Euler integration over definable functions

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We extend the theory of Euler integration from the class of constructible functions to that of $\mathbb{R}$-valued functions, definable with respect to an o-minimal structure. The corresponding integral operator has some unusual defects (it is not a linear operator); however, it has a compelling Morse-theoretic interpretation. In addition, we show that it is an appropriate setting in which to do numerical analysis of Euler integrals, with applications to incomplete and uncertain data in sensor networks.

motivic integration | sensor networks | euler characteristic

Integration with respect to Euler characteristic is a homomorphism $\int_X \cdot d\chi : CF(X) \to \mathbb{Z}$ from the ring of constructible functions $CF(X)$ ("tame" integer-valued functions on a topological space X) to the integers $\mathbb{Z}$. It is an elegant topological integration theory which uses as a measure the venerable Euler characteristic χ. Euler characteristic, when suitably defined, satisfies the fundamental property of a measure:

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B), \qquad [1]$$

for A and B "tame" subsets of X. We shall extend the theory to $\mathbb{R}$-valued integrands and demonstrate its utility in managing incomplete data in sensor networks.

Constructible integrands

Because the Euler characteristic is only finitely additive, one must continually invoke the word "tame" to ensure that χ is well-defined. The best means by which to do so it via an O-MINIMAL STRUCTURE [21], a sequence $\mathcal{O} = (\mathcal{O}_n)$ of Boolean algebras of subsets of $\mathbb{R}^n$ satisfying a small list of axioms. Elements of $\mathcal{O}$ are called DEFINABLE sets and these are "tame" for purposes of integration theory. Examples of o-minimal structures include (1) piecewise-linear sets; (2) semi-algebraic sets; and (3) globally subanalytic sets. Definable functions between spaces are those whose graphs are in $\mathcal{O}$. For X and Y definable spaces, let $\mathsf{Def}(X, Y)$ denote the class of compactly supported definable functions $h : X \to Y$, and fix as a convention $\mathsf{Def}(X) = \mathsf{Def}(X, \mathbb{R})$. Note that definable functions are not necessarily continuous.

We briefly recall the theory of Euler integration, established as an integration theory in the constructible setting in [13, 18, 19, 22] and anticipated by a combinatorial version in [2, 10, 17]. Fix an o-minimal structure $\mathcal{O}$ on a space X. The geometric Euler characteristic is the function $\chi : \mathcal{O} \to \mathbb{Z}$ which takes a definable set $A \in \mathcal{O}$ to $\chi(A) = \sum_i (-1)^i \dim H_i^{BM}(A; \mathbb{R})$, where H_*^{BM} is the Borel-Moore homology (equivalently, singular compactly supported homology) of A. This also has a combinatorial definition: if A is definably homeomorphic to a finite disjoint union of (open) simplices $\coprod_j \sigma_j$, then $\chi(A) = \sum_j (-1)^{\dim \sigma_j}$. Algebraic topology asserts that χ is independent of the decomposition into simplices. The Mayer-Vietoris principle asserts that χ is a measure on $\mathcal{O}$, as expressed in [1].

Let $CF(X)$ denote the ring of CONSTRUCTIBLE FUNCTIONS: compactly supported $\mathbb{Z}$-valued functions all of whose level sets are definable. The EULER INTEGRAL is the pushforward of the trivial map $X \mapsto \{pt\}$ to $\int_X d\chi : CF(X) \to CF(\{pt\}) \cong \mathbb{Z}$ satisfying $\int_X \mathbf{1}_A \, d\chi = \chi(A)$ for $\mathbf{1}_A$ the characteristic function over a definable set A. From the definitions and a telescoping sum one easily obtains:

$$\int_X h \, d\chi = \sum_{s=-\infty}^{\infty} s\chi\{h = s\} = \sum_{s=0}^{\infty} \chi\{h > s\} - \chi\{h < -s\}. \qquad [2]$$

Because the Euler integral is a pushforward, any definable map $F : X \to Y$ induces $F_* : CF(X) \to CF(Y)$ satisfying $\int_X h \, d\chi = \int_Y F_* h \, d\chi$. Explicitly,

$$F_* h(y) = \int_{F^{-1}(y)} h \, d\chi, \qquad [3]$$

as a manifestation of the Fubini Theorem.

The Euler integral has been found to be an elegant and useful tool for explaining properties of algebraic curves [3] and stratified Morse theory [20, 4], for reconstructing objects in integral geometry [19], for target counting in sensor networks [1], and as an intuitive basis for the more general theory of motivic integration [7, 6].

