$$\varphi(x) = \sum_{k=0}^{N-1} c_k \varphi(2x - k).$$

- The HRT is related to the Zero Divisor Conjecture.

Some tools for simplification

Remark

Suppose $\#\Lambda \geq 3$. *Using time-frequency shifts, rotations, shears, we can always assume that each of the points* $(0, 0), (0, 1)$, *and* $(a, 0)$ $a \neq 0$ *belongs to* Λ.

In particular, when $\#\Lambda = 3$, *we can assume that*

$$\Lambda = \{(0, 0), (0, 1), (a, 0)\}$$

where $a \neq 0$.

Some known results

Remark

The known results fall into two categories.

- *Arbitrary* $g \in L^2$ *and special sets of points.*
- *Arbitrary set of points in* $\mathbb{R}^2$ *and special classes of* $g \in L^2$.

Other results include

- *Special sets of points and special functions.*
- *Perturbation arguments on either* g *or* Λ.
- *Spectral version of the HRT.*

A sample of settled cases

Proposition ($0 \neq g \in L^2$ is arbitrary)

(i) *Collinear points (Edgar and Rosenblatt).*

(ii) $N-1$ *collinear and equi-spaced points, and the last point located off this line (Heil, Ramanathan, Topiwala).*

(iii) *Any subset of a lattice (Linnell). In particular, when* $\#\Lambda \leq 3$, *(Heil, Ramanathan, Topiwala).*

Proposition ($\Lambda \subset \mathbb{R}^2$ is arbitrary)

(i) g *Compactly supported, (Heil, Ramanathan, Topiwala).*

(ii) g *is a Hermite function, i.e.,* $g(x) = p(x)e^{-\pi x^2}$: p *polynomial (Heil, Ramanathan, Topiwala).*

(iii) g *with some fast decay, e.g.,* $\lim_{x\to\infty} |g(x)|e^{cx \log x} = 0$ *for all* $c > 0$ *(Bownik, Speegle).*

Known results: Special g and Λ

Proposition (Special g and Λ)

(i) *g is ultimately positive, and $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ is such that $\{b_k\}_{k=1}^N$ are independent over the rationals $\mathbb{Q}$ (Benedetto, Bourouihiya).*

(ii) *When $\#\Lambda = 4$, g is ultimately positive, and $g(x)$ and $g(-x)$ are ultimately decreasing (Benedetto, Bourouihiya).*

(iii) *When $\#\Lambda = 4$ is a $(2,2)$-configuration and $g \in L^2(\mathbb{R})$ (Demeter, Zaharescu).*

(iv) *When $\#\Lambda = 4$ is a $(1,3)$-configuration and $g \in S(\mathbb{R})$ (Demeter; Liu).*

Question

Is the HRT true for any Λ with $\#\Lambda = 4$ and any $g \in L^2$?

The proof of the HRT for 3 points

Remark

Suppose $g \in L^2$ *and* $\Lambda = \{(0,0),(0,1),(a,0)\}$.

- *Suppose the system was linearly dependent and write*

$$g(x-a) = (c_1 + c_2 e^{2\pi i x}) g(x) = p(x) g(x).$$

$$\begin{cases} g(x-na) = g(x) \prod_{j=0}^{n-1} P(x-ja) = g(x) P_n(x) \\ g(x+na) = g(x-a) \prod_{j=0}^{n} P(x+ja)^{-1} = g(x) Q_n(x) \end{cases}$$

$$Q_n(x) = P_n(x+na)^{-1}$$

- *A different proof by Linnell (90s) using Von Neumann algebra techniques to prove HRT for all lattices and any three points lie on a lattice.*

A different approach

Fact

Let $\|g\|_2 = 1$ *, and* $\Lambda = \{(0,0),(0,1)\} \cup \{(a,b)\}$. *For* $\{g(x), e^{2\pi i x} g(x), g(x-a)e^{2\pi i b x}\}$ *is linearly independent*

$$\Updownarrow$$

$$G_{g,\Lambda} = (\langle e^{2\pi i b_k \cdot} g(\cdot - a_k), e^{2\pi i b_\ell \cdot} g(\cdot - a_\ell)\rangle)_{k,\ell=1}^{3}$$

is positive definite.

