TARGET ENUMERATION VIA EULER CHARACTERISTIC INTEGRALS *

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Abstract. We solve the problem of counting the total number of observable targets (*e.g.*, persons, vehicles, landmarks) in a region using local counts performed by a network of sensors, each of which measures the number of targets nearby but neither their identities nor any positional information. We formulate and solve several such problems based on the types of sensors and mobility of the targets. The main contribution of this paper is the adaptation of a topological sheaf integration theory — integration with respect to Euler characteristic — to yield complete solutions to these problems.

1. Topological enumeration.

1.1. Sensors.

Sensor networks are poised to impact society in fundamental ways analogous to the impact of the personal computer and the internet. The rapid development of small-scale sensor devices coupled with wireless ad hoc networking capability is giving birth to a wide variety of sensor networks for applications to agriculture, defense, environmental monitoring, and more.

At present, there are severe limits on sizes and types of implementable networks. In particular, there are trade-offs between power consumption, sensing complexity (how much data is gathered and processed on-board), sensor size, sensor range (how large a neighborhood of the sensor is scanned), and communication bandwidth. Current implementations of sensor networks in environmental and agricultural monitoring are limited to dozens or at most hundreds of sensors (see *e.g.*, [15] for a typical deployment). This constraint will not persist. Ubiquitous sensing — the saturation of physical environments with networked sensors — is a future scenario whose possibility is strongly suggested by Moore's law, by advances in micro- and nano-scale electronic device fabrication, and by advances in wireless networking capabilities [8]. The potential to integrate sensors into building materials, roads, soil, and other media portend an extraordinary increase in the ability to monitor street traffic, crowd dynamics, wildlife habitats, and crop development.

This paper initiates a mathematical (and in particular *topological*) approach to target-counting problems in sensor networks. Counting is a fundamental application of sensors, both in present and potential settings. Scenarios where counting is critical include: agriculture (crop/weed/insect populations, herd size); border security (people, vehicles, shipping containers); commerce (customers, stock, mail parcels); ecology (wildlife population, particulate pollutants); situational awareness (troops, vehicles, artillery); traffic control (vehicles, pedestrians); and more. Note that objects to be counted span several orders of magnitude in scale, depending on the particular application domain. In many instances, targets may be in motion.

At present, there are many sensor modalities which admit counting. The most common in

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current applications are what one might call *one-dimensional* sensors, *e.g.*, people-counters installed at entrances and exits of retail stores. Such sensors act via beam-interruption and are used to enumerate inflow and outflow. More relevant to the setting of this paper are sensors whose 'support' is of dimension higher than one. For example, the motion-detectors common in security applications range over a broad (but localized) swath of space. Infrared, acoustic, optical, and radio frequency are a few of the relevant sensor modalities. On the horizon are a variety of sensor types under rapid development, many at very small physical scales. These include smart optical arrays and biophotonic sensors used for counting individual DNA strands, molecules, or photons.

Following the paradigm of integration theory, we begin with the 'continuum limit' where sensors are assumed to be very small and distributed with very high density. We then pass to the setting of networks (dense or sparse, grids or ad hoc) from the perspective of discretizing the domain. The sensors in our models require very few higher-order processing capabilities: no clocks, time-stamps, or synchronization; no range-finding or directional sensing capabilities; and very little memory. On-board computational needs are flexible. For a centralized scheme, nodes need no computational ability at all and can be scanned by a central data collector and processor. Our methods allow for decentralized computation, assuming the ability to share counting data with neighbors and perform integer arithmetic. Such computational abilities are well within the range of small-scale processors — communication complexity becomes the limiting factor.

1.2. A simple example: geometric enumeration. The following simple example of target enumeration has a trivial geometric solution that motivates our topological techniques. Consider a field of (infinitesimally small) sensors, one at each point x in a planar domain $\mathbb{R}^2$. Assume there is a finite set of points acting as observables or TARGETS $\{\mathcal{O}_\alpha\} \subset \mathbb{R}^2$, and that the sensor at $x \in \mathbb{R}^2$ returns a quantized count $h(x) \in \mathbb{N}$ equal to the number of targets which are 'nearby' — which can be sensed by whatever modality (optical, acoustic, infrared) is used. The sensors have no information about target location or identity.

How can sensors merge their local counts into a global count of the number of targets? If the targets are sufficiently separated, then this becomes a simple problem of counting the number of connected 'on' clusters in the network [9]. If we want to allow for more complex target interactions, the following critical (if not entirely realistic) assumption makes the problem very simple. Assume that each target $\mathcal{O}_\alpha$ impacts its environment in such a way that $\mathcal{O}_\alpha$ is detected precisely on those sensors within Euclidean distance R of $\mathcal{O}_\alpha$. This is a reasonable assumption if all targets are identical on a homogeneous domain. Then the total number of targets may be computed as an integral:

$$\#\{\mathcal{O}_\alpha\} = \frac{1}{M} \int_{\mathbb{R}^2} h(x)\, dx, \tag{1.1}$$

where $M = \pi R^2$ is the 'mass' of the support set U_α on which the target $\mathcal{O}_\alpha$ is detected. This formula is trivial: the count returned by the sensor field is of the form $h = \sum_\alpha \mathbb{1}_{U_\alpha}$, where $\mathbb{1}$ denotes the characteristic function. As integration is linear, one observes:

$$\int_{\mathbb{R}^2} h(x)\, dx = \int_{\mathbb{R}^2} \sum_\alpha \mathbb{1}_{U_\alpha}\, dx = \sum_\alpha \int_{\mathbb{R}^2} \mathbb{1}_{U_\alpha}\, dx = \sum_\alpha M = M\#\alpha.$$

This method is immediately applicable to arbitrary dimensions as well as to more general target supports U_α, so long as all targets have supports with identical mass. In addition, one can discretize the domain, sampling h on a finite set, say, a grid. Then a total count of targets can be realized as a discrete integral, assuming the grid is neither too coarse nor degenerate with respect to target supports.

This paper generalizes this simple idea to problems whose target supports may have different (and unknown) sizes but whose topology is fixed. Because we replace geometric assumptions with topological ones, our methods are adaptable to a wide range of enumeration problems in sensor networks, particularly situations where one wants to count many different types of targets. For example, if one wants to count the number of vehicles in an area via a field of counting sensors, our methods allow for different sizes of 'footprints' — SUVs and subcompacts alike can be counted.

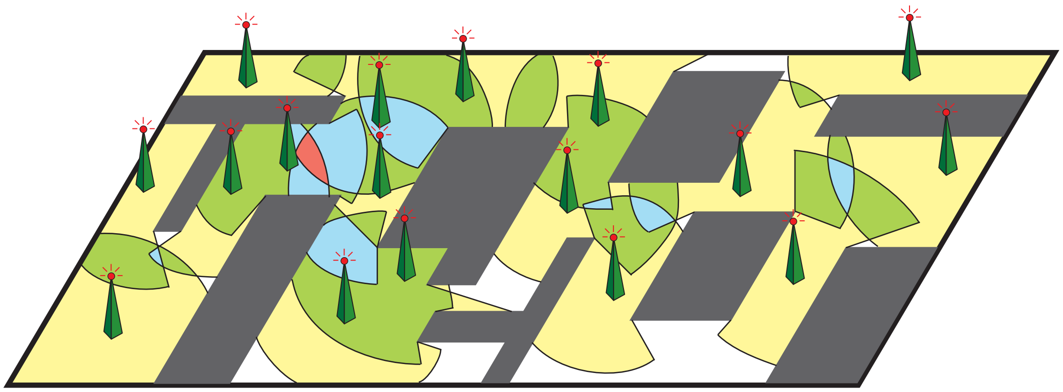

FIG. 1.1. *A sensor field counts the number of in-range visible beacons based on line-of-sight. Target supports (the region in the sensor space from which the target is sensed) may vary greatly in size and shape depending on beacon strength and target space obstacles; however, target supports are star-convex with respect to the target and thus all topologically trivial.*

1.3. Other problem statements. There are numerous interesting and challenging problems which involve determining a global count based on a local count. Some variants we address in this paper include the following.

