Variation bounds for spherical averages

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joint work with
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Spherical maximal function: $L^p \to L^p$ bounds

Consider the family of spherical averages $A = \{A_t\}_{t>0}$, defined by

$$A_t f(x) = \int_{S^{d-1}} f(x - ty)\, \mathrm{d}\sigma(y)$$

where $\mathrm{d}\sigma$ denotes the normalized surface measure on the unit sphere S^{d-1}.

Define the spherical maximal function as

$$Sf(x) = \sup_{t>0} |A_t f(x)|.$$

We have the bounds

$$\|Sf\|_p \lesssim \|f\|_p \qquad \text{for} \quad \tfrac{d}{d-1} < p \leqslant \infty,$$

- Stein (1976) for $d \geqslant 3$,
- Bourgain (1986) for $d = 2$.

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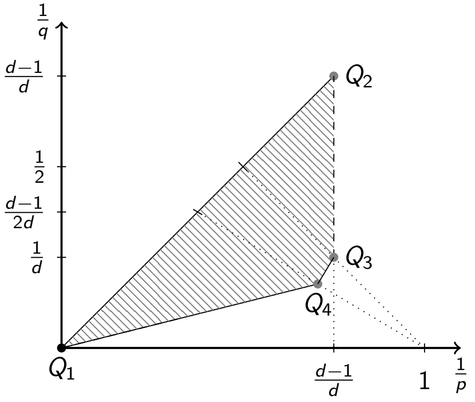
Spherical maximal function: $L^p \to L^q$ bounds ($d \geqslant 3$)

$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for the local variant

$$S^l f(x) = \sup_{1 \leqslant t \leqslant 2} |A_t f(x)|$$

established by Schlag (1997), Schlag–Sogge (1997) and Lee (2003) (endpoint).



$d \geqslant 3$

$Q_1 = (0,0)$, $\quad Q_2 = (\frac{d-1}{d}, \frac{d-1}{d})$,

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$S^l : L^p \to L^q$ on $(Q_3, Q_4), (Q_4, Q_1]$

sharp except RWT Q_3, Q_4

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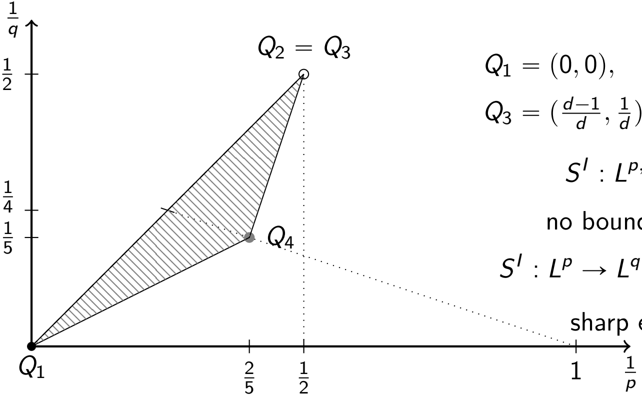

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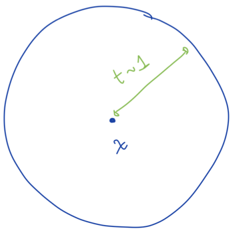
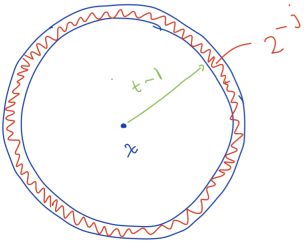
Sobolev embedding

$$\sup_{1\leqslant t\leqslant 2}|A_t f| \leqslant \sup_{1\leqslant t\leqslant 2}\underbrace{|A_t f_0|}_{|\xi|\lesssim 1} + \sum_{j>0}\sup_{1\leqslant t\leqslant 2}\underbrace{|A_t f_j|}_{|\xi|\sim 2^j}$$

Heuristic: if $|t_1 - t_2| \lesssim 2^{-j}$, then $|A_{t_1} f_j| \sim |A_{t_2} f_j|$





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More precisely: FTC + Hölder,

$$\sup_{1\leqslant t\leqslant 2}|A_t f_j|^q \leqslant |A_1 f_j|^q + q\Big(\int_1^2 |A_t f_j|^q \,\mathrm{d}t\Big)^{(q-1)/q}\Big(\int_1^2 |\partial_t A_t f_j|^q \,\mathrm{d}t\Big)^{1/q}.$$

$$\Longrightarrow \|S^I f_j\|_{L^q(\mathbb{R}^d)} \lesssim \|A_1 f_j\|_{L^q(\mathbb{R}^d)} + \|A_t f_j\|^{1-1/q}_{L^q(\mathbb{R}^d\times[1,2])}\|\partial_t A_t f_j\|^{1/q}_{L^q(\mathbb{R}^d\times[1,2])}$$

Fixed-time estimates ($t \sim 1$)

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where

$$|\partial_r^\gamma b_\pm(r)| \lesssim (1+|r|)^{-\frac{(d-1)}{2}-\gamma}, \qquad \gamma \in \mathbb{N}_0.$$

Plancherel: $\|A_t f_j\|_2 \lesssim 2^{-j\frac{(d-1)}{2}}\|f\|_2 \Longrightarrow \begin{cases} \|A_t f_j\|_p \lesssim 2^{-j\frac{(d-1)}{p}}\|f\|_p, & 2 \leqslant p \leqslant \infty, \\ \|A_t f_j\|_p \lesssim 2^{-j\frac{(d-1)}{p'}}\|f\|_p, & 1 \leqslant p \leqslant 2. \end{cases}$

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Annulus avg: $|A_t f_j(x)| \lesssim \int_{\mathbb{R}^d} \frac{2^j}{(1+2^j\big||x-y|-t\big|)^N}|f(y)|\,\mathrm{d}y \Longrightarrow \|A_t f_j\|_\infty \lesssim 2^j\|f\|_1$

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Derivative: $|\partial_t A_t f_j(x)| \sim |A_t f_j(x)| + 2^j|A_t f_j(x)|$

Local smoothing estimates

Recall:

$$\|S^l f_j\|_p \lesssim 2^{-j\frac{(d-2)}{p}} \|f\|_p, \qquad 2 \leqslant p \leqslant \infty$$

$$\|S^l f_j\|_p \lesssim 2^{-j(d-1-\frac{d}{p})} \|f\|_p, \qquad 1 \leqslant p \leqslant 2$$

- good for $d \geqslant 3$.
- need extra gain $2^{-j\varepsilon}$ for $d = 2$.

Local smoothing estimates

Are there better estimates for

$$\|A_t f_j\|_{L^q(\mathbb{R}^d \times [1,2])} \lesssim 2^{je(p,q)} \|f\|_p$$

than those implied just by $\|A_t f_j\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_p$ and a trivial t-integration?

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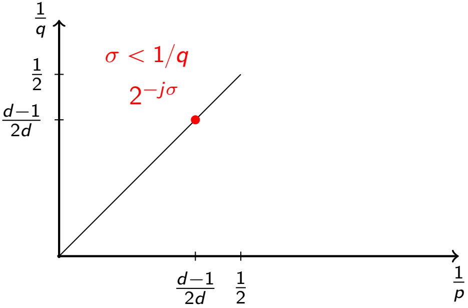

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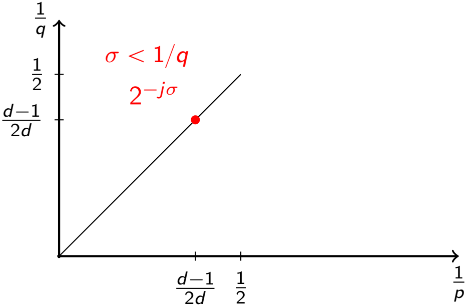

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- Upgrade from S^l to S by LP theory.