Real-valued integrands

We extend the definition of Euler integration to integrands in $\mathsf{Def}(X)$.

A Riemann-sum definition.

Definition 1. *Given $h \in \mathsf{Def}(X)$, define:*

$$\int_X h \lfloor d\chi \rfloor = \lim_{n \to \infty} \frac{1}{n} \int_X \lfloor nh \rfloor d\chi. \qquad [4]$$

$$\int_X h \lceil d\chi \rceil = \lim_{n \to \infty} \frac{1}{n} \int_X \lceil nh \rceil d\chi. \qquad [5]$$

We establish that these limits exist and are well-defined, though not equal.

Lemma 1. *Given an affine function $h \in \mathsf{Def}(\sigma)$ on an open k-simplex σ,*

$$\int_\sigma h \lfloor d\chi \rfloor = (-1)^k \inf_\sigma h \, ; \int_\sigma h \lceil d\chi \rceil = (-1)^k \sup_\sigma h. \qquad [6]$$

Proof: For h affine on σ, $\chi\{\lfloor nh \rfloor > s\} = (-1)^k$ for all $s < n \inf_\sigma h$, and 0 otherwise. One computes

$$\lim_{n \to \infty} \frac{1}{n} \int_\sigma \lfloor nh \rfloor d\chi = \lim_{n \to \infty} \frac{1}{n} \sum_{s=0}^{\infty} \chi\{\lfloor nh \rfloor > s\} = (-1)^k \inf_\sigma h.$$

The analogous computation holds with $\chi\{\lceil nh \rceil > s\} = (-1)^k$ for all $s < n \sup_\sigma h$, and 0 otherwise. ■

This integration theory is robust to changes in coordinates.

Lemma 2. *Integration on* $\mathsf{Def}(X)$ *with respect to* $\lfloor d\chi\rfloor$ *and* $\lceil d\chi\rceil$ *is invariant under the right action of definable bijections of* X.

Proof: This is true for Euler integration on $CF(X)$; thus, it holds for $\int_X \lfloor nh\rfloor\, d\chi$ and $\int_X \lceil nh\rceil\, d\chi$. ■

Lemma 3. *The limits in Definition 1 are well-defined.*

Proof: The TRIANGULATION THEOREM for $\mathsf{Def}(X)$ [21] states that to any $h \in \mathsf{Def}(X)$, there is a definable triangulation (a definable bijection to a disjoint union of open affine simplices in some Euclidean space) on which h is affine on each open simplex. The result now follows from Lemmas 1 and 2. ■

Integrals with respect to $\lfloor d\chi\rfloor$ and $\lceil d\chi\rceil$ are related to total variation (in the case of compactly supported continuous functions).

Corollary 1. *If* M *is a 1-dimensional manifold and* $h \in \mathsf{Def}(M)$ *is continuous, then*

$$\int_M h\,\lfloor d\chi\rfloor = -\int_M h\,\lceil d\chi\rceil = \frac{1}{2}\mathsf{totvar}(h). \qquad [7]$$

Proof: Apply Lemma 1 to an affine triangulation of h which triangulates M with the maxima $\{p_i\}$ and minima $\{q_j\}$ as 0-simplices and the intervals between them as 1-simplices. To each minimum q_j is associated two open 1-simplicies, since M is a 1-manifold. Thus:

$$\int_M h\,\lfloor d\chi\rfloor = \sum_i h(p_i) + \sum_j h(q_j) - 2\sum_j h(q_j) = \frac{1}{2}\mathsf{totvar}(h).$$

This equals $-\int_M h\,\lceil d\chi\rceil$ via an analogous computation. ■

One notes that $\lfloor d\chi\rfloor$ and $\lceil d\chi\rceil$ give different though related integrals. They are conjugate in the following sense.

Lemma 4.

$$\int h\lceil d\chi\rceil = -\int -h\lfloor d\chi\rfloor. \qquad [8]$$

Proof: Apply Lemma 1 to an affine triangulation of h, and note that $\sup_\sigma h = -\inf_\sigma -h$. ■

Computation. Definition 1 has a Riemann-sum flavor which extends to certain computational formulae. The following is a definable analogue of [2].