1. Each sensor sweeps a finite length beam or cone over all bearings, recording the number of targets sensed as a function of bearing and location.
2. Each sensor increments an internal counter whenever a moving target comes within range.
3. Each sensor increments an internal counter whenever a moving wavefront initiated by a target event passes within range.

In these situations, assume that the sensor field is parameterized by some topological space X. The sensor field returns a 'counting function' $h : X \to \mathbb{N}$. The goal is to aggregate the redundant data of h and normalize in such a way as to yield a global count of targets/events.

1.4. Results and related work. The major result of this paper is a topological integral calculus for a variety of target enumeration problems.

Although our applications are novel, all of the mathematical tools used in this paper are fairly elementary and known to experts. Our key tool is the theory of integration with respect to Euler characteristic, a calculus based on the Euler characteristic. Though the ideas date back to Blaschke [3] and Hadwiger [13] as a form of inclusion-exclusion, we follow the more modern approach, based on ideas from (1) SHEAF THEORY, as in [30, 23, 24, 25]; and (2) MOTIVIC INTEGRATION, as in [6, 29]

Integration with respect to Euler characteristic is not well-known to applied mathematicians, though see [24] for an application to computational geometry, as well as [18] for recent work on using Euler characteristic integrals to compute volumes. Integrals involving Euler characteristic appear frequently in the literature on integral geometry [3, 13] and convex geometry [11, 20]. There is also a significant literature in geometric combinatorics concerning Euler characteristic as a measure [5, 21, 22]. More recently, integrals involving Euler characteristic have arisen in analyses of the geometry of Gaussian random fields, as in [1, 2, 31, 28]. Many of these papers appear to use integration with respect to Euler characteristic without the formal machinery. An algebraic cousin to Euler characteristic integrals appears in the body of literature on generalized Bonferroni inequalities [10] and tube formulae [7], all generalizations of the inclusion-exclusion principle.

There are few if any similar approaches to problems in target estimation or tracking, the literature on which almost always assumes the ability to identify different targets (along with other high-level functions, including distance estimation, bearing estimation, and sensor localization). For example, the large-scale wireless system implemented in [15] assumes an aggregation phase based on strict spatial separation of targets. Jung and Sukhatme [16] implement a multi-target robotic tracking system where the targets are labeled with colored lights. The survey paper of Guibas [12] pointing to the broader literature on geometric range-searching assumes the ability to aggregate target identities and concerns itself with computational complexity issues. The paper by Li *et al.* [19] on multi-target tracking via sensor networks notes that, *"target classification is arguably the most challenging signal processing task in the context of sensor networks."*

One significant solution to a target enumeration problem is found in the work of Fang, Zhao, and Guibas [9], which gives a distributed algorithm for target enumeration without any target-identification capabilities on the part of the sensors. Their work assumes that all target supports are round balls in $\mathbb{R}^2$; that each sensor reads a $\mathbb{R}$-valued signal proportional to the inverse square of distance-to-target; and that target impacts are additive. Their algorithm counts the number of local maxima in the sensor signal field and therefore gives an accurate count so long as the target supports overlap minimally or not at all. Our work is complementary to this in that the theory we introduce accommodates very complex target support overlaps.

The other example of target counting without identification or localization arises in work of Singh *et al.*, who consider a network of sensors which return a value in $\{0, 1\}$ depending on target proximity [27]. Their technique involves using time-series data in the case of moving targets/sensors, since a target count in the stationary case is too difficult, even if all target supports are convex, round, fixed, etc.

The sensors in our model are *minimal* in that they use nothing more than local counts without geometric data about the target identity, distance, or bearing. That being said, we have ignored for the moment many of the important technical issues associated with network implementation of our methods. Much of the work in aggregation of data by a network concerns network protocols for signal processing [19], managing constraints on bandwidth and energy [4], and dealing with errors or node failures [32]. This introductory paper does not treat these important issues. We also assume noise-free sensor readings over a continuum field of sensors (or at least a sufficiently dense network). In particular, certain degenerate target configurations require knowing sensor readings on sets of Lebesgue measure zero: an unrealistic demand. The present paper assumes an idealized setting to develop and highlight the mathematical tools.

2. Integration with respect to Euler characteristic.

Our results follow from the classical and elegant theory of integration with respect to Euler characteristic [30, 23]. We restrict attention to subsets of $\mathbb{R}^n$.

2.1. The simplicial approach.

For simplicity, we begin our treatment of topological integration with simplicial complexes (triangulated piecewise-linear sets) in $\mathbb{R}^n$. For $k \geq 0$, a k-SIMPLEX, σ, is the convex hull of a set of $k+1$ affinely independent VERTICES $\{v_i\}_0^k$ in $\mathbb{R}^n$: $\sigma = \left\{\sum_i t_i v_i \,:\, t_i \in (0,1), \sum_i t_i = 1\right\}$. A FACE of σ is a simplex spanned by a proper nonempty subset of the vertex set of σ. The closure $\mathsf{cl}(\sigma)$ of a simplex σ is the (disjoint) union of σ and all its faces. A SIMPLICIAL COMPLEX X is a finite collection of simplices such that each pair σ, τ of simplices satisfies $\mathsf{cl}(\sigma) \cap \mathsf{cl}(\tau)$ is either empty or the closure of some simplex in the complex. A complex is called CLOSED if it contains all its faces.

See, *e.g.*, [14] for elementary definitions and examples. Throughout this paper, all simplicial complexes will be implicitly assumed finite. The above definition differs from the usual in that we do not assume that a simplicial complex is closed: most authors do.

DEFINITION 2.1. *The* GEOMETRIC EULER CHARACTERISTIC *of a simplicial complex X is*

$$\chi(X) = \sum_{\sigma} (-1)^{\dim \sigma} = \sum_{k=0}^{\infty} (-1)^k \# \left\{k\text{-simplices in } X\right\}. \tag{2.1}$$

EXAMPLE 2.2.

1. Euler characteristic generalizes cardinality: for a finite set X, $\chi(X) = |X|$.
2. If X is a compact contractible set — if it can be deformed continuously within itself to a single point — then $\chi(X) = 1$.
3. For a finite graph Γ, $\chi(\Gamma) = \#V - \#E$: vertices minus edges.
4. For X a triangulated orientable surface of genus g, $\chi(X) = 2(1-g)$.
5. For $X \subset \mathbb{R}^2$ a compact connected set with N disjoint discs (open or closed) removed, $\chi(X) = 1 - N$.

As one could guess from the above examples, the Euler characteristic is a topological invariant: $\chi(h(X)) = \chi(X)$ for h a homeomorphism. It is therefore independent of the simplicial structure imposed on X. That this is so follows from a homological interpretation: $\chi(X) = \sum_{k=0}^{\infty} (-1)^k \dim(H_k(X))$, where $H_k(X) = H_k^{BM}(X;\mathbb{R})$ denotes the BOREL-MOORE HOMOLOGY of X with $\mathbb{R}$ coefficients.[1] For X a closed complex, $H_k(X) \cong H_k^{\Delta}(X;\mathbb{R})$, the more familiar k^{th} simplicial homology group of X with $\mathbb{R}$ coefficients, a vector space whose dimension measures the number of 'holes' in X that a k-dimensional subcomplex can detect. The MAYER-VIETORIS PRINCIPLE [14], a homological inclusion-exclusion principle, yields the following: for A and B subcomplexes of X,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{2.2}$$

[1] This is isomorphic to the singular relative homology $H_k(\mathsf{cl}(X), \mathsf{cl}(X) - X; \mathbb{R})$. A detailed understanding of this definition is not required of the reader.

The above equation evokes the definition of a measure and allows one to interpret Eqn. (2.1) as giving a 'topological volume' of a complex. As many authors have implicitly or explicitly observed [3, 13, 21, 23, 30], the measure $d\chi$ derived form χ is well-behaved when restricted to the appropriate classes of integrands and domains. In the setting of $\mathbb{Z}$-valued functions over simplicial complexes, this measure theory has a simple combinatorial definition.

DEFINITION 2.3. *Let X denote a finite simplicial complex and $CF(X)$ the abelian group of functions $X \to \mathbb{Z}$ with basis $\mathbb{1}_\sigma$, where σ is a simplex of X. The* EULER INTEGRAL *is the homomorphism $\int_X d\chi : CF(X) \to \mathbb{Z}$ which sends $\mathbb{1}_\sigma \mapsto \chi(\sigma) = (-1)^k$, for σ a k-simplex.*

The following is obvious from Definitions 2.1 and 2.3.