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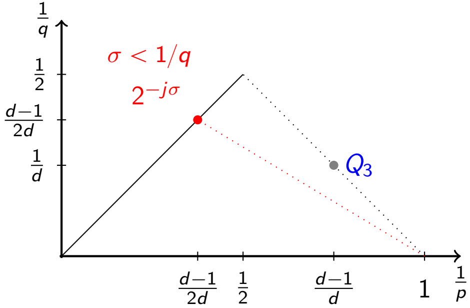

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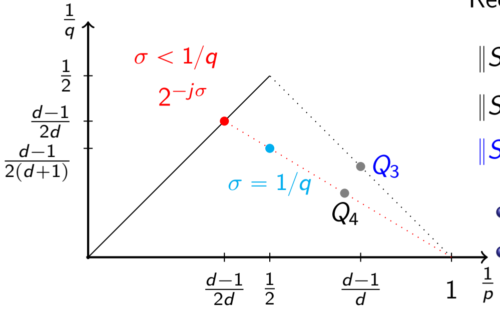

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- even worse for $d=2$.

Spherical maximal function: $L^p \to L^q$ bounds

$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for the local variant

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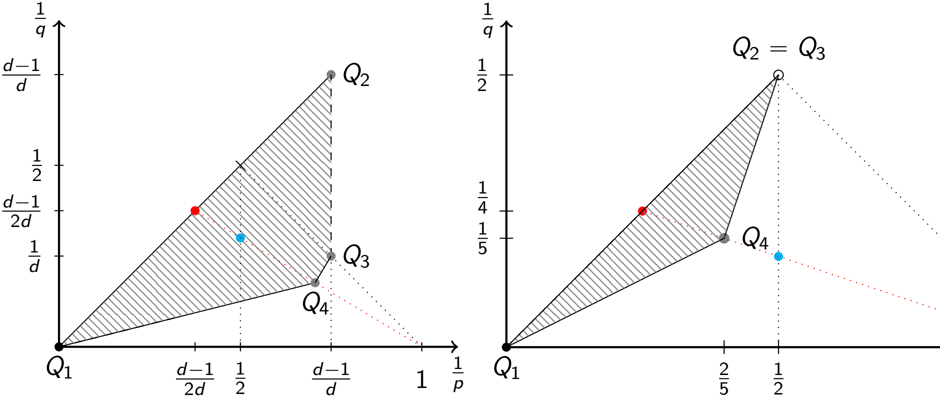
established by Schlag (1997), Schlag–Sogge (1997) and Lee (2003) (endpoint).



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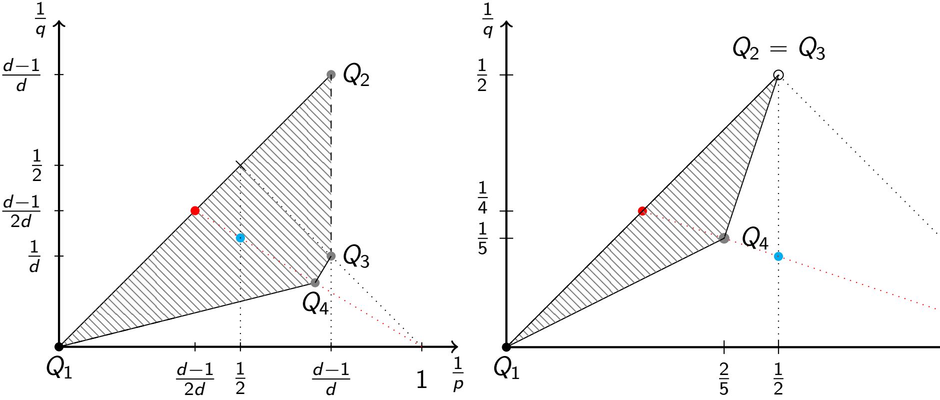
The endpoint relies on Tao (2001) endpoint bilinear restriction thm for the cone.



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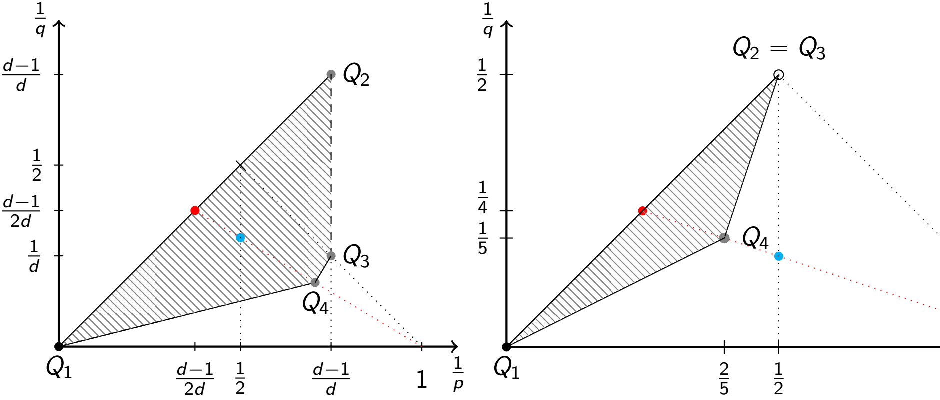
All non-endpoint local-smoothing are known for $d = 2$ Guth–Wang–Zhang (2019)



Variation norm

Given a subset $E \subset \mathbb{R}$ and a family of complex valued functions $t \mapsto a_t$ defined on E, the r-variation of $a = \{a_t\}_{t \in E}$ is defined by

$$|a|_{V_r(E)} := \sup_{N \in \mathbb{N}} \sup_{\substack{t_1 < \cdots < t_N \\ t_j \in E}} \Big(\sum_{j=1}^{N-1} |a_{t_{j+1}} - a_{t_j}|^r \Big)^{1/r}$$

for all $1 \leqslant r < \infty$, and replacing the ℓ^r-sum by a sup in the case $r = \infty$.

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Lépingle (1976): $F = \{\mathbb{E}_n f\}_{n=1}^{\infty}$ martingale, $\|V_r F\|_p \lesssim \|f\|_p$ $1 < p < \infty$, $r > 2$.

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Bourgain (1989): $\|V_r A f\|_{L^2(\mathbb{Z})} \lesssim \|f\|_{L^2(\mathbb{Z})}$, $r > 2$, where $A = \{A_N\}_{N \in \mathbb{N}}$ is given by

$$A_N f(m) = \frac{1}{N} \sum_{n=1}^{N} f(m+n).$$

Given dyn system (X, μ, T), implies bounds on $V_r \widetilde{A}$ for $\widetilde{A} = \{\widetilde{A}_N\}_{N \in \mathbb{N}}$ given by

$$\widetilde{A}_N f(x) = \frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

which yield an alternative proof of Birkhoff's pointwise ergodic theorem.

Global variation operators for spherical averages

Given the family of spherical averages $\{A_t\}_{t>0}$ consider

$$V_r Af(x) \equiv V_r[Af](x) := |Af(x)|_{V_r((0,\infty))}.$$

Jones–Seeger–Wright (2008):

$$\|V_r Af\|_p \lesssim \|f\|_p \qquad \text{for} \quad \begin{cases} r > 2 & \text{if } \frac{d}{d-1} < p \leqslant 2d \\ r > p/d & \text{if } 2d < p \end{cases}$$

Moreover, $V_r A : L^{\frac{d}{d-1},1} \to L^{\frac{d}{d-1},\infty}$ for $r > 2$, $d \geqslant 3$.