The Gramian

Consider the Gram matrix of

$$\{g(x), e^{2\pi i x} g(x), g(x-a)e^{2\pi i b x}\}$$

$$\begin{aligned} G_{a,b} &= \begin{bmatrix} 1 & \alpha & V_g g(a,b) \\ \overline{\alpha} & 1 & V_g g(a,b-1) \\ \overline{V_g g(a,b)} & \overline{V_g g(a,b-1)} & 1 \end{bmatrix} \\ &= \begin{bmatrix} A & u(a,b) \\ u(a,b)^* & 1 \end{bmatrix} \end{aligned}$$

$V_g g(x,y) = \int_{\mathbb{R}} g(t)\, \overline{g(t-x)}\, e^{-2\pi i y t}\, dt.$

The key function

$$G_{a,b} \sim \begin{bmatrix} A & 0 \\ 0 & 1 - \langle A^{-1}u(a,b), u(a,b)\rangle \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 - F(a,b) \end{bmatrix}$$

Lemma

Let

$$F(a,b) = \langle A^{-1}u(a,b), u(a,b)\rangle.$$

The HRT holds for g *and* $\{(0,0),(0,1),(a,b)\}$ *if and only if*

$$F(a,b) < 1.$$

Examples of F with 3 points

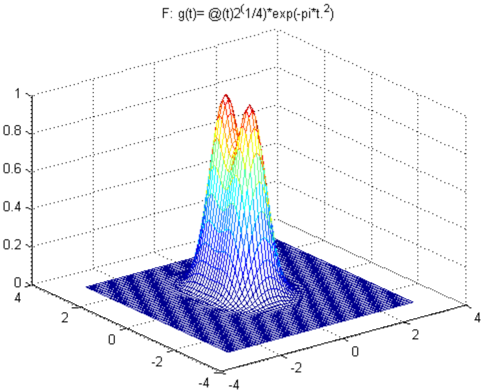

Figure: $g(x) = ce^{-\pi x^2}, \quad \Lambda = \{(0,0),(0,1),(a,b)\}$

Three points HRT for real-valued functions

Remark

Let $g : \mathbb{R} \to \mathbb{R}$ *with* $\|g\|_2 = 1$ *, and* $\Lambda = \{(0,0),(0,1),(a,b)\}$*, where* $(a,b) \in \mathbb{Z}^2$*. Then* $F(a,b) < 1$ *for all* $(a,b) \notin \{(0,0),(0,1)\}$*.*

This follows from the following observation

$$F(a,b) = F(a,1-b)$$

and connects the HRT with three points to the HRT with four points in a special position: a $(2,2)$ configuration.

A different formulation

Question

Suppose that the HRT conjecture holds for a given function $g \in L^2(\mathbb{R})$ and a given set $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$. Characterize the set of all points $(a, b) \in \mathbb{R}^2 \setminus \Lambda$ such that the conjecture remains true for the same function g, and the new set $\Lambda' = \Lambda \cup \{(a, b)\}$.

A recursive formulation for the HRT conjecture

Suppose that HRT holds for $\|g\|_2 = 1$, and $\Lambda = \{(a_k, b_k)\}_{k=1}^N$. Let $\Lambda' = \{(a_k, b_k)\}_{k=1}^N \cup \{(a, b)\}$, and
$\mathcal{G}(g, \Lambda') = \{e^{2\pi i b_k \cdot} g(\cdot - a_k)\}_{k=1}^N \cup \{e^{2\pi i b \cdot} g(\cdot - a)\}$

$$G_g(a, b) = \begin{bmatrix} G_{g,N} & u(a, b) \\ u(a, b)^* & 1 \end{bmatrix}$$

Proposition (K.O. (2019))

Let $0 \neq g \in L^2(\mathbb{R})$, *and* $\Lambda = \{(a_k, 0)\}_{k=1}^N \subset \mathbb{R}^2$ *with* $a_0 = 0$. *Then* $\mathcal{G}(g, \Lambda) = \{g(\cdot - a_k)\}_{k=1}^N$ *is linearly independent.*

Proof.