LEMMA 2.4. *The Euler integral satisfies $\int_X \mathbb{1}_A \, d\chi = \chi(A)$ for $A \subset X$ a subcomplex.*

2.2. The definable approach. We extend the definitions to a wide class of spaces which are not obviously simplicial. The reader who is in a hurry may skip ahead to §3 and remain in the class of simplicial complexes.

Not all sets are measurable in $d\chi$: Cantor sets, Hawaiian earrings, and other 'wild' sets do not possess a well-defined Euler characteristic. In many respects, the best definitions for what should be called a 'tame' set are to be found in the literature on O-MINIMAL STRUCTURES [29].

Briefly, an o-minimal system $\mathcal{A} = \{\mathcal{A}_n\}$ is a sequence of Boolean algebras $\mathcal{A}_n$ of subsets of $\mathbb{R}^n$ (families of sets closed under the operations of intersection and complement) which satisfies certain natural properties (the system is closed under products and projections, and $\mathcal{A}_1$ consists of all finite unions of points and open intervals). Sets belonging to $\mathcal{A}_n$ are called "tame" or, more properly, DEFINABLE. A (not necessarily continuous) function is called definable if its graph (in the product of domain and range) is a definable set in the system. The interested reader is encouraged to see the text of van den Dries [29], which requires little background. Canonical examples of o-minimal systems include semilinear sets (sets defined by a finite number of affine inequalities), semialgebraic sets (sets defined by a finite number of polynomial inequalities) and subanalytic sets (sets which are the local image of real-analytic manifolds under real-analytic mappings).

One of the principal results from the theory of o-minimal structures is the TRIANGULATION THEOREM: any definable set A can be expressed as the image under a definable homeomorphism of a simplicial complex. From this, and the fact that χ is a homeomorphism invariant, it follows that all definable sets possess a well-defined Euler characteristic. One proceeds in the obvious manner:

DEFINITION 2.5. *Denote by $CF(X)$ the group of* CONSTRUCTIBLE *functions*[2]*, functions $h : X \to \mathbb{Z}$ with finite range and definable level sets $h^{-1}(c)$.*

It follows from the Triangulation Theorem that any $f \in CF(X)$ admits a decomposition as $f = \sum_\alpha c_\alpha \mathbb{1}_{\sigma_\alpha}$ for $c_\alpha \in \mathbb{Z}$ and $\sigma_\alpha \subset X$ a simplex (in a definable triangulation).

DEFINITION 2.6. *The* EULER INTEGRAL *is the homomorphism $\int_X d\chi : CF(X) \to \mathbb{Z}$ which sends $\sum_\alpha c_\alpha \mathbb{1}_{\sigma_\alpha} \mapsto \sum_\alpha c_\alpha \chi(\sigma_\alpha)$.*

The analogue of Lemma 2.4 clearly holds.

[2]In order to avoid complications concerning proper functions, we will assume for the remainder of the paper that $CF(X)$ is restricted to compactly-supported functions.

2.3. The sheaf approach. There is, as intimated earlier, a deeper treatment in the literature on sheaves. There are significant and deep perspectives from sheaf theory which can be easily interpreted in the setting of Euler integration.

A sheaf is, roughly speaking, a means of assigning to open sets of a topological space some algebraic object (a group, *e.g.*) in a manner that respects the operations of (1) restriction; and (2) gluing. The canonical example of a sheaf on a space X is $C(X)$, the continuous functions $X \to \mathbb{R}$. This forms a sheaf since (1) the restriction of a continuous function is continuous; and (2) two continuous functions on subsets which agree on the intersection extend to a continuous function on the union. Sheaf theory allows one to extend some of the more familiar concepts implicit in $C(X)$ — germs, sections, analytic continuation, etc. — to a vast degree of generality.

As the definition of constructible function is local (depends only on a system of open sets covering X), $CF(X)$ can be thought of a sheaf over X with respect to the standard topology. Following the general formalism of sheaf operations ([17, 23, 25]), one obtains an interpretation of $\int_X d\chi$ via the pushforward (or direct image, since we assume compactly supported integrands) construction.

DEFINITION 2.7. *Let $F : X \to Y$ be a definable map for a fixed o-minimal structure.*[3] *The* PUSHFORWARD *of F is the induced homomorphism $F_* : CF(X) \to CF(Y)$ defined, for $h \in CF(X)$ via*

$$F_* h(y) = \int_{F^{-1}(y)} h(x)\, d\chi(x). \tag{2.3}$$

Since we assume definability and compact supports, h is integrable on $F^{-1}(y)$ for all $y \in Y$. Formal methods imply that the pushforward is functorial, meaning that things which ought to commute do. This is no mere formality: in the present context, this result is extremely important, leading to the following.

THEOREM 2.8 (Fubini Theorem [30, 23]). *For $F : X \to Y$ as above and $h \in CF(X)$,*

$$\int_X h(x)\, d\chi(x) = \int_Y F_* h(y)\, d\chi(y). \tag{2.4}$$

Proof. In the case of the trivial map $X \to \{pt\}$, $CF(\{pt\}) \cong \mathbb{Z}$ and one checks that the induced pushforward $CF(X) \to \mathbb{Z}$ *is* precisely the integral with respect to Euler characteristic. Given $F : X \to Y$, functoriality says that the composition commutes:

$$CF(X) \xrightarrow{F_*} CF(Y) \xrightarrow{\int_Y d\chi} \mathbb{Z}\,, \qquad CF(X) \xrightarrow{\int_X d\chi} \mathbb{Z}. \tag{2.5}$$

□

The fact that $\int_X d\chi$ is the pushforward of $X \mapsto \{pt\}$ implies that this operation respects the gluings and restriction operations implicit in sheaves, and it is thus proper to call it by the

[3] Concrete examples include piecewise-linear maps between simplicial complexes, algebraic maps between real semi-algebraic sets, and analytic maps between real-analytic manifolds.

term *integral*.[4] This brief aside on sheaves hints at a wealth of available tools: convolution operators, integral transforms, etc., to be utilized in future work.

3. A simple enumeration theorem. We give an immediate application of integration with respect to Euler characteristic in a simple target enumeration problem. The following generalizes the problem and setting of §1.2.

Let W denote the TARGET SPACE, a topological space which models the domain in which the targets $\mathcal{O}_\alpha$ lie. In the setting of §1.2, the sensors fill $W = \mathbb{R}^2$. More generally, one can parameterize the collection of sensors into a SENSOR SPACE, denoted X. The sensor space is a topological space which may differ from W. For example, one may count targets in a 3-dimensional room via a network of sensors located along a 2-d wall. Each target $\mathcal{O}_\alpha$ has a TARGET SUPPORT, defined to be $U_\alpha = \{x \in X : \text{the sensor at } x \text{ detects } \mathcal{O}_\alpha\}$.

PROBLEM 3.1. **(Fixed targets)** Assume a sensor modality in which each sensor $x \in X$ records the number of targets in range. This yields a (constructible) HEIGHT FUNCTION,

$$h(x) := \#\{\alpha : x \in U_\alpha\}, \tag{3.1}$$

and represents the count that a collection of sensors on X gives of the targets in W. The problem is to compute the number of targets, given only $h : X \to \mathbb{N}$.

If we assume that the target supports all have the same nonzero Euler characteristic, then Problem 3.1 is immediately solved.

THEOREM 3.2. *Given $h : X \to \mathbb{N}$ the counting function for $\{U_\alpha\}$ a collection of compact definable target supports in X satisfying $\chi(U_\alpha) = N \neq 0$ for all α. Then*

$$\#\alpha = \frac{1}{N} \int_X h \, d\chi. \tag{3.2}$$

Proof.

$$\int_X h \, d\chi = \int_X \left(\sum_\alpha \mathbb{1}_{U_\alpha} \right) d\chi = \sum_\alpha \int_X \mathbb{1}_{U_\alpha} \, d\chi = \sum_\alpha \chi(U_\alpha) = N \, \#\alpha. \tag{3.3}$$

□

This result solves Problem 3.1 and greatly generalizes the simple example of §1.2, in that we do not need to assume that the target supports are round or even convex. In the case of convex, or even contractible compact sets, $N = 1$ and the formula is especially simple. Note however that if $N = 0$, no solution is possible in general: see Fig. 3.1.