All ranges of p and r are sharp, except for the critical case $r = p/d$, $p > 2d$ which was left open.

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Theorem (BORSS, 2020)

Let $d \geqslant 3$, $p > 2d$. Then the operator $V_{p/d}A$ is of restricted weak type (p, p), i.e. maps $L^{p,1}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$.

One writes

$$V_r Af(x) \leqslant V_r^{\mathrm{dyad}} Af(x) + V_r^{\mathrm{sh}} Af(x)$$

where

$$V_r^{\mathrm{dyad}} Af(x) := \sup_{N \in \mathbb{N}} \sup_{k_1 < \cdots < k_N} \Big(\sum_{i=1}^{N-1} |A_{2^{k_{i+1}}} f(x) - A_{2^{k_i}} f(x)|^r \Big)^{1/r}$$

is the *dyadic or long variation operator* and

$$V_r^{\mathrm{sh}} Af(x) := \Big(\sum_{k \in \mathbb{Z}} |V_r^{I_k} Af(x)|^r \Big)^{1/r}$$

is the *short variation operator*, using variation within dyadic intervals $I_k = [2^k, 2^{k+1}]$.

One writes

$$V_r Af(x) \leqslant V_r^{\mathrm{dyad}} Af(x) + V_r^{\mathrm{sh}} Af(x)$$

where

$$V_r^{\mathrm{dyad}} Af(x) := \sup_{N\in\mathbb{N}} \sup_{k_1<\cdots<k_N} \Big(\sum_{i=1}^{N-1} |A_{2^{k_{i+1}}} f(x) - A_{2^{k_i}} f(x)|^r \Big)^{1/r}$$

is the *dyadic or long variation operator* and

$$V_r^{\mathrm{sh}} Af(x) := \Big(\sum_{k\in\mathbb{Z}} |V_r^{I_k} Af(x)|^r \Big)^{1/r}$$

is the *short variation operator*, using variation within dyadic intervals $I_k = [2^k, 2^{k+1}]$.

- For $V_r^{\mathrm{dyad}} A$ one uses Lépingle's inequality $(r > 2)$; holds for $1 < p < \infty$.
- The condition $r > 2$ does not seem to enter in V_r^{sh}; which restricts p-range.

Local variation operators for spherical averages

We explore the existence of $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for

$$V_r^I Af(x) := |Af(x)|_{V_r([1,2])}$$

for $1 \leqslant r \leqslant \infty$, which are meant to refine the bounds on

$$S^I f(x) = \sup_{1 \leqslant t \leqslant 2} |A_t f(x)|; \qquad \text{recall} \quad \|A_t f_j\|_{L^p \to L^q(L^\infty)} \lesssim 2^{j/q} \|A_t f_j\|_{L^p \to L^q(L^q)}.$$

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Embedding (Plancherel–Polya inequality):

$$B^{1/r}_{r,1} \hookrightarrow V_r \hookrightarrow B^{1/r}_{r,\infty},$$

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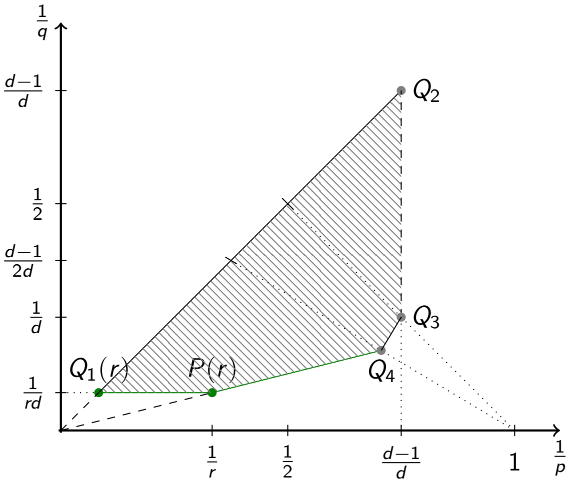
Space-time FT of $A_t f(x)$: $e^{-it(\tau\pm|\xi|)} \Longrightarrow \|A_t f_j\|_{L^p\to L^q(B^{1/r}_{r,1})} \lesssim 2^{j/r}\|A_t f_j\|_{L^p\to L^q(L^r)}$.

$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for $V_r^l A$ if $d \geqslant 3$

Theorem (BORSS, 2020)

If $r > \frac{d^2+1}{d(d-1)}$, $d \geqslant 3$: sharp except RWT Q_3, Q_4



$$P(r) = \left(\tfrac{1}{r}, \tfrac{1}{rd}\right),$$
$$Q_1(r) = \left(\tfrac{1}{rd}, \tfrac{1}{rd}\right),$$
$$Q_2 = \left(\tfrac{d-1}{d}, \tfrac{d-1}{d}\right)$$
$$Q_3 = \left(\tfrac{d-1}{d}, \tfrac{1}{d}\right),$$
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$V_r^l A : L^{p,1} \to L^{q,\infty}$ at Q_3, Q_4

$V_r^l A : L^{p,1} \to L^q$ at $[Q_2, Q_3)$

$V_r^l A : L^p \to L^q$ at (Q_3, Q_4)

$V_r^l A : L^p \to L^q$ at $[Q_1(r), P(r)], [P(r), Q_4(r)), (Q_1(r), Q_2(r))$

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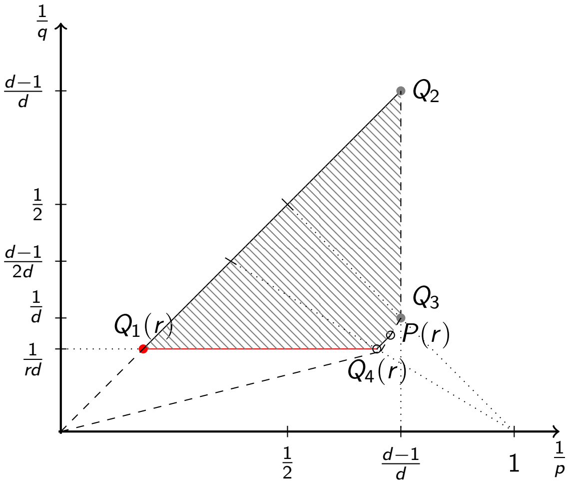
Theorem (BORSS, 2020)

If $\frac{d}{d-1} < r \leqslant \frac{d^2+1}{d(d-1)}$, $d \geqslant 3$: sharp except RWT Q_3, left open $(Q_3, Q_4(r)]$



$$Q_1(r) = \left(\tfrac{1}{rd}, \tfrac{1}{rd}\right),$$
$$Q_2 = \left(\tfrac{d-1}{d}, \tfrac{d-1}{d}\right),$$
$$Q_3 = \left(\tfrac{d-1}{d}, \tfrac{1}{d}\right)$$
$$P(r) = \left(\tfrac{1}{r}, \tfrac{d+1-r(d-1)}{r(d-1)}\right),$$
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$V_r^I A : L^{p,1} \to L^{q,\infty}$ at Q_3

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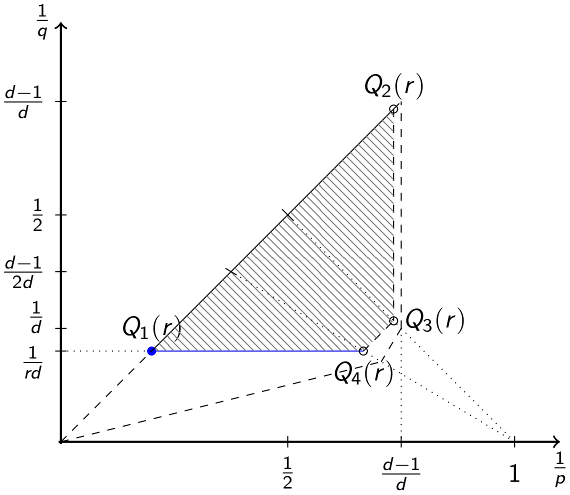
$V_r^I A : L^{p} \to L^{q}$ at