Proposition 2. *For* $h \in \mathsf{Def}(X)$,

$$\int_X h\,\lfloor d\chi\rfloor \;=\; \int_{s=0}^{\infty} \chi\{h \geq s\} - \chi\{h < -s\}\, ds \qquad [9]$$

$$\int_X h\,\lceil d\chi\rceil \;=\; \int_{s=0}^{\infty} \chi\{h > s\} - \chi\{h \leq -s\}\, ds. \qquad [10]$$

Proof: For $h \geq 0$ affine on an open k-simplex σ,

$$\int_\sigma h\,\lfloor d\chi\rfloor = (-1)^k \inf_\sigma h = \int_0^\infty \chi(\sigma \cap \{h \geq s\})ds,$$

and for $h \leq 0$, the equation holds with $-\chi(\sigma \cap \{h < -s\})$. The result for $\int \lceil d\chi\rceil$ follows from Lemma 4. ■

It is not true that $\int_X h\,\lfloor d\chi\rfloor = \int_0^\infty s\chi\{h = s\}ds$: the proper Lebesgue generalization of [2] is the following:

Proposition 3. *For* $h \in \mathsf{Def}(X)$,

$$\int_X h\,\lfloor d\chi\rfloor \;=\; \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_{\mathbb{R}} s\,\chi\{s \leq h < s+\epsilon\}\, ds \qquad [11]$$

$$\int_X h\,\lceil d\chi\rceil \;=\; \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_{\mathbb{R}} s\,\chi\{s < h \leq s+\epsilon\}\, ds. \qquad [12]$$

Proof: For h affine on an open k-simplex σ, and $0 < \epsilon$ sufficiently small, $\int_{\mathbb{R}} s\,\chi\{s \leq h < s+\epsilon\}\, ds = \epsilon\,(-1)^k \left(-\frac{\epsilon}{2} + \inf_\sigma h\right)$ and $\int_{\mathbb{R}} s\,\chi\{s < h \leq s+\epsilon\}\, ds = \epsilon\,(-1)^k \left(-\frac{\epsilon}{2} + \sup_\sigma h\right)$. ■

Morse theory. One important indication that the definition of $\int \lfloor d\chi\rfloor$ is correct for our purposes is the natural relation to Morse theory: the integrals with respect to $\lfloor d\chi\rfloor$ and $\lceil d\chi\rceil$ are Morse index weighted sums of critical values of the integrand. This is a localization result, reducing from an integral over all of X to an integral over an often discrete set of critical points.

Recall that a C^2 function $h : M \to \mathbb{R}$ on a smooth manifold M is MORSE if all critical points of h are nondegenerate, in the sense of having a nondegenerate Hessian matrix of second partial derivatives. Denote by $\mathcal{C}(h)$ the set of critical points of h. For each $p \in \mathcal{C}(h)$, the MORSE INDEX of p, $\mu(p)$, is defined as the number of negative eigenvalues of the Hessian at p, or, equivalently, the dimension of the unstable manifold $W^u(p)$ of the vector field $-\nabla h$ at p.

Stratified Morse theory [9] is a powerful generalization to triangulable spaces, including definable sets with respect to an o-minimal structure [4, 20]. We may interpret $\lfloor d\chi\rfloor$ and $\lceil d\chi\rceil$ in terms of the weighted stratified Morse index of the graph of the integrand.

Definition 2. *For* $X \subset \mathbb{R}^n$ *definable and* $h \in \mathsf{Def}(X)$, *define the co-index of* h, $\mathcal{I}^*h$ *to be the stratified Morse index of the graph of* h, $\Gamma_h \subset X\times\mathbb{R}$, *with respect to the projection* $\pi : X\times\mathbb{R} \to \mathbb{R}$:

$$(\mathcal{I}^*h)(x) = \lim_{\epsilon' \ll \epsilon \to 0^+} \chi\left(\overline{B_\epsilon(x)} \cap \{h < h(x) + \epsilon'\}\right), \qquad [13]$$

where $\overline{B_\epsilon(x)}$ *is the closed ball of radius* ϵ *about* $x \in X$. *The index* $\mathcal{I}_*$ *is the stratified Morse index with respect to height function* $-\pi$ *and is defined similarly using* $\{h > h(x) - \epsilon'\}$ *in Eqn. [13].*

Note that $\mathcal{I}_*, \mathcal{I}^* : \mathsf{Def}(X) \to CF(\overline{X})$, and the restriction of these operators to $CF(X)$ is the identity (every point of a constructible function is a critical point). The two types of integration on $\mathsf{Def}(X)$ correspond to the Morse indices of the graph with respect to the two orientations of the graph axis — the projections π and $-\pi$.