Prove that the Gramian is positive definite using Bochner's Theorem □

Extending the HRT one point at the time

Definition

Define the extension function $F : \mathbb{R}^2 \to [0, \infty]$ by

$$F(a,b) = \langle G_{g,N}^{-1} u(a,b), u(a,b) \rangle.$$

Proposition (K. O. (2019))

Let $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$ and $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$. Assume that $\mathcal{G}(g, \Lambda)$ is linearly independent. Let $\Lambda' = \{(a_k, b_k)\}_{k=1}^N \cup \{(a,b)\}$. Then $\mathcal{G}(g, \Lambda')$ is linearly independent if and only if $F(a,b) < 1$. Furthermore, there exists $R > 0$ such that for all $(a,b) \in \mathbb{R}^2$ with $|(a,b)| > R$, then $\mathcal{G}(g, \Lambda')$ is linearly independent where $\Lambda' = \Lambda \cup \{(a,b)\}$

Properties of the extension function

Theorem (K. O. (2019))

The following statements hold

(i) $0 \leq F(a,b) \leq 1$ *for all* $(a,b) \in \mathbb{R}^2$, *and moreover,* $F(a_k, b_k) = 1$ *for each* $k = 1, \ldots, N$.

(ii) F *is uniformly continuous and* $\lim_{|(a,b)|\to\infty} F(a,b) = 0$.

(iii) $\iint_{\mathbb{R}^2} F(a,b)dadb = N$.

(iv) *The Fourier transform* $\widehat{F} : \mathbb{R}^2 \to \mathbb{C}$ *of* F *is strictly positive definite, and integrable.*

(v) $\det G_g(a,b) = (1 - F(a,b)) \det G_{g,N}$.

Definition of (n, m) configurations

Definition

An (n, m) configuration is a collection of $n + m$ distinct points: $\exists\, 2$ distinct parallel lines such that one of them contains exactly n of the points and the other one contains exactly m of the points.
$\Lambda_{(n,m)}$ = set of (n, m) configurations.

Fact

Any set of four points in $\mathbb{R}^2$ can be transformed into either: • a $(1, 3)$ configuration; • a $(2, 2)$ configuration; • or neither.

The HRT for $(1,3)$ and $(2,2)$ configurations

Remark

- *The HRT conjecture holds for* $g \in L^2(\mathbb{R})$*, and all* $(2,2)$ *configurations: Demeter and Zaharescu (2012).*
- *The HRT conjecture holds for* $g \in \mathcal{S}(\mathbb{R})$*, and all* $(1,3)$ *configurations: Demeter (2010).*
- *The HRT conjecture holds for* $g \in L^2(\mathbb{R})$*, and almost all* $(1,3)$ *configurations: Liu (2020).*
- *Less is known on general configuration, unless additional restrictions are imposed on* g *or on the points in the configuration.*

HRT For $(1, n)$ configurations

Theorem (K. O. (2019))

Let $n \geq 3$ and $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$. Suppose that the HRT conjecture holds for g and any $(1, n-1)$ configuration. Then there exists at most one (equivalence class of) $(1, n)$ configuration $\Lambda_0 \in \Lambda_{(1,n)}$ such that $\mathcal{G}(g, \Lambda_0)$ is linearly dependent.