$[Q_1(r), Q_4(r)), (Q_1(r), Q_2)$

$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for $V_r^l A$ if $d \geqslant 3$

Theorem (BORSS, 2020)

If $1 \leqslant r \leqslant \frac{d}{d-1}$ and $d \geqslant 4$ or $\frac{4}{3} < r \leqslant \frac{3}{2}$ and $d = 3$:



sharp, left open
$[Q_2(r), Q_3(r)], [Q_3(r), Q_4(r)]$

$$Q_1(r) = \left(\tfrac{1}{rd}, \tfrac{1}{rd}\right),$$
$$Q_2(r) = \left(\tfrac{r(d-1)-1}{r(d-1)}, \tfrac{r(d-1)-1}{r(d-1)}\right),$$
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$V_r^l A : L^p \to L^q$ at

$[Q_1(r), Q_4(r)), (Q_1(r), Q_2(r))$

Why partial results for $d = 3$?

- for $d = 3$ we only obtain sharp results in the partial range $\frac{4}{3} < r \leqslant \frac{d}{d-1}$.
- for $d \geqslant 4$, sharp results are obtained for all $1 \leqslant r \leqslant \frac{d}{d-1}$.

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Banach space condition $1 \leqslant r \leqslant \infty$ for the variation norm.

One can extend, with modifications, V_r to the range $0 < r < 1$ (Bergh–Peetre). In this context:

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- One can formulate conjectural results for $V_r^l A$ for $\frac{2}{d-1} < r < 1$ for $d \geqslant 4$.

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Our positive results for $d \geqslant 3$ pivot around "Stein–Tomas": more sophisticated restriction technology should give further partial results between $\frac{2}{d-1} < r \leqslant \frac{2(d+1)}{d(d-1)}$.

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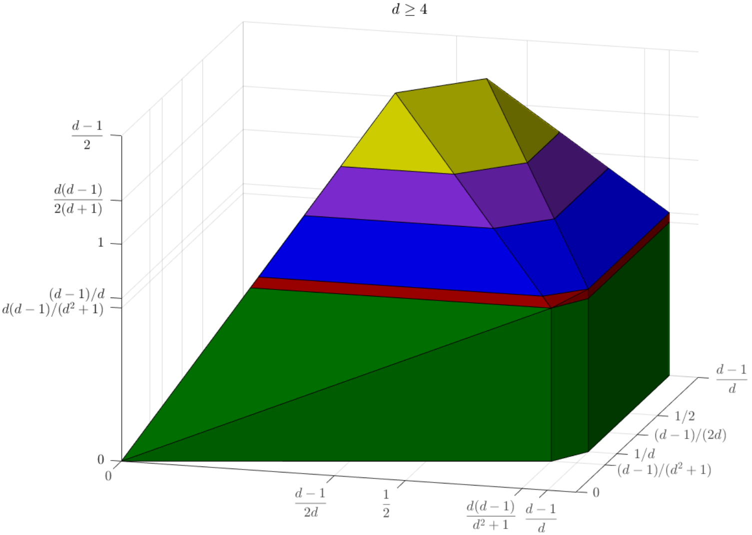
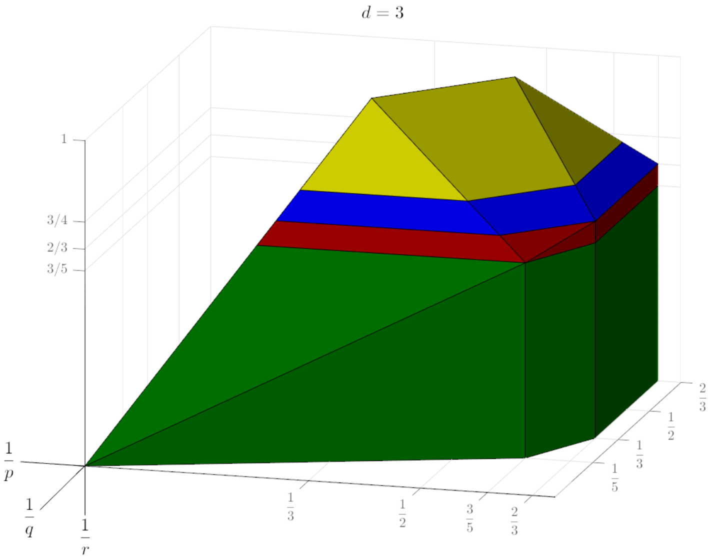
The conjectured bounds seem to be slightly weaker than the local smoothing or Bochner–Riesz conjecture, but stronger than Kakeya conjecture (work in progress).

$(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-bounds for $d \geqslant 4$



$(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-bounds for $d = 3$



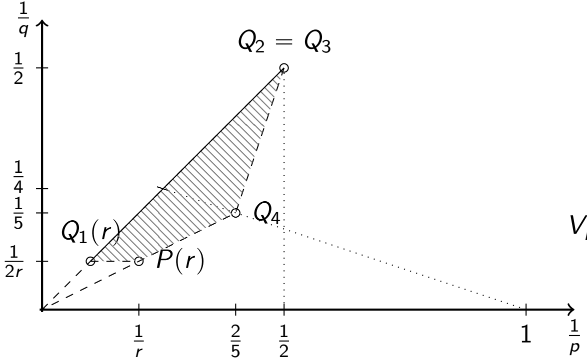
$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for $V_r^l A$ if $d = 2$

Optimal result, up to endpoints, due to the recent full resolution of Sogge's problem in $2+1$ dimensions by Guth, Wang and Zhang, that is,

$$\partial_t^{1/2-\varepsilon} A : L^4 \to L^4(L^4).$$

Theorem (BORSS, 2020)

If $r > 5/2$, $d = 2$: sharp but no endpoints



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$V_r^l : L^p \to L^q$ on $(Q_1(r), Q_2(r))$

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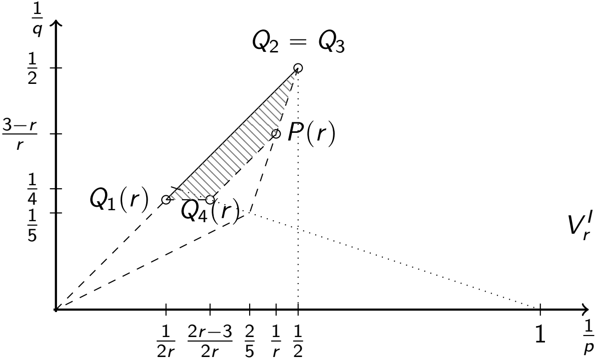
Theorem (BORSS, 2020)

If $2 < r \leqslant 5/2$, $d = 2$: sharp but no endpoints



$Q_1(r) = \left(\frac{1}{2r}, \frac{1}{2r}\right)$,

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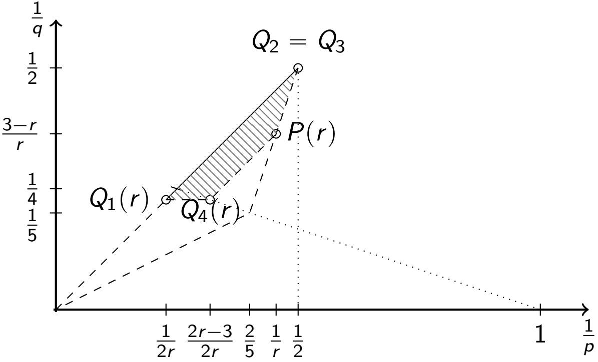
If $r < 2 = \frac{2}{d-1}$, $d = 2$, unbounded.