4. Computation. Integration with respect to Euler characteristic, being partly combinatorial in nature, permits simple, clear computations.

4.1. Excursion sets. We employ the following convenient notation. For functions $h : X \to \mathbb{Z}$, the set $\{h = s\}$ is the LEVEL SET $\{x \in X : h(x) = s\}$, and the set $\{h > s\}$ is the UPPER EXCURSION SET $\{x \in X : h(x) > s\}$. Lower excursion sets are likewise defined.

[4] One could prove the Fubini theorem directly by using the lemma that χ is multiplicative under Cartesian products and invoking the definable Hardt theorem [29] if the sheaf formalism is to be avoided.

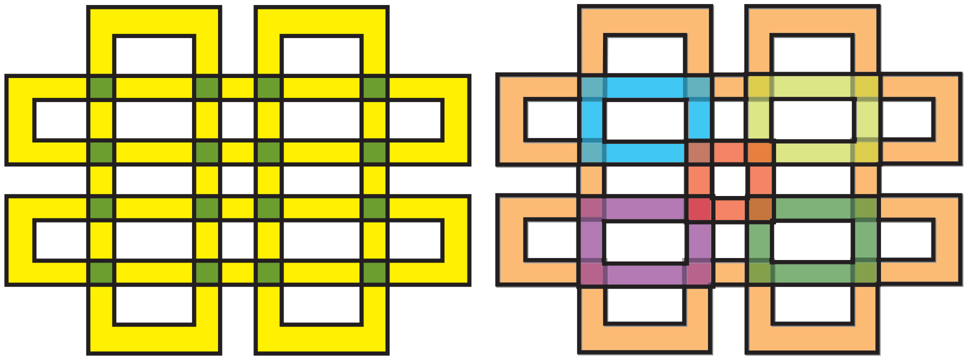

FIG. 3.1. *Given $h = \sum_\alpha \mathbb{1}_{U_\alpha}$ with $\chi(U_\alpha) = 0$, one has $\int h\,d\chi = 0$. It is not possible in general to determine $\#\alpha$. [left] The height function of a collection of annuli. Four? or [right] six? Any even number between two and twelve is possible with embedded annuli.*

PROPOSITION 4.1. *Given $h \in CF(X)$, the integral of h with respect to $d\chi$ may be computed as follows:*

$$\int_X h\,d\chi = \sum_{s=-\infty}^{\infty} s\,\chi\{h = s\} \tag{4.1}$$

$$= \sum_{s=0}^{\infty} \chi\{h > s\} - \chi\{h < -s\}\ . \tag{4.2}$$

Proof.

$$h = \sum_{s=-\infty}^{\infty} s\mathbb{1}_{\{h=s\}} = \sum_{s=0}^{\infty} s(\mathbb{1}_{\{h\geq s\}} - \mathbb{1}_{\{h>s\}}) + \sum_{s=0}^{-\infty} s(\mathbb{1}_{\{h\leq s\}} - \mathbb{1}_{\{h<s\}}) = \sum_{s=0}^{\infty} \mathbb{1}_{\{h>s\}} - \mathbb{1}_{\{h<-s\}},$$

where the last equality comes from telescoping sums. Lemma 2.4 completes the proof. □

Eqn. (4.2) is preferable to Eqn. (4.1) in practice, since, for h a sum of indicator functions over compact sets, the excursion sets of h are compact for all s, whereas $h^{-1}(s)$ is only relatively compact.

EXAMPLE 4.2. **Fixed targets.** In the example of Fig. 4.2, seven contractible sets are displayed, along with a height function on connected components. In the intended application, only h is known, not the target supports. Decomposing h into upper excursion sets (Fig. 4.2) allows one to compute $\int h\,d\chi$ via Eqn. (4.2):

$$\int h\,d\chi = \sum_{s=0}^{\infty} \chi\{h > s\} = \overbrace{2}^{s=3} + \overbrace{3}^{s=2} + \overbrace{3}^{s=1} + \overbrace{-1}^{s=0} = 7. \tag{4.3}$$

The levels sets of h in Fig. 4.2 are 'fragile' in the sense that a perturbation of $h^{-1}(s)$ (in the Hausdorff metric) usually changes the topology and hence the Euler characteristic of the level set. The upper excursion sets are less likely to exhibit such instability. This will become more important when sampling integrands over discrete sets.

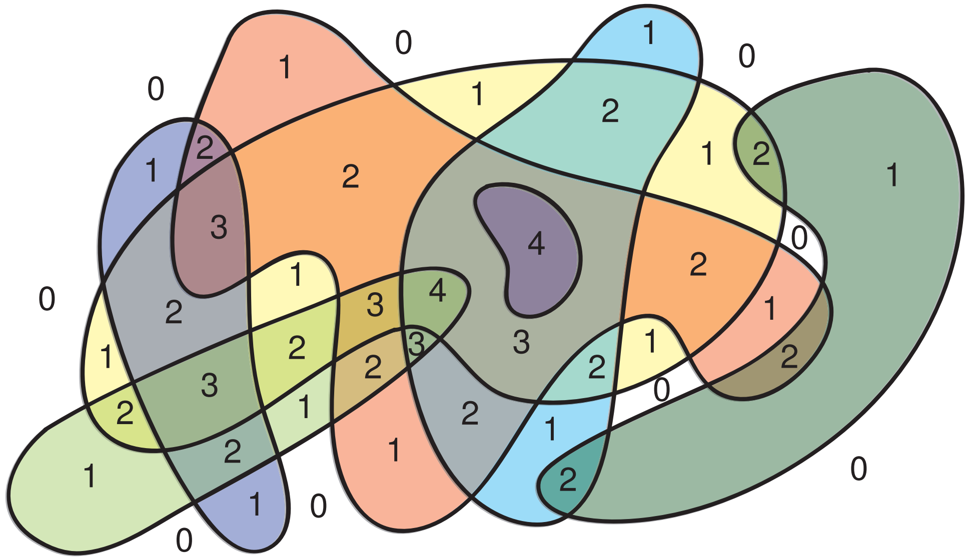

FIG. 4.1. *A collection of contractible patches $\{U_\alpha\}$ in $\mathbb{R}^2$ corresponding to the supports or 'visibility regions' of seven targets. The collection decomposes $\mathbb{R}^2$ into cells labeled according to the height function h returned by a dense sensor network.*

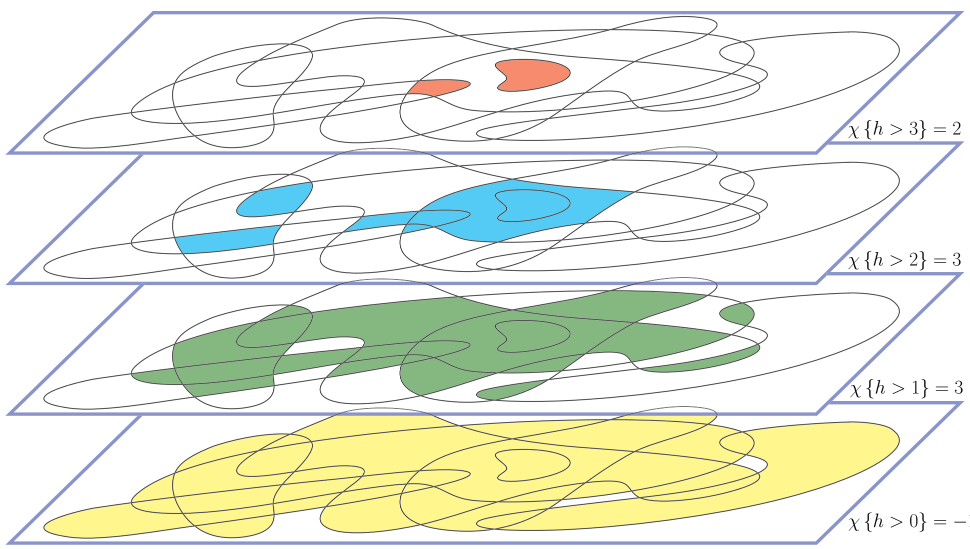

FIG. 4.2. *Decomposing h into upper excursion sets and computing χ yields the integral $\int h\,d\chi$.*

There are other means of computing integrals with respect to $d\chi$, many of which are related to Morse theory. We detail these in subsequent work.