Theorem 4. *For any continuous* $h \in \mathsf{Def}(X)$,

$$\int_X h\,\lfloor d\chi\rfloor = \int_{\overline{X}} h\mathcal{I}^*h\, d\chi \quad ; \quad \int_X h\,\lceil d\chi\rceil = \int_{\overline{X}} h\mathcal{I}_*h\, d\chi. \qquad [14]$$

Proof: On an open k-simplex $\sigma \subset X \subset \mathbb{R}^n$ in an affine triangulation of h, the co-index $\mathcal{I}^*h$ equals $(-1)^{\dim(\sigma)}$ times the characteristic function of the closed face of σ determined by $\inf_\sigma h$. Since h is continuous, $\int_{\overline{\sigma}} h\mathcal{I}^*h\, d\chi = (-1)^{\dim(\sigma)} \inf_\sigma h$. Lemma 1 and additivity completes the proof; the analogous proof holds for $\mathcal{I}_*$ and $\lceil d\chi\rceil$. ■

Corollary 5. *If* h *is a Morse function on a closed* n*-manifold* M, *then*

$$\int_M h\,\lfloor d\chi\rfloor = \sum_{p\in\mathcal{C}(h)} (-1)^{n-\mu(p)} h(p) \qquad [15]$$

$$\int_M h\,\lceil d\chi\rceil = \sum_{p\in\mathcal{C}(h)} (-1)^{\mu(p)} h(p) = (-1)^n \int_M h\,\lfloor d\chi\rfloor. \qquad [16]$$

Proof: For p a nondegenerate critical point on an n-manifold, $\mathcal{I}^*(p) = (-1)^{n-\mu(p)}$ and $\mathcal{I}_*(p) = (-1)^{\mu(p)}$. ■

Corollary 1 on total variation thus generalizes in a fundamental manner. Corollary 5 can also be proved directly using classical handle-addition techniques or in terms of the Morse complex, using the fact that the restriction of the integrand to each unstable manifold of each critical point is unimodal with a unique maximum at the critical point. It is also possible to express the stratified Morse index — and thus the integral here considered — in terms of integration against a characteristic cycle, *cf.* [9, 20].

The generalization from continuous to general definable integrands is simple, but requires, weighting $\mathcal{I}^*h$ by h directly. To compute $\int_X h\lfloor d\chi\rfloor$, one integrates the weighted co-index

$$\lim_{\epsilon'\ll\epsilon\to 0^+} h(x+\epsilon')\chi\left(\overline{B_\epsilon(x)}\cap\{h<h(x)+\epsilon'\}\right) \qquad [17]$$

with respect to $d\chi$.

The integral operator

We consider properties of the integral operator(s) on $\mathsf{Def}(X)$.

Linearity. One is tempted to apply all the standard constructions of sheaf theory (as in [18, 19]) to $\int_X : \mathsf{Def}(X) \to \mathbb{R}$. However, there are some serious complications in the $\mathbb{R}$-valued theory. Our formulation of the integral on $\mathsf{Def}(X)$ has the glaring disadvantage that $\int_X$ is no longer a homomorphism: *e.g.*,

$$1=\int_{[0,1]} 1\,\lfloor d\chi\rfloor \neq \int_{[0,1]} x\,\lfloor d\chi\rfloor + \int_{[0,1]}(1-x)\,\lfloor d\chi\rfloor = 1+1=2.$$

This loss of functoriality can be seen as due to the fact that $\lfloor f+g\rfloor$ agrees with $\lfloor f\rfloor+\lfloor g\rfloor$ only up to a set of Lebesgue measure zero, not χ-measure zero. The nonlinear nature of the integral is also clear from the Morse formulation in Eqn. [14].

The Fubini Theorem. In one sense, the change of variables formula trivializes (Lemma 2). The more general change of variables formula encapsulated in the Fubini theorem does not, however, hold for non-constructible integrands. Let $F : X = Y \coprod Y \to Y$ be the projection map with fibers $\{p\}\coprod\{p\}$. Any $h \in \mathsf{Def}(X)$ is expressible as $f\coprod g$ for $f, g \in \mathsf{Def}(Y)$. The Fubini theorem applied to F is equivalent to the statement

$$\int_Y f + \int_Y g = \int_X h = \int_Y F_*h = \int_Y f+g,$$

where the integration is with respect to $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$ as desired. As noted above, this fails for certain f, g. There are, however, conditions under which Fubini holds.