$$Q_1(2) = Q_4(2) = (\tfrac{1}{4}, \tfrac{1}{4})$$

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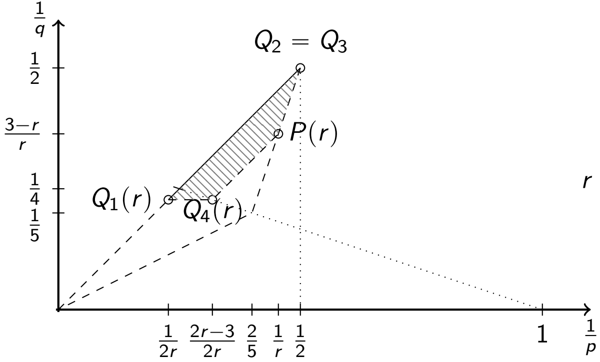

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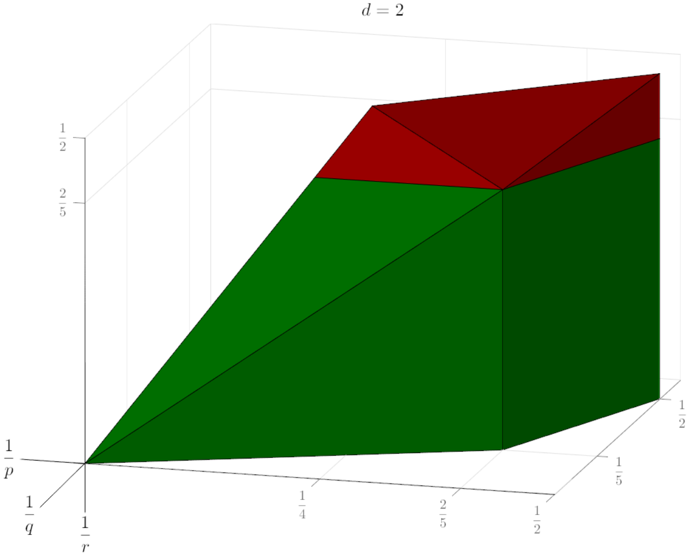
$r = 2 = \frac{2}{d-1}$ also unbounded (work in progress)

$(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$-bounds for $d = 2$



Key single scale estimates $\|\mathcal{A}_j\|_{L^p \to L^q(L^r)}$

Let $\mathcal{A}_j f(x,t) = A_t f_j(x)$. Recall

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For $p = q$, fixed-time estimates: $\begin{cases} \|\mathcal{A}_j f\|_{L^2(L^2)} \lesssim 2^{-j\frac{(d-1)}{2}} \|f\|_2 \\ \|\mathcal{A}_j f\|_{L^\infty(L^\infty)} \lesssim \|f\|_\infty \\ \|\mathcal{A}_j f\|_{L^1(L^1)} \lesssim \|f\|_1 \end{cases}$

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The Stein–Tomas estimate

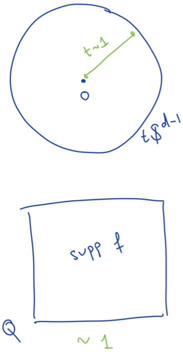
$$\|\mathcal{A}_j f\|_{L^q(L^q)} \lesssim 2^{-j\frac{d}{q}+j(\frac{1}{2}-\frac{1}{q})}\|f\|_2, \qquad q=\frac{2(d+1)}{d-1},$$

can be improved into a square function Stein–Tomas estimate

$$\|\mathcal{A}_j f\|_{L^q(L^2)} \lesssim 2^{-j\frac{d}{q}}\|f\|_2, \qquad q=\frac{2(d+1)}{d-1}.$$

$\|\mathcal{A}_j\|_{L^r(\mathbb{R})}$ essentially local

- If Q cube of $|Q| \sim 1$, $A_t(f\mathbb{1}_Q) = A_t(f\mathbb{1}_Q)\mathbb{1}_{10Q}$ if $t \in [1,2]$.



$f * \sigma_t$ supp in $10Q$.

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$$|K_{j,t}(x)| \lesssim_N \frac{2^j}{(1+2^j\big||x|-t\big|)^N} \implies |K_{j,t}(x)| \lesssim_N (2^j|x|)^{-N}, \quad |x| \geqslant 10,\ t \in [1,2]$$

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  By Hölder's inequality, if $p_0 \leqslant p_1$

$$\|\mathbb{1}_{10Q}\mathcal{A}_j(f\mathbb{1}_Q)\|_{L^q(L^r)} \lesssim \|\mathcal{A}_j\|_{L^{p_0}\to L^q(L^r)}\|f\mathbb{1}_Q\|_{L^{p_0}} \lesssim \|\mathcal{A}_j\|_{L^{p_0}\to L^q(L^r)}\|f\mathbb{1}_Q\|_{L^{p_1}}$$

  and one can sum over a tiling of $\mathbb{R}^d$ provided $p_0 \leqslant p_1 \leqslant q$.

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Increase the exponent p on the right, keeping r and q fixed.

Localization

For $p_0 \leqslant p_1 \leqslant q_0$, $1 \leqslant r \leqslant \infty$, and every $N \in \mathbb{N}$,

$$\|\mathcal{A}_j\|_{L^{p_1} \to L^{q_0}(L^r)} \lesssim \|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^r)} + C_N 2^{-jN}.$$

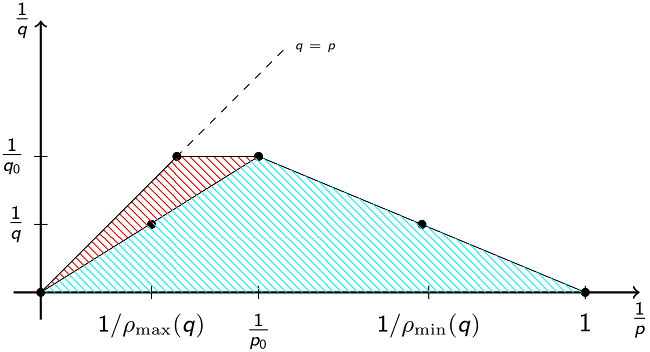
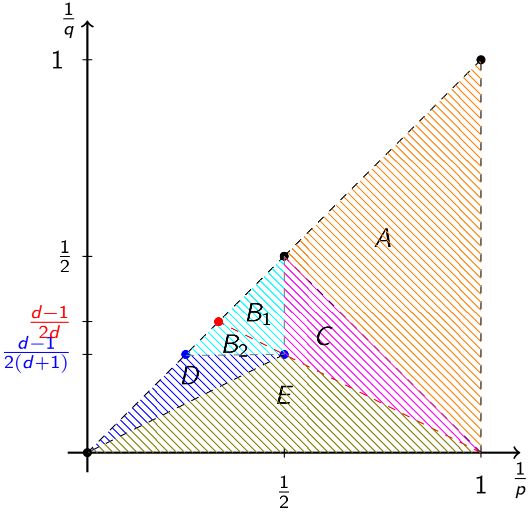


Figure: If $\|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^{p_0})} \lesssim 2^{-jd/q_0}$, then $\|\mathcal{A}_j\|_{L^p \to L^q(L^p)} \lesssim 2^{-jd/q}$ in the blue triangle and $\|\mathcal{A}_j\|_{L^p \to L^q(L^{\rho_{\max}(q)})} \lesssim 2^{-jd/q}$ in the red triangle.