4.2. Homology and duality. Since the Euler characteristic has a homological as well as a combinatorial definition, we can switch perspectives at will, playing off strengths for computational purposes. We augment Proposition 4.1 with a specialized formula for certain integrands on the plane.

Theorem 4.3. *For $h : \mathbb{R}^2 \to \mathbb{N}$ constructible and upper semi-continuous,*

$$\int_{\mathbb{R}^2} h\, d\chi = \sum_{s=0}^{\infty} \left(\beta_0\{h > s\} - \beta_0\{h \leq s\} + 1\right), \tag{4.4}$$

where β_0 denotes the zeroth Betti number*; equivalently, the number of connected components of the set.*

Proof. Let A be a compact nonempty subset of $\mathbb{R}^2$. From the homological definition of the Euler characteristic, $\chi(A) = \sum_{s=0}^{\infty}(-1)^s \dim H_s(A)$, where, by compactness, H_s is the singular homology. However, since $A \subset \mathbb{R}^2$, $H_s(A) = 0$ for all $s \geq 2$. Thus, it suffices to compute $\chi(A) = \dim H_0(A) - \dim H_1(A)$. By Alexander duality [14],

$$\dim H_1(A) = \dim H^0(\mathbb{R}^2 - A, A) = \dim H_0(\mathbb{R}^2 - A) - 1,$$

and $\dim H_0 = \beta_0$, the number of connected components. One substitutes into Eqn. (4.2) the above computations for $A = \{h > s\}$ and $\mathbb{R}^2 - A = \{h \leq s\}$. □

This formulation is extremely important to numerical implementation of this integration theory to planar sensor networks: see §5.2.

Example 4.4. The duality formula (4.4) applied to the integrand of Example 4.2 yields

$$\int_{\mathbb{R}^2} h\, d\chi = \overbrace{(2-1+1)}^{s=3} + \overbrace{(3-1+1)}^{s=2} + \overbrace{(4-2+1)}^{s=1} + \overbrace{(1-3+1)}^{s=0} = 7.$$

5. From fields to networks. The mathematical tools formulated here for enumeration problems depend on having a sensor *field* with counting data at all points in a continuum of the sensor space. Any realistic implementation must occur over a discrete collection of sensors: a *network*, where nodes $\mathcal{N}$ (typically within the target space $\mathcal{N} \subset W$) record values.

Trying to parameterize the sensor space X as a discrete set based on the nodes $\mathcal{N}$ is doomed to failure, as the target supports will be likewise discrete and of unknown and non-uniform Euler characteristic. Likewise, if the communication links between nodes give the sensor network the structure of a graph, then the network graph is an equally bad candidate for the sensor space X: cycles in the graph can make the Euler characteristics of the target supports vary greatly. In fact, the denser the network in W, the more negative the Euler characteristics of the target supports become (*e.g.*, in Fig. 5.3, the target supports intersected with the network are graphs with negative Euler characteristic).

5.1. Numerical analysis and Euler integrals. A more sensible parametrization is to model the sensor space X as a simplicial approximation to the target space W, using the nodes $\mathcal{N}$ as vertices. Assume that enough structure is known about $\mathcal{N}$ to give a simplicial structure that has $\mathcal{N}$ as the vertex set. In analogy with the problem of computing a numerically approximate Riemann integral of an integrand h based on a discrete sampling, we propose that the integral of the piecewise-linear (PL) interpolation h_{PL} of h based on the sampled values at $\mathcal{N}$ is a good approximation.

Unfortunately, the integration theory of §2 does not take continuous $\mathbb{R}$-valued functions as integrands. In a sequel to this paper, we extend the integration theory to $\mathbb{R}$-valued (definable) integrands and show that the PL approximation is, indeed, correct; furthermore, when the sampling is too coarse, one can construct an "expected value" for the integral which contains useful data. For the present, we limit ourselves to computing the constructible upper semi-continuous approximation $\lfloor h_{PL} \rfloor$. The following result shows that this gives an accurate approximation to $\int h \, d\chi$ upon refinement.

THEOREM 5.1. *Let $h : \mathbb{R}^n \to \mathbb{N}$ be an upper semi-continuous constructible function satisfying $\{h \geq s\} = \mathsf{cl}(\mathsf{int}(\{h \geq s\}))$ — every upper excursion set of h is the closure of its interior in $\mathbb{R}^n$. Then, for a sufficiently dense and regular triangulation of $\mathbb{R}^n$, the PL interpolation h_{PL} of h over the vertex set of the triangulation satisfies*

$$\int_{\mathbb{R}^n} \lfloor h_{PL} \rfloor \, d\chi = \int_{\mathbb{R}^n} h \, d\chi. \tag{5.1}$$

This is the first step toward the development of numerical analysis for Euler integration.

Proof. For h as above, a sufficiently fine and regular triangulation has the following features. Let σ be a simplex of the triangulation and $\Delta = \mathsf{cl}(\sigma)$ its closure. For each Δ, (1) $\max h|\Delta$ is attained at some vertex of Δ; and (2) $h|\Delta$ has all upper excursion sets contractible. Thus, $\int_\Delta h \, d\chi = \max_\Delta h = \int_\Delta \lfloor h_{PL} \rfloor \, d\chi$. Additivity of $\int d\chi$ completes the proof. ☐

EXAMPLE 5.2. Fig. 5.1 gives an example of an integrand sampled on a uniform hexagonal grid. The integral of $\lfloor h_{PL} \rfloor$ with respect to Euler characteristic is:

$$\int \lfloor h_{PL} \rfloor \, d\chi = \overbrace{1}^{s=3} + \overbrace{3}^{s=2} + \overbrace{0}^{s=1} = 4. \tag{5.2}$$

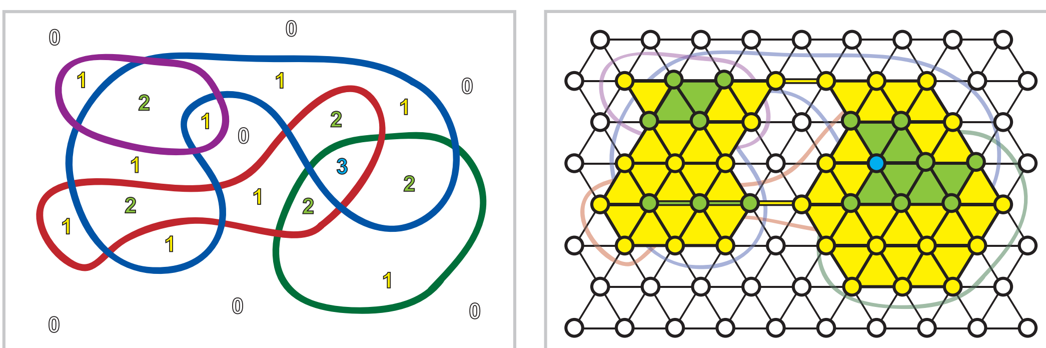

FIG. 5.1. *The height function of a collection of target supports [left] is sampled on a regular mesh [right].*

Applying Theorem 5.1 can be problematic. For target supports which are nearly tangent, a given sensor network may or may not sample the chamber correctly. See Fig. 5.2 for a sampling of errors that can arise from small chambers.