Theorem 6. *For $h \in \mathsf{Def}(X)$, let $F : X \to Y$ be definable and h-preserving (h is constant on fibers of F). Then $\int_Y F_*h\lfloor d\chi\rfloor = \int_X h\lfloor d\chi\rfloor$, and $\int_Y F_*h\lceil d\chi\rceil = \int_X h\lceil d\chi\rceil$.*

Proof: An application of the o-minimal Hardt theorem [21] implies that Y has a partition into definable sets Y_α such that $F^{-1}(Y_\alpha)$ is definably homeomorphic to $U_\alpha \times Y_\alpha$ for U_α definable, and that $F : U \times Y_\alpha \to Y_\alpha$ acts via projection. Since h is constant on fibers of F, one computes

$$\int_{Y_\alpha} F_*h\lfloor d\chi\rfloor = \int_{Y_\alpha} h\,\chi(U_\alpha)\lfloor d\chi\rfloor = \int_{U_\alpha\times Y_\alpha} h\lfloor d\chi\rfloor.$$

The same holds for $\int\lceil d\chi\rceil$. ■

Corollary 7. *For $h \in \mathsf{Def}(X)$, $\int_X h = \int_{\mathbb{R}} h_*h$. In other words,*

$$\int_X h\,\lfloor d\chi\rfloor = \int_{\mathbb{R}} s\,\chi\{h=s\}\lfloor d\chi\rfloor, \qquad [18]$$

and likewise for $\lceil d\chi\rceil$.

Continuity. Though the integral operator is not linear on $\mathsf{Def}(X)$, it does retain some nice properties. All properties below stated for $\int\lfloor d\chi\rfloor$ hold for $\int\lceil d\chi\rceil$ via duality.

Lemma 5. *The integral $\int\lfloor d\chi\rfloor : \mathsf{Def}(X) \to \mathbb{R}$ is positively homogeneous.*

Proof: For $f \in \mathsf{Def}(X)$ and $\lambda \in \mathbb{R}^+$, the change of variables variables $s \mapsto \lambda s$ in [9] gives $\int \lambda f\,\lfloor d\chi\rfloor = \lambda\int f\,\lfloor d\chi\rfloor$. ■

Integration is not continuous on $\mathsf{Def}(X)$ with respect to the C^0 topology. An arbitrarily large change in $\int h\lfloor d\chi\rfloor$ may be effected by small changes to h on a (large) finite point set. However, integration does itself define a natural "L^1"-topology on $\mathsf{Def}(X)$ on which integration is continuous.

Definition 3. *The L^1 ϵ-neighborhood of $h \in \mathsf{Def}(X)$ is the intersection of the C^0 ϵ-neighborhood (definable functions with ϵ-close graphs) with those functions $g \in \mathsf{Def}(X)$ satisfying $\int_X f-g\,\lfloor d\chi\rfloor < \epsilon$.*

This provides a basis for what we call the L^1 topology on $\mathsf{Def}(X)$. As a consequence of Lemma 4, the definition is independent of the use of $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$. The interested reader can speculate on other function space topologies on $\mathsf{Def}(X)$ defined using $\int\cdot\lfloor d\chi\rfloor$.

Duality and links. There is an integral transform on $CF(X)$ that is the analogue of Poincaré-Verdier duality [20]. It extends seamlessly to integrals on $\mathsf{Def}(X)$ by means of the following definition.

Definition 4. *The* DUALITY OPERATOR *is the integral transform $\mathcal{D} : CF(X) \to CF(X)$ given by*

$$\mathcal{D}h(x) = \lim_{\epsilon\to 0^+}\int_X h\mathbb{1}_{B_\epsilon(x)}d\chi. \qquad [19]$$

We extend the definition to $\mathcal{D} : \mathsf{Def}(X) \to \mathsf{Def}(X)$ by integrating with respect to $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$.