$\|\mathcal{A}_j\|_{L^p \to L^q(L^r)}$ bounds



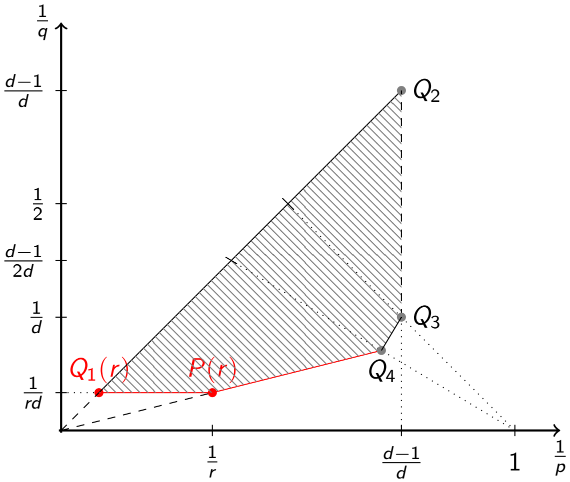
Combined with the $2^{j/r}$ loss:

- all claimed interior bounds for $V_r^l A$.
- maximal function-type endpoints implied by Bourgain's interpolation trick.

The hard endpoint

Theorem (BORSS, 2020)

If $r > \frac{d^2+1}{d(d-1)}$, $d \geqslant 3$:



$P(r) = (\frac{1}{r}, \frac{1}{rd})$,

$Q_1(r) = (\frac{1}{rd}, \frac{1}{rd})$,

$Q_2 = (\frac{d-1}{d}, \frac{d-1}{d})$

$Q_3 = (\frac{d-1}{d}, \frac{1}{d})$,

$Q_4 = (\frac{d(d-1)}{d^2+1}, \frac{d-1}{d^2+1})$.

$V_r^l A : L^p \to L^q$ at

$[Q_1(r), P(r)] \cup [P(r), Q_4(r))$

and analogous boundary segment $[Q_1(r), Q_4(r))$ for $1 \leqslant r \leqslant \frac{d^2+1}{d(d-1)}$.

The hard endpoint

Instead of $\|V_r^I A f_j\|_q \lesssim 2^{-j\varepsilon}\|f\|_p$, we need to keep the frequency scales together.

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Besov reduction for the endpoint: the bound

$$\|V_r^I A f\|_{L^q} \lesssim \|\mathcal{A} f\|_{L^q(B^{1/r}_{r,1})} \lesssim \|f\|_p$$

follows from

$$\Big\| \sum_{j \geqslant 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\sum_{j \geqslant 0} 2^{-jq/r} \|f_j\|_p^q \Big)^{1/q}$$

provided $1 \leqslant r < \infty$, $2 \leqslant q < \infty$, $1 < p < \infty$ satisfy $r, p \leqslant q$.

Space-time FT of $A_t f(x)$: $e^{-it(\tau \pm |\xi|)}$ + LP-theory ($q \geqslant 2$).

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Space-time FT of $A_t f(x)$: $e^{-it(\tau \pm |\xi|)}$ + LP-theory ($q \geqslant 2$).

The single-scale bound $\|\mathcal{A}_j f\|_{L^q(L^r)} \lesssim 2^{-j/r}\|f\|_p$ only yields

$$\Big\| \sum_{j \geqslant 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{j \geqslant 0} 2^{-j/r} \|f_j\|_p,$$

we want to upgrade the RHS to ℓ^q.

The hard endpoint

Theorem

Let $1 < p_0 \leqslant q_0 < \infty$. Assume that

$$\sup_{j\geqslant 0} 2^{jd/q_0} \|\mathcal{A}_j\|_{L^{p_0}\to L^{q_0}(L^{p_0})} \leqslant C_0 \leqslant \infty. \quad \text{ST sq fn}: r_0 = p_0 = 2,\ q_0 = \tfrac{2(d+1)}{d-1}$$

Let $q_0 < q < \infty$ and define $\frac{1}{\rho_{\max}(q)} = \frac{q_0}{q}\frac{1}{p_0}$ and $\frac{1}{\rho_{\min}(q)} = 1 - \frac{q_0}{q}(1 - \frac{1}{p_0})$. Assume that p, r satisfy

$$\rho_{\min}(q) < p \leqslant q \text{ and } \begin{cases} r \leqslant p & \text{if } \rho_{\min}(q) < p < \rho_{\max}(q), \\ r < \rho_{\max}(q) & \text{if } \rho_{\max}(q) \leqslant p \leqslant q. \end{cases}$$

Then for all $\{f_j\}_{j\geqslant 0}$,

$$\Big\| \sum_{j\geqslant 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \leqslant C(p,q)(1 + C_0)\Big(\sum_{j\geqslant 0} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}.$$

Take $r = q/d$, so that $2^{-jd} = 2^{-jq/r}$.

The hard endpoint

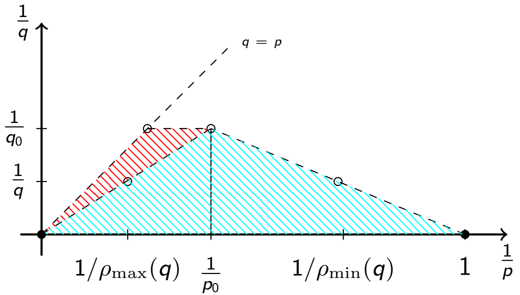

Figure: Bounds for multi-scale frequency sums with $2^{-jd/q}$ smoothness hold for $r = p$ in the interior of the blue triangle.

The hard endpoint

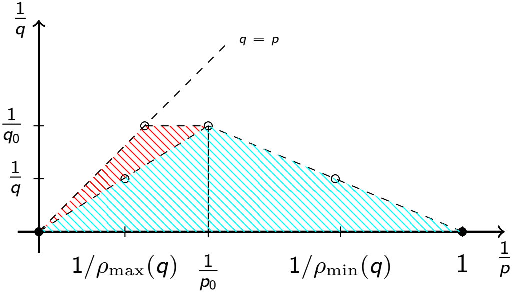

Figure: Bounds for multi-scale frequency sums with $2^{-jd/q}$ smoothness hold for $r = p$ in the interior of the blue triangle.

- $d \geqslant 3$: Stein–Tomas square function as input, and setting $r = q/d$, gives
  - strong bounds on $[Q_1(r), P(r)]$ (horizontal) and $[P(r), Q_4)$ for $r > \frac{d^2+1}{d(d-1)}$.
  - strong bounds on $[Q_1(r), Q_4(r))$ for $1 \leqslant r \leqslant \frac{d^2+1}{d(d-1)}$.

The hard endpoint

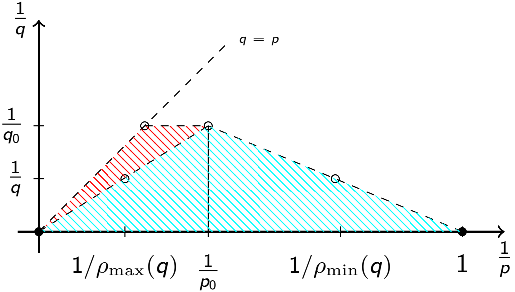

Figure: Bounds for multi-scale frequency sums with $2^{-jd/q}$ smoothness hold for $r = p$ in the interior of the blue triangle.