5.2. Ad hoc planar networks. We note that the strategy of converting the sampling of the true impact function h over $\mathcal{N}$ to a PL interpolation h_{PL} does not necessarily require knowing the coordinates of the nodes. Indeed, the evaluation of $\int_X \cdot \, d\chi$ is conspicuous in its

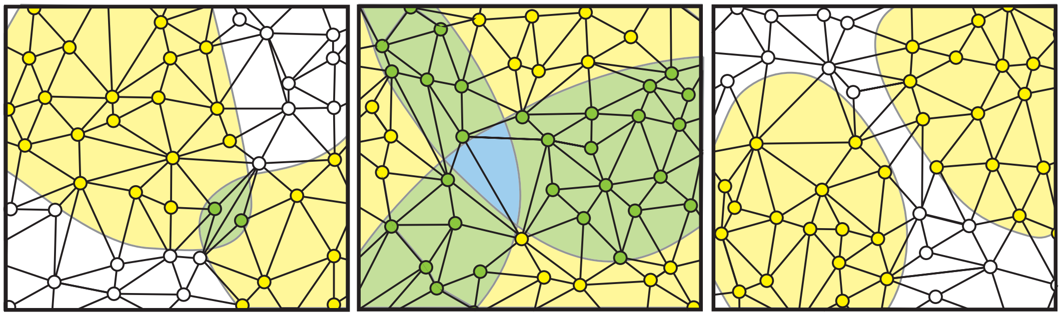

FIG. 5.2. *Errors can arise from sampling an integrand with geometrically small chambers.*

freedom from coordinate geometry: it is a topological integral. If one is given a triangulation, the extension of the counting function h on vertices over the domain is automatic. However, if no geometry associated to $\mathcal{N}$ is known, it may not be possible to determine a canonical extension h_{PL} over the domain. Such a situation is not uncommon in sensor networks based on *ad hoc* wireless communications, an increasingly common protocol for distributed sensor networks and robotics.

Assume that one is given a network in the form of an abstract graph $\mathcal{G}$. By *abstract* we mean that the projection of the 1-d cell complex $\mathcal{G}$ to the workspace is unknown. Edges should possess some coarse proximity data. One popular (if rigid) assumption is that $\mathcal{G}$ is a UNIT DISC GRAPH, in which edges exist between nodes if and only if they are within unit distance in the workspace. For this and other proximity-based networks, the duality results of §4.2 allow us to compute integrals based on coordinate-free ad hoc network sampling.

COROLLARY 5.3. *Assume an upper semi-continuous constructible integrand $h : \mathbb{R}^2 \to \mathbb{N}$, and let $\mathcal{G}$ be a network graph with nodes $\mathcal{N} \subset \mathbb{R}^2$, where the only thing known is the restriction of h to $\mathcal{N}$ (in particular, the coordinates of $\mathcal{N}$ in $\mathbb{R}^2$ are unknown). If the network $\mathcal{G}$ correctly samples the connectivity of the upper and lower excursion sets of h, then Eqn. (4.4) returns the exact number of targets.*

An example appears in Fig. 5.3. Note that in this example, the *topology* of the excursion sets of h is *not* always sampled correctly: sparsity leads to holes in the network. Nevertheless, since the connectivity of the upper and lower excursion sets is sampled faithfully, the integral is correct. Although the example drawn is a unit disc graph, this is by no means necessary for the result.

REMARK 5.4. The situation in higher-dimensional workspaces is not as convenient as in the planar case, since duality does not pair completely to connected components alone. The next most natural domain in which to work is $\mathbb{R}^3$. Here, duality is effective in mitigating spurious generators of H_2 — voids in the network. However, holes appearing as generators of H_1 are dual to H_1 generators in the complement, and one must deal with homology in dimension one. This is by no means as straightforward as in the planar case, and a simple clustering algorithm will not suffice.

5.3. Distributed computation. Since our methods are based on an integration theory, the enumeration of targets detailed in this paper is a local computation. To wit:

$$\int_{A\cup B} h\,d\chi = \int_A h\,d\chi + \int_B h\,d\chi - \int_{A\cap B} h\,d\chi. \tag{5.3}$$

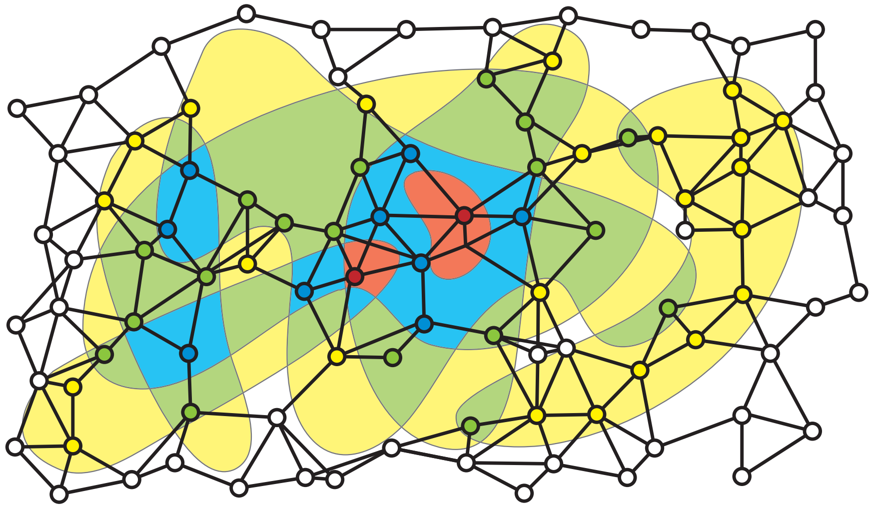

FIG. 5.3. *A sparse sampling over an* ad hoc *network retains enough connectivity data to evaluate the integral exactly.*

Thus, enumeration can be performed in a distributed manner easily. This is particularly easy when the network is a lattice, as one can employ standard distributed protocols for localization and merging of target counts.

6. Enumeration. In this section, we solve a variety of target enumeration problems in terms of integrals with respect to Euler characteristic.

6.1. Moving targets. The lack of a convexity assumption in our theory makes it ideal for applications in which target supports are generated by moving vehicles.

PROBLEM 6.1. **(Moving targets)** In this setting, one has a finite collection of targets $\mathcal{O}_\alpha$ which move along continuous paths $\mathcal{O}_\alpha(t)$ in the domain $W \subset \mathbb{R}^n$ over some fixed time interval, see 6.1. Assume that sensor nodes can detect when some target comes within proximity range (a time-dependent set $U_\alpha(t)$ containing $\mathcal{O}_\alpha(t)$), and that each such detection produces an increment in its internal counter: such increments occur only when the node detects an increase in the number of targets within range. One obtains a height function $h : W \to \mathbb{N}$ of the form:

$$h(x) := \#\left\{(t, \alpha) \,:\, x \in U_\alpha(t+\epsilon) \text{ and } x \notin U_\alpha(t-\epsilon) \text{ for } \epsilon \to 0^+\right\} \tag{6.1}$$

The problem is to compute the number of targets based solely on the function h. Note in particular the absence of temporal data: there are no clocks.

One interesting feature in this setting is the possibility that the "trace" — the union of temporal supports $\cup_t U_\alpha(t)$ — can be a non-contractible set. In this setting, the Fubini theorem is invaluable.

THEOREM 6.2. *Assume moving targets as per Problem 6.1. Then the number of targets is equal to* $\#\alpha = \int_W h \, d\chi$*, where* h *is the height function of Eqn. (6.1).*

Proof. Consider the sensor space $X = W \times \mathbb{R}$ as the product of the target space W with time, and let $F : X \to W$ be temporal projection. The target supports in X are the traces $U_\alpha := \cup_t U_\alpha(t) \times \{t\}$: this is a contractible set, as illustrated in Figure 6.1[right]. Let $g : X \to \mathbb{N}$ be $g = \sum_\alpha \mathbb{1}_{U_\alpha}$. From Theorem 3.2, the number of targets is $\int_X g\, d\chi$. By the Fubini theorem, this equals $\int_W F_* g\, d\chi$, where $(F_*g)(w) = \int_{F^{-1}(w)} g\, d\chi$. The intersection $F^{-1}(w) \cap U_\alpha$ is a finite number of compact intervals — one for each time w goes from being outside $U_\alpha(t)$ to inside it as t increases. Thus, $h = F_*g$ and

$$\int_W h\, d\chi = \int_W F_* g\, d\chi = \int_X g\, d\chi = \#\alpha.$$

□

The fact that integration with respect to Euler characteristic admits a Fubini theorem is thus not merely a curiosity but rather a crucial feature.