We do not use different symbols for the $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$ versions of the dual, thanks to the following:

Lemma 6. *$\mathcal{D}h$ is well-defined on $\mathsf{Def}(X)$ and independent of whether the integration in (19) is with respect to $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$.*

Proof: The limit is always well-defined thanks to the Conic Theorem in o-minimal geometry [21]. To show that it is independent of the upper- or lower-semicontinuous approximation, take $\epsilon > 0$ sufficiently small. Note that by triangulation, h can be assumed to be piecewise-affine on open simplices. Pick a point x in the support of h and let $\{\sigma_i\}$ be the set of open simplices whose closures contain x. Then for each i, the limit $h_i(x) := \lim_{y\to x} h(y)$ for $y \in \sigma_i$ exists. Then, applying [19], one computes

$$\mathcal{D}h(x) = \lim_{\epsilon\to 0^+}\sum_i (-1)^{\dim\sigma_i}h_i(x), \qquad [20]$$

independent of the measure $\lfloor d\chi\rfloor$ or $\lceil d\chi\rceil$. ■

For a continuous definable function h on a manifold M, $\mathcal{D}h = (-1)^{\dim M}h$, as one can verify by combining Eqns. [9] and [19]. This is commensurate with the result of Schapira [18] that $\mathcal{D}$ is an involution on $CF(X)$.

Theorem 8. *Duality is involutive on $\mathsf{Def}(X)$: $\mathcal{D}\circ\mathcal{D}h = h$.*

Proof: Given h, fix a triangulation on which h is affine on open simplices. From [20], we see that the dual of h at x is completely determined by the 'linearization' of h at x. Let $\tilde{h}$ be the constructible function on $B_\epsilon(x)$ which takes on the value $h_i(x)$ on strata $\sigma_i \cap B_\epsilon(x)$. (Though this is not necessarily an integer-valued function, the range is discrete and therefore is constructible.) Then $\mathcal{D}h(x) = \mathcal{D}\tilde{h}(x)$ and $\mathcal{D}^2h(x) = \mathcal{D}^2\tilde{h}(x) = \tilde{h}(x) = h(x)$. ■

One can define related integral transforms. For example, the LINK of $h \in CF(X)$ is defined as

$$\Lambda h(x) = \lim_{\epsilon\to 0^+}\int_X h\mathbb{1}_{\partial B_\epsilon(x)}d\chi. \qquad [21]$$

The link of a continuous function on an n-manifold M is multiplication by $1+(-1)^n$, as a simple computation shows. In general, $\Lambda = \mathsf{Id}-\mathcal{D}$, where Id is the identity operator.

Convolution. On a vector space V (or Lie group, more generally), a convolution operator with respect to Euler characteristic is straightforward. Given $f, g \in CF(V)$, one defines

$$(f * g)(x) = \int_V f(t)g(x - t)\, d\chi. \qquad [22]$$

Convolution behaves as expected in $CF(X)$. By reversing the order of integration, one has immediately that $\int_V f * g\, d\chi = \int_V f\, d\chi \int_V g\, d\chi$. There is a close relationship between convolution and the Minkowski sum, as observed in, *e.g.*, [10]: for A and B convex, $\mathbf{1}_A * \mathbf{1}_B = \mathbf{1}_{A+B}$, cf. [22, 18]. Convolution is a commutative, associative operator providing $CF(V)$ with the structure of an (interesting [3]) algebra.

Convolution is well-defined on $\mathsf{Def}(V)$. However, the product formula for $\int f * g$ fails, since one relies on the Fubini theorem to prove it in $CF(V)$.

Numerical integration

The mélange of combinatorial, analytic, and homological features of $\lfloor d\chi \rfloor$ and $\lceil d\chi \rceil$ permits a wealth of computational formulae. The following result is a simple generalization of an argument in [1] in the constructible category.

Proposition 9. *Let $h \in \mathsf{Def}(\mathbb{R}^2)$. Then $\int h \lfloor d\chi \rfloor$ equals*

$$\int_{s=0}^{\infty} \beta_0\{h \geq s\} + \beta_0\{h \geq -s\} - \beta_0\{h < s\} - \beta_0\{h < -s\}\, ds, \qquad [23]$$

where $\beta_0(\cdot)$ denotes the zeroth Betti number, the rank of $H_0(\cdot; \mathbb{R})$.

Proof: Apply the homological definition of χ to [9]; then, use Alexander duality in the plane to reduce all terms to β_0 quantities. ■

The value of Proposition 9 is that it allows for computation based on β_0 quantities. Such connectivity data are easily obtained via clustering algorithms, even from a discrete sampling. The Lebesgue integral is easily discretized and well-behaved; we have implemented this formula in software.

Refinement. The utility of Euler integration compels questions of efficient, accurate computation given a discrete sampling of the integrand. The continuity of the integral operator implies the following.