Corollary (K. O. (2019))

Let $g \in L^2(\mathbb{R})$, $\|g\|_2 = 1$ be a <u>real-valued function</u>. Let $\Lambda = \{(0,0), (0,1), (a,0), (b,0)\}$ where $b \neq a \neq 0$ be any $(1,3)$ configuration. Then, Conjecture 1 holds for Λ and g.

Proof of the Corollary

A restriction theorem for the HRT with four points

Proposition (K. O. (2019))

Let $g \in L^2(\mathbb{R})$, $\|g\|_2 = 1$ be real-valued. Suppose that $\Lambda = \{(0,0),(0,1),(s,0),(a,b)\} \subset \mathbb{R}^2$ is a subset of four distinct points. The HRT Conjecture holds for Λ and g, whenever:

(i) $a, b \in \mathbb{Q}$,

(ii) $a \in \mathbb{Q}$ *but* $b \notin \mathbb{Q}$.

(iii) $a, b \notin \mathbb{Q}$ *but* $ab \in \mathbb{Q}$

(iv) $a \notin \mathbb{Q}$, $b \in \mathbb{Q}$ *and we assume in addition,* $g \in S(\mathbb{R})$.

HRT for symmetric $(2,3)$ configurations

Theorem (K. O. (2019))

Let $g \in L^2(\mathbb{R})$, with $\|g\|_2 = 1$. Suppose Λ is a $(3,2)$ configuration given by $\Lambda = \{(0,0),(0,1),(0,-1),(a,b),(a,-b)\}$ where $b \neq 0$. Then, the HRT conjecture holds for Λ and g whenever

(i) $a, b \in \mathbb{Q}$,

(ii) $a \in \mathbb{Q}$ *but* $b \notin \mathbb{Q}$.

(iii) $a, b \notin \mathbb{Q}$ *but* $ab \in \mathbb{Q}$.

(iv) $a, b, ab \notin \mathbb{Q}$ *and in addition* $g \in S(R)$.

Proof of the HRT for symmetric $(2,3)$ configurations

Proof.

Suppose that
$\{g(x), e^{2\pi aix}g(x), e^{-2\pi aix}g(x), e^{2\pi ibx}g(x-1), e^{-2\pi ibx}g(x-1)\}$
was LD. Then, $P(x)g(x) = Q(x)g(x-1)$ where

$$P(x) = c_1 + c_2e^{-2\pi aix} + c_3e^{2\pi iax}, \quad Q(x) = d_1e^{-2\pi bix} + d_2e^{2\pi ibx}$$

$c_k, d_k \neq 0$, $c_1 \in \mathbb{R}$. Then extend Demeter's "conjugate trick" argument. □

Back to our examples

Question

Let $g \in L^2(\mathbb{R})$, *or even* $\mathcal{S}(\mathbb{R})$. *Define*

$$\mathcal{G}_1(g) = \{g(x), g(x-1), e^{2\pi i x}g(x), g(x-\sqrt{2})\}$$
$$\mathcal{G}_2(g) = \{g(x), g(x-1), e^{2\pi i x}g(x), e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\}$$
$$\mathcal{G}_3(g) = \{g(x), g(x-1), e^{2\pi i x}g(x), e^{2\pi^2 i x}g(x-\sqrt{2})\}.$$

Is $\mathcal{G}_i(g)$ *linearly independent for* $i = 1, 2, 3$*?*

Solution to the question

Example

(1) $\mathcal{G}_1(g) = \{g(x), g(x-1), e^{2\pi i x}g(x), g(x-\sqrt{2})\}$ is generated from a $(1,3)$ configuration. Linearly independent if $g \in \mathcal{S}(\mathbb{R})$ (Demeter), or $g \in L^2$ and real-valued (K.O.).

(2) $\mathcal{G}_2$ is linearly independent if $g \in L^2$ and real-valued (K.O.).

(3) $\mathcal{G}_2$ is linearly independent if $g \in \mathcal{S}(\mathbb{R})$ and real-valued (K.O.).

Thank You!

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