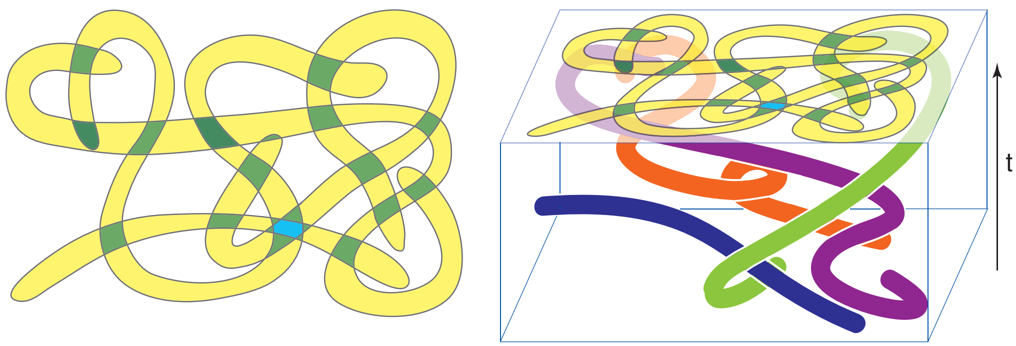

FIG. 6.1. *Vehicles moving in a planar environment activate sensors along regions which intersect over time and accumulate a larger height function there. The resulting integrand is the pushforward of a temporal projection map.*

EXAMPLE 6.3. **Moving targets.** Fig. 6.1 illustrates a height function for a moving-target situation as in Problem 6.1. Note that some traces self-intersect. One computes:

$$\int h\, d\chi = \sum_{s=0}^{\infty} \chi\,\{h > s\} = \overbrace{1}^{s=2} + \overbrace{16}^{s=1} + \overbrace{-13}^{s=0} = 4. \tag{6.2}$$

Theorem 6.2 is applicable to the problem of counting vehicles which move over a region with acoustic sensors embedded. The advantage of the Euler integration method is that one can count vehicles of different 'size' — large or small vehicle traces are irrelevant. A challenge for implementation lies in (cusp) singularities generated by a vehicle that turns too sharply, leading to an integrand that it not upper semi-continuous, as in Fig. 6.2. Such a singularity does not invalidate Theorem 6.2. Indeed, the integral of this height function with respect to Euler characteristic is equal to

$$\int_{\mathbb{R}^2} h\, d\chi = \chi\{h > 1\} + \chi\{h > 0\} = (1 - 1) + (1) = 1.$$

The upper excursion set $h > 1$ is *not* a compact disc, but rather has a closed interval in the boundary removed. Thus $\chi\{h > 1\} = 1 - 1 = 0$. Any sensor network, no matter how dense, will fail to see the higher codimension piece of boundary that is set to 1 instead of 2.

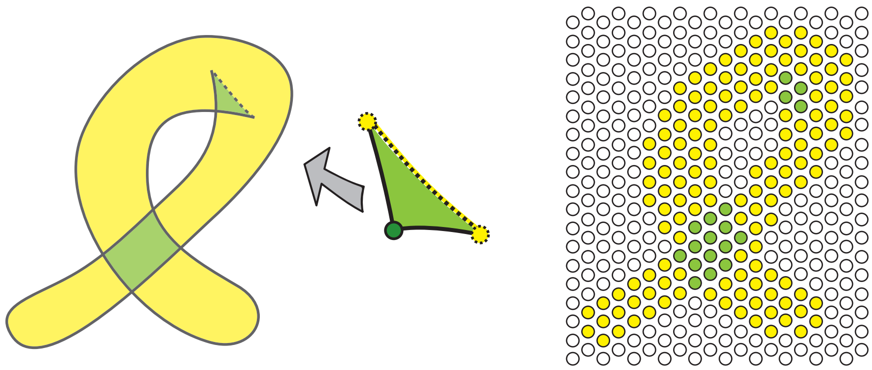

FIG. 6.2. *A cusp singularity in the trace of a moving vehicle; no matter how densely it is sampled, one will never recover the correct Euler integral.*

6.2. Wavefronts. The following problem is motivated by enumerating events whose effects propagate in time.

PROBLEM 6.4. **(Wave fronts)** Consider a finite collection of points $\mathcal{O}_\alpha$ in $W \subset \mathbb{R}^n$. Each $\mathcal{O}_\alpha$ represents an event which occurs at some time and which triggers a wavefront that propagates for a finite extent. Assume that each sensor has the ability to record the presence of a wavefront which passes through its vicinity. Nodes have a simple counter memory which allows them to store the number of wavefronts that have passed over as a counting function $h : W \to \mathbb{N}$. The problem is to determine the number of source events $\mathcal{O}_\alpha$.

Again, there is no temporal data associated to the sensors. With a particular assumption on the sensing modality, this problem is solved as a corollary of Theorem 6.2, using the same Fubini argument. Assume that the 'wavefront' associated to each event $\mathcal{O}_\alpha$ induces a continuous definable map F_α from a compact ball D^n to W whose restriction to rays from the origin are geodesic rays in W based at $\mathcal{O}_\alpha$. It is not enough to model sensors which count wavefronts by recording the number of 'fronts' that have passed, as one must account for singularities. To that end, the cleanest assumption for the counting sensors is the following:

COROLLARY 6.5. *In the context of Problem 6.4, assume that each sensor at $w \in W$ increments its internal counter by $\chi(F_\alpha^{-1}(w))$ whenever the wavefront of $\mathcal{O}_\alpha$ passes over. Under this assumption, the number of triggering events is $\#\alpha = \int_W h\, d\chi$.*

Proof. Apply the Fubini theorem to $h = \sum_\alpha (F_\alpha)_* \mathbb{1}_{D^n}$, where F_α is the mapping of the wavefront into W. ∎

Since $n = \dim(W) = \dim(D^n)$, the inverse image $F_\alpha^{-1}(w)$ is generically discrete, and the assumption on the sensor modality boils down to counting the number of passing wavefronts. However, certain complications can arise in practice. For example, very coarse binary sensors may not be able to distinguish between one wavefront and several wavefronts passing over simultaneously: this can lead to positive-codimension defects in the counting function h.

A similar loss of upper semi-continuity occurs when there is reflection of wavefronts along the boundary ∂W. For a compact domain $W \subset \mathbb{R}^n$ with smooth boundary ∂W. Consider a wavefront-counting integrand $h = \sum_\alpha (F_\alpha)_* \mathbb{1}_{D^n}$ whose projection maps F_α may have fold singularities (reflections) along ∂W. Let $h^+ : W \to \mathbb{N}$ be the upper semi-continuous

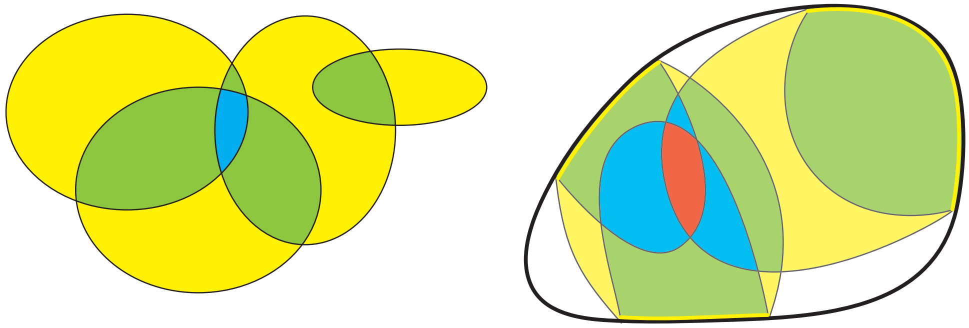

FIG. 6.3. *[left] Events trigger wavefronts which increment counting sensors as the front passes over. [right] Reflections along the boundary can be accounted for accurately.*

extension of h. Then it is easy to show using the techniques of this paper that

$$\#\alpha = \int_W h\, d\chi = \int_W h^+ d\chi - \frac{1}{2}\int_{\partial W} h^+\, d\chi. \tag{6.3}$$

6.3. Beam sensors. Instead of having the targets (or their wavefronts) move, we can consider situations in which the sensors have some internal degree of freedom. The following is a mathematical abstraction of the idea of a sensor which uses a bi- (or multi-) directional beam to count targets.