Theorem 10. *For $h \in \mathsf{Def}(X)$ continuous, let h_{PL} be the piecewise-linear function obtained from sampling h on the vertex set of a triangulation $\mathcal{T}$ of X. As the sampling and triangulation are refined,*

$$\lim_{|\mathcal{T}| \to 0^+} \int_X h_{PL} \lfloor d\chi \rfloor = \int_X h \lfloor d\chi \rfloor. \qquad [24]$$

What one really desires, however, is a measure of how far a given sampling is from the true integral. This seems to be a challenging problem.

Toward applications

The Euler calculus on CF is quite useful; the extension to Def deepens these and opens new potential applications. We highlight a few below, omitting details for the time being.

Sensor networks. The application of Euler integration over $CF(X)$ to sensor networks problems was initiated in [1]. Consider a space X whose points represent target-counting sensors that scan a workspace W. Target detection is encoded in a SENSING RELATION $\mathcal{S} \subset W \times X$ where $(w, x) \in \mathcal{S}$ iff a target at w is detected by a sensor at x. Assume that sensors count the number of sensed targets, but do not locate or identify the targets. The sensor network therefore induces a TARGET COUNTING FUNCTION $h : X \to \mathbb{N}$ of the form $h = \sum_\alpha \mathbf{1}_{U_\alpha}$, where U_α is the TARGET SUPPORT — the set of sensors which detect target α. Euler integration allows for simple enumeration:

Theorem 11. ([1]) *Assume $h \in CF(X)$ and $\chi(U_\alpha) = N \neq 0$ for all α. Then the number of targets in W is precisely $\frac{1}{N} \int_X h\, d\chi$.*

Since the target count is presented as an integral, it is possible to accurately estimate the answer when the integrand h is known not on all of X (a continuum of sensors being highly unrealistic in practice) but rather on a sufficiently dense grid of sample points (physical sensors in a network).

The $\mathbb{R}$-valued theory aids in establishing expected values of target counts in the presence of confidence measures on sensor readings. Let $\mathcal{N} = \{x_i\}$ denote a discrete set of sensor nodes in $\mathbb{R}^n$, and assume each sensor returns a target count $h(x_i) \in \mathbb{N}$ and a fluctuation measure $c(x_i) \in [0, 1]$ obtained, say, by stability of the reading over a time average. View h as a sampling over $\mathcal{N}$ of the true target count $f = \sum_\alpha \mathbf{1}_{U_\alpha}$. Assume that nodes with fluctuation reading 0 have perfect information ($h = f$ at x_i) and that c correlates with error $|f - h|$. Assume that sensor nodes $\mathcal{N}$ are part of a network whose edges $\mathcal{E}$ are based roughly on proximity.

The integral of an extension of f over a triangulation gives a terrible approximation to $\int h\, d\chi$: an error of ± 1 on K nodes can cause a change in the integral of order K. More specifically, if $h = f + e$, where $e : \mathcal{N} \to \{-1, 0, 1\}$ is an error function that is nonzero on a sparse subset $\mathcal{N}' \subset \mathcal{N}$, then, for certain infelicitous choices of $\mathcal{N}'$, $\left|\int h - \int f\right| = |\mathcal{N}'|$.

A $\mathbb{R}$-valued (and in particular, a harmonic) relaxation can mitigate errors by using fluctuation c as a weight. Let $\tilde{h}$ be the result of averaging the value at $x_i \in \mathcal{N}$ over all neighbors, with c as a weight. Specifically,

$$\tilde{h}(x_i) = \frac{\sum_j c(x_j) h(x_j)}{\sum_j c(x_j)}, \qquad [25]$$

where the sums are over all j such that x_j is no more than one edge away from x_i. Since averaging damps out local variations, the resulting integral will tend to mitigate point-errors, thanks to the Morse-theoretic formula.

Such averaging naturally leads to non integer-valued integrands. By using integration with respect to $\lfloor d\chi \rfloor$ or $\lceil d\chi \rceil$ for upper/lower semi-continuous integrands associated to such an averaged signal $\tilde{h}$, one obtains an expected value of $\int h\, d\chi$. This can be particularly illuminating when a network has incomplete information, *e.g.*, a hole.

Statistics and mode counting. The previous application lends itself to more general statistical ends. Consider a distribution $f : X \to [0, \infty)$ of compact support and bounded variation. The statistical problem of mode-counting — of decomposing f into a convex combination of unimodal summands — bears no small resemblance to the problem of target enumeration.