- $d \geqslant 3$: Stein–Tomas square function as input, and setting $r = q/d$, gives
  - strong bounds on $[Q_1(r), P(r)]$ (horizontal) and $[P(r), Q_4)$ for $r > \frac{d^2+1}{d(d-1)}$.
  - strong bounds on $[Q_1(r), Q_4(r))$ for $1 \leqslant r \leqslant \frac{d^2+1}{d(d-1)}$.
- $d = 2$, need sharp $\|\mathcal{A}_j\|_{L^{p_0} \to L^{q_0}(L^{p_0})} \lesssim 2^{-jd/q_0}$ beyond ST. Currently $2^{-jd/q_0 + j\varepsilon}$.

Fefferman–Stein sharp maximal function

Goal:

$$\Big\| \sum_{j\geqslant 0} \|\mathcal{A}_j f_j\|_{L^r(\mathbb{R})} \Big\|_{L^q(\mathbb{R}^d)} \leqslant C(p,q)(1+C_0)\Big(\sum_{j\geqslant 0} 2^{-jd}\|f_j\|_p^q\Big)^{1/q}.$$

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Given $G \in L^{q_0}(\mathbb{R}^d)$, the Fefferman–Stein sharp maximal function is defined as

$$G^{\#}(x) := \sup_{x\in Q} ⨍_Q \Big| G(y) - ⨍_Q G(w)\,\mathrm{d}w \Big| \,\mathrm{d}y$$

which satisfies

$$\|G\|_q \leqslant c(q)\|G^{\#}\|_q \quad \text{for } q_0 < q < \infty.$$

Use this with our function on the LHS:

$$G(x) = \sum_{j\geqslant 0} \Big(\int_1^2 |\mathcal{A}_j f_j(x,t)|^r \,\mathrm{d}t \Big)^{1/r}.$$

We estimate

$$G^\sharp(x) \lesssim \mathcal{G}_I(x) + \mathcal{G}_{II}(x) + \mathcal{G}_{III}(x)$$

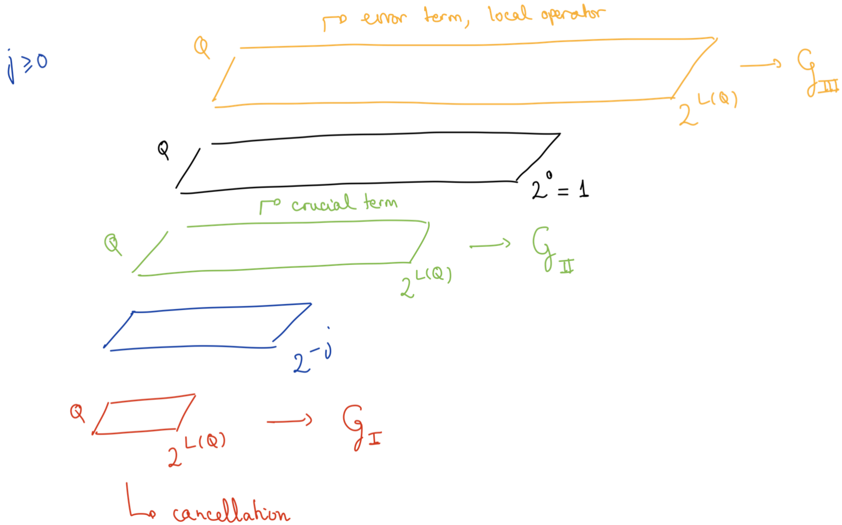
where,

$$\mathcal{G}_I(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \fint_Q \Big| \sum_{0\leqslant j\leqslant -L(Q)} \Big(\|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\,\mathrm{d}w \Big) \Big| \,\mathrm{d}y,$$

$$\mathcal{G}_{II}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \fint_Q \sum_{j\geqslant -L(Q)} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}\,\mathrm{d}y,$$

$$\mathcal{G}_{III}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)> 0}} \fint_Q \sum_{j\geqslant 0} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}\,\mathrm{d}y$$

and $\mathcal{Q}(x)$ is the collection of all cubes containing x and $2^{L(Q)} \sim \ell(Q)$.


error term, local operator
$j \geq 0$
Q
$\mathcal{G}_{III}$
$2^{L(Q)}$
Q
$2^0 = 1$
crucial term
Q
$\mathcal{G}_{II}$
$2^{L(Q)}$
2^{-j}
Q
$\mathcal{G}_{I}$
$2^{L(Q)}$
cancellation

Estimate for $\mathcal{G}_I$

$$\mathcal{G}_I(x) := \sup_{\substack{Q \in \mathcal{Q}(x) \\ L(Q) \leqslant 0}} ⨍_Q \Big| \sum_{0 \leqslant j \leqslant -L(Q)} \Big(\|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - ⨍_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\, \mathrm{d}w \Big) \Big| \, \mathrm{d}y$$

Estimate for $\mathcal{G}_l$

$$\mathcal{G}_l(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \fint_Q \Big| \sum_{0\leqslant j\leqslant -L(Q)} \Big(\|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\,\mathrm{d}w \Big) \Big| \,\mathrm{d}y$$

Fubini:

$$\sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \sum_{0\leqslant j\leqslant -L(Q)} |a_j| = \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \sum_{n=0}^{-L(Q)} |a_{-L(Q)-n}| \leqslant \sum_{n=0}^{\infty} \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant -n}} |a_{-L(Q)-n}|$$

$$= \sum_{n=0}^{\infty} \sup_{j\geqslant 0} \sup_{Q\in\mathcal{Q}_{-n-j}(x)} |a_j|.$$

Estimate for $\mathcal{G}_I$

$$\mathcal{G}_I(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \fint_Q \Big| \sum_{0\leqslant j\leqslant -L(Q)} \Big(\|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\, \mathrm{d}w \Big) \Big| \mathrm{d}y$$

Fubini:

$$\sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \sum_{0\leqslant j\leqslant -L(Q)} |a_j| = \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \sum_{n=0}^{-L(Q)} |a_{-L(Q)-n}| \leqslant \sum_{n=0}^{\infty} \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant -n}} |a_{-L(Q)-n}|$$

$$= \sum_{n=0}^{\infty} \sup_{j\geqslant 0} \sup_{Q\in\mathcal{Q}_{-n-j}(x)} |a_j|.$$

Then $\mathcal{G}_I(x) \leqslant \sum_{n\geqslant 0} \mathcal{G}_{I,n}(x)$, where

$$\mathcal{G}_{I,n}(x) := \sup_{j\geqslant 0} \sup_{Q\in\mathcal{Q}_{-n-j}(x)} \fint_Q \Big| \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r} - \fint_Q \|\mathcal{A}_j f_j(w,\cdot)\|_{L^r}\, \mathrm{d}w \Big| \mathrm{d}y$$

and one uses cancellation and the single-scale estimate to obtain

$$\|\mathcal{G}_{I,n}\|_q \lesssim 2^{-n} \Big(\sum_{j\geqslant 0} 2^{-jd} \|f_j\|_p^q \Big)^{1/q} \quad \text{for } p, q, r \text{ in the desired range.}$$

Estimate for $\mathcal{G}_{II}$

$$\mathcal{G}_{II}(x) := \sup_{\substack{Q \in \mathcal{Q}(x) \\ L(Q) \leqslant 0}} \fint_Q \sum_{j \geqslant -L(Q)} \|\mathcal{A}_j f_j(y, \cdot)\|_{L^r} \, \mathrm{d}y,$$