PROBLEM 6.6. **(Beam sensors)** Fix a Euclidean target space in $\mathbb{R}^n$ and consider a variant of Problem 3.1 in which each sensor node at $x \in \mathbb{R}^n$ senses targets via a "beam" that is a round k-dimensional ball in $\mathbb{R}^n$ centered at x (the term 'beam' evoking the case $k = 1$). Each target $\mathcal{O}_\alpha$ has a spatially extended region of brightness over some convex neighborhood V_α of $\mathcal{O}_\alpha$ in $\mathbb{R}^n$. The sensor at $x \in \mathbb{R}^n$ performs a sweep of its k-ball beam over all possible bearings. At each such bearing, the sensor counts the number of intensity regions V_α within the beam. Problem: compute the number of targets $\#\alpha$.

Note that each target has a spatially extended range over which it is sensed. The sensor field is parameterized over the Grassmannian bundle $\mathrm{Gr}_k(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{Gr}_k^n$, where Gr_k^n is the Grassmannian of k-planes in $\mathbb{R}^n$. (For example, the projective space $\mathbb{R}P^n$ is Gr_1^n.) Thus, the sensor field returns a counting function $h : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{N}$.

THEOREM 6.7. *Under the assumptions of Problem 6.6 and the additional assumption that if n if even then so is k, the number of targets is equal to*

$$\#\alpha = \frac{\lfloor \frac{n-k}{2} \rfloor ! \lfloor \frac{k}{2} \rfloor !}{\lfloor \frac{n}{2} \rfloor !} \int_{\mathrm{Gr}_k(\mathbb{R}^n)} h\, d\chi, \tag{6.4}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. Target supports $U_\alpha \subset \mathrm{Gr}_k(\mathbb{R}^n)$ are computed as follows. Fix an α and fix a 'bearing' k-plane in the Grassmannian Gr_k^n. The set of nodes in $\mathbb{R}^n$ at which this k-ball intersects V_α is star-convex with respect to (the centroid of) V_α. Thus, the target support U_α is topologically equivalent to $V_\alpha \times \mathrm{Gr}_k^n$ and thus to Gr_k^n. The Euler characteristic of Gr_k^n is:

$$\chi(\mathrm{Gr}_k^n) = \begin{cases} 0 & : n \text{ even}, k \text{ odd} \\ \begin{pmatrix} \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{k}{2} \rfloor \end{pmatrix} & : \text{ else} \end{cases} .$$

The result follows from Theorem 3.2. ☐

In the case of the plane ($n = 2$) with linear beams ($k = 1$), the theorem fails, since target supports have the homotopy type of a circle. Fig. 3.1 suggests that counting may be impossible in this setting. See, however, §6.4.

6.4. Sweeping sensors. We consider a variant on Problem 6.6 in which sensors sweep cones instead of beams.

PROBLEM 6.8. **(Sweeping sensors)** Fix a Euclidean target space in $\mathbb{R}^n$ and consider a variant of Problem 3.1 in which sensor nodes do not return merely a count of the number of targets within range, but rather a parameterized count of targets as the sensor performs a 'sweep' over its visual sphere. For example, in dimension 2, a sensor at $x \in \mathbb{R}^2$ returns a piecewise-constant function $h_x : S^1 \to \mathbb{N}$ which indicates how many targets are seen as a function of bearing. Assume, for simplicity, that a sensor at location x with bearing $\boldsymbol{v} \in T^1_x(\mathbb{R}^n)$ scans a compact cone at x centered on $\boldsymbol{v}$ whose aperture and length are independent of x and $\boldsymbol{v}$. The problem is how to compute the target count given the collection of functions $h = \{h_x : x \in X\}$, where $h_x(\boldsymbol{v})$ returns the number but not identity of targets within the cone at $(x, \boldsymbol{v})$.

Note that it is not assumed that the size, shape, or volume of the scanning cone C is known, only that it is a cone. For a very thin scanning cone, intersections with the targets rarely overlap, and $\chi(h_x^{-1}(1))$ yields the correct number of targets within range of x, reducing the problem to that of Problem 3.1. For more general cones, however, sweeping does not return an immediate count at x.

THEOREM 6.9. *Under the assumptions of Problem 6.8 and a general-position assumption on the targets $\{\mathcal{O}_\alpha\}$, the number of targets is equal to*

$$\#\alpha = \int_{\mathbb{R}^n} \Phi_n \left(\int_{T^1_x} h_x \, d\chi(\boldsymbol{v}) \right) d\chi(x), \tag{6.5}$$

where Φ_n is the operator that replaces a function with its upper-continuous (for n even) or lower-continuous (for n odd) extension over 0-dimensional discontinuities.

Proof. The rationale: the inner integral has the effect of aggregating all targets visible at x during a complete sweep over the unit tangent sphere $T^1_x = T^1_x\mathbb{R}^n \cong S^{n-1}$. The outer integral would give $\sum_\alpha \chi(U_\alpha)$ in accordance with Theorem 3.2.

Unfortunately, the target supports U_α are not contractible. Fix a bearing direction and consider the intersection of the target support in $T^1_*\mathbb{R}^n$: it is a contractible set (translated copies of the sensing cone). Hence, each U_α is a bundle over the $(n-1)$-dimensional sphere of tangent directions with compact contractible fiber. This has Euler characteristic $\chi(U_\alpha) = (-1)^{n-1}$. For n even, this vanishes.

If one considers $T^1\mathbb{R}^n$ as a bundle over $\mathbb{R}^n$, the resulting target support is the function $\mathbb{1}_{S_\alpha} - (-1)^n \mathbb{1}_{\mathcal{O}_\alpha}$, where S_α is the set of points $x \in \mathbb{R}^n$ such that there exists a bearing placing the sensing cone at x over $\mathcal{O}_\alpha$. At the single point $\mathcal{O}_\alpha$ this function is equal to $\chi(S^{n-1})$: 'filling in' to a 1 would make the target support a compact contractible set.

The operator Φ_n wipes out these defects by, in effect, gluing in an open unit disk into each U_α, so that the resulting tame set $\tilde{U}_\alpha$ is contractible. By the general position assumption, the defects caused by sensors on top of targets are the only such strata on which Φ_n acts. It remains to note that the operator Φ_n is linear on upper (or lower) semi-continuous functions, and thus acts independently on each singularity. □

Note that it is not necessary to assume the functions h_x have a rigid parametrization — one need not interrogate how many targets are seen at a given bearing. The combinatorial type suffices. In the case of a discrete network of sensors as opposed to a continuum sensor field, one can assume that the sensors are not at the same location as the targets and the defects will be hidden. (Though it should also be noted that in practice a sensor's support region can 'diffuse' somewhat, yielding small codimension-0 defects in the height function.) A discrete network of sweeping sensors actually makes the computation of the integral easier, as one can dispense with the 0-dimensional singularities by a general position argument.

7. A concluding introduction. The goal of this paper is to introduce integration with respect to Euler characteristic as a powerful and computable tool to perform data aggregation in networks of extremely weak sensors. As with many problems in sensor networks, the fundamental issue is the passage from local data to global information: an ideal target for algebraic topology.

This is an introduction to the mathematical techniques, and we have ignored many of the complications present in physical sensor networks and important to implementation. We have not dealt with communication and signal protocol issues, the difficulty of localizing nodes, and the stochastic nature of real sensors. We are optimistic that these topological methods will robustly adapt to many of these issues.

We stress one point: by solving these target enumeration problems via an *integration* theory, we can leverage techniques (*e.g.*, the Fubini theorem) and perspectives (*e.g.*, numerical integration) forcefully. Due to limitations of space, this paper provides a bare introduction to the mechanics and utility of Euler integration. Much more is knowable and known. Future work will contain the following highlights:

1. The integration theory extends from constructible functions $CF(X)$ to definable $\mathbb{R}$-valued functions. Though the corresponding integral operator is not linear (!) it does have an attractive expression in terms of Morse theory.
2. The analogue of Theorem 5.1 holds for the $\mathbb{R}$-valued interpolant h_{PL}.
3. The $\mathbb{R}$-valued theory gives a natural approach to expected target counts, confidence measures on sensor readings, and harmonic extensions of integrands over holes.
4. Euler integration admits a variety of integral transforms which are often invertible [24]. In the context of sensor networks, inverse transforms can be used to localize the targets, based only on counting sensors.

Open questions abound, including the impact of noise, the confidence of a discretized sampling, and the challenge of integrating more complex type of data (such as logical statements).

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