Estimate for $\mathcal{G}_{II}$

$$\mathcal{G}_{II}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)\leqslant 0}} \int_Q \sum_{j\geqslant -L(Q)} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}\,\mathrm{d}y,$$

As before, we can rewrite it as

$$\mathcal{G}_{II}(x) \leqslant \sum_{n\geqslant 0} \mathfrak{M}_{r,n}F(x)$$

where, for a sequence $F = \{f_j\}_{j\geqslant 0}$,

$$\mathfrak{M}_{r,n}F(x) = \sup_{j\geqslant n} \sup_{Q\in\mathcal{Q}_{n-j}(x)} \int_Q \big\|\mathcal{A}_j f_j(y,\cdot)\big\|_{L^r}\,\mathrm{d}y.$$

It suffices to show

$$\|\mathfrak{M}_{r,n}F\|_q \leqslant C_{p,q,r} 2^{-n\varepsilon(p,q,r)} \Big(\sum_{j\geqslant n} 2^{-jd} \|f_j\|_p^q\Big)^{1/q}.$$

$$\mathfrak{M}_{r,n}F(x) = \sup_{j\geqslant n} \sup_{Q\in\mathcal{Q}_{n-j}(x)} \fint_Q \big\|\mathcal{A}_j f_j(y,\cdot)\big\|_{L^r}\,\mathrm{d}y.$$

$$\mathfrak{M}_{r,n}F(x) = \sup_{j \geqslant n} \sup_{Q \in \mathcal{Q}_{n-j}(x)} \fint_Q \big\| \mathcal{A}_j f_j(y, \cdot) \big\|_{L^r} \, \mathrm{d}y.$$

<u>Uniform estimate in n</u>: single-scale estimate, via Hardy–Littlewood and $\ell^q \subseteq \ell^\infty$

$$\|\mathfrak{M}_{r,n}F\|_q \lesssim \Big(\sum_{j \geqslant n} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}.$$

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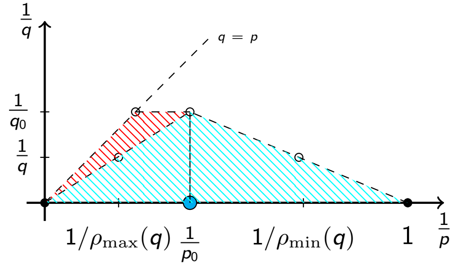
$$\|\mathfrak{M}_{r,n}F\|_q \lesssim \Big(\sum_{j \geqslant n} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}.$$

Crucial gain in n if $r = p = p_0$, $q = \infty$:

$$\|\mathfrak{M}_{p_0,n}F\|_\infty \lesssim 2^{-nd/q_0} \sup_{j \geqslant n} \|f_j\|_{p_0}.$$



The gain in n at $r = p = p_0$, $q = \infty$, is interpolated with the uniform estimates for $r = p = \rho_{\min}(p)$ and $r = p = \rho_{\max}(q)$ on the boundary of the blue triangle to yield summable bounds in the interior.

$$\mathfrak{M}_{r,n}F(x) = \sup_{j \geqslant n} \sup_{Q \in \mathcal{Q}_{n-j}(x)} \fint_Q \big\| \mathcal{A}_j f_j(y,\cdot) \big\|_{L^r} \, \mathrm{d}y.$$

<u>**Uniform estimate in** n</u>: single-scale estimate, via Hardy–Littlewood and $\ell^q \subseteq \ell^\infty$

$$\|\mathfrak{M}_{r,n}F\|_q \lesssim \Big(\sum_{j \geqslant n} 2^{-jd} \|f_j\|_p^q \Big)^{1/q}.$$

<u>**Crucial gain in** n **if** $r = p = p_0$**,** $q = \infty$</u>:

$$\|\mathfrak{M}_{p_0,n}F\|_\infty \lesssim 2^{-nd/q_0} \sup_{j \geqslant n} \|f_j\|_{p_0}.$$

Just use Hölder's and the single-scale estimate:

$$\begin{aligned}
\mathfrak{M}_{p_0,n}F(x) &\leqslant \sup_{j \geqslant n} \sup_{Q \in \mathcal{Q}_{n-j}(x)} \Big(\frac{1}{|Q|} \int \|\mathcal{A}_j f_j(y,\cdot)\|_{p_0}^{q_0} \, \mathrm{d}y \Big)^{1/q_0} \\
&\lesssim \sup_{j \geqslant n} \sup_{Q \in \mathcal{Q}_{n-j}(x)} |Q|^{-1/q_0} 2^{-jd/q_0} \|f_j\|_{p_0} \\
&\lesssim 2^{-nd/q_0} \sup_{j \geqslant n} \|f_j\|_{p_0}.
\end{aligned}$$

Estimate for $\mathcal{G}_{III}$

$$\mathcal{G}_{III}(x) := \sup_{\substack{Q \in \mathcal{Q}(x) \\ L(Q) > 0}} \int_Q \sum_{j \geqslant 0} \|\mathcal{A}_j f_j(y, \cdot)\|_{L^r} \, \mathrm{d}y$$

Essentially local operator at unit scale. Large cubes are just an error term.

Follows from the case $L(Q) = 0$, i.e., from $\mathcal{G}_{II}$.

Estimate for $\mathcal{G}_{III}$

$$\mathcal{G}_{III}(x) := \sup_{\substack{Q \in \mathcal{Q}(x) \\ L(Q) > 0}} \int_Q \sum_{j \geqslant 0} \|\mathcal{A}_j f_j(y, \cdot)\|_{L^r} \, \mathrm{d}y$$

Essentially local operator at unit scale. Large cubes are just an error term.

Follows from the case $L(Q) = 0$, i.e., from $\mathcal{G}_{II}$. Let

$$U(y) = \sum_{j \geqslant 0} \|\mathcal{A}_j f_j(y, \cdot)\|_{L^r}, \qquad U_*(w) = \sup_{\substack{Q \in \mathcal{Q}(w) \\ L(Q) = 0}} \int_Q U(y) \, \mathrm{d}y.$$

Given a cube $\widetilde{Q} \in \mathcal{Q}(x)$ with $L(\widetilde{Q}) > 0$ we may tile $\widetilde{Q}$ into cubes of side length 1 and get

$$\int_{\widetilde{Q}} U(y) \, \mathrm{d}y \leqslant \int_{\widetilde{Q}} U_*(w) \, \mathrm{d}w \leqslant M_{HL}[U_*](x).$$

Estimate for $\mathcal{G}_{III}$

$$\mathcal{G}_{III}(x) := \sup_{\substack{Q\in\mathcal{Q}(x)\\ L(Q)>0}} \fint_Q \sum_{j\geqslant 0} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}\,\mathrm{d}y$$

Essentially local operator at unit scale. Large cubes are just an error term.

Follows from the case $L(Q)=0$, i.e., from $\mathcal{G}_{II}$. Let

$$U(y) = \sum_{j\geqslant 0} \|\mathcal{A}_j f_j(y,\cdot)\|_{L^r}, \qquad U_*(w) = \sup_{\substack{Q\in\mathcal{Q}(w)\\ L(Q)=0}} \fint_Q U(y)\,\mathrm{d}y.$$

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$$\fint_{\widetilde{Q}} U(y)\,\mathrm{d}y \leqslant \fint_{\widetilde{Q}} U_*(w)\,\mathrm{d}w \leqslant M_{HL}[U_*](x).$$

By a very crude estimate we can replace U_* by $\mathcal{G}_{II}$ and get

$$\mathcal{G}_{III}(x) \leqslant M_{HL}[\mathcal{G}_{II}](x).$$

Thanks!