Indeed, the nonlinearity of the integral operator with respect to $\lfloor d\chi \rfloor$ and $\lceil d\chi \rceil$ mirrors the nonlinear interaction of unimodal summands in a distribution. Just as two modes can interfere, creating an artificial local maximum when an increasing and a decreasing portion of the modes are summed, the Euler integral over Def loses linearity when increasing and decreasing integrands are combined.

The problem of mode-counting leads naturally to an extension of the Lyusternik-Schnierlmann category from sets to continuous definable functions [**?**]. Euler integration with continuous integrands thus relates to this unimodal Lyusternik-

Schnirelmann category and yields bounds on this topological invariant.

Integral transforms. Integration with respect to Euler characteristic over $CF(X)$ has a well-defined and well-studied class of integral transforms, expressed beautifully in Schapira's work on inversion formulae for the generalized Radon transform in $d\chi$ [19]. Integral transforms with respect to $\lfloor d\chi \rfloor$ and $\lceil d\chi \rceil$ are similarly appealing, with applications to signal processing as a primary motivation. Inversion formulae for, *e.g.*, convolution involve the duality operator of §.

The nonlinearily of the integration operator prevents most straightforward applications of inversion formulae à la Schapira. It is an interesting open problem to adapt current techniques from inverse problems and integral transforms to integrals with respect to Euler characteristic.

Hadwiger measures. Integration with respect to Euler characteristic is the 'zeroth-order' Hadwiger measure from integral geometry; there is one such measure on tame sets of $\mathbb{R}^n$ for each $j = 0, \ldots, n$ the n^{th}-order measure being Lebesgue. These are related to curvature measures [3, 8] and are of contemporary use in, among other things, coarsening of three-dimensional microstructures in crystals [14].

The Hadwiger measures can be defined in terms of Euler measure over CF via standard integral-geometric formulae. It would appear to be a simple matter to modify these definitions to accommodate the Hadwiger measures from definable sets to definable functionals. Connections between these definable Hadwiger measures and Morse-theoretic properties of the integrands are particularly enticing.

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1. Y. Baryshnikov and R. Ghrist, "Target enumeration via Euler characteristic integration," SIAM J. Appl. Math., in press.
2. W. Blaschke, Vorlesungen über Integralgeometrie, Berlin, 1955.
3. L. Bröcker, "Euler integration and Euler multiplication," Adv. Geom. 5(1), 2005, 145–169.
4. L. Bröcker and M. Kuppe, "Integral geometry of tame sets," Geom. Dedicata 82, 2000, 285–323.
5. B. Chen, "On the Euler measure of finite unions of convex sets," Discrete and Computational Geometry 10, 1993, 79–93.
6. R. Cluckers and M. Edmundo, "Integration of positive constructible functions against Euler characteristic and dimension," J. Pure Appl. Algebra, 208, no. 2, 2006, 691–698.
7. R. Cluckers and F. Loeser, "Constructible motivic functions and motivic integration," Invent. Math., 173(1), 2008, 23–121.
8. J. Fu, "Curvature measures of subanalytic sets," Amer. J. Math. 116, 1994, 819-880.
9. M. Goresky and R. MacPherson, Stratified Morse Theory, Springer-Verlag, Berlin, 1988.
10. H. Groemer, "Minkowski addition and mixed volumes," Geom. Dedicata 6, 1977, 141–163.
11. H. Hadwiger, "Integralsätze im Konvexring," Abh. Math. Sem. Hamburg, 20, 1956, 136–154.
12. D. Klain, K. Rybnikov, K. Daniels, B. Jones, C. Neacsu, "Estimation of Euler Characteristic from Point Data," preprint 2006.
13. R. MacPherson, "Chern classes for singular algebraic varieties," Ann. of Math. 100, 1974, 423-432.
14. R. MacPherson and D. Srolovitz, "The von Neumann relation generalized to coarsening of three-dimensional microstructures," Nature 446, 1053-1055, 26 April 2007.
15. C. McCrory and A. Parusiński, "Algebraically constructible functions: real algebra and topology," Proc. RAAG'01, Univ. Rennes and CNRS, 2001.
16. J. Milnor, Morse Theory, Princeton University Press, 1963.
17. G.-C. Rota, "On the combinatorics of the Euler characteristic," Studies in Pure Mathematics, Academic Press, London, 1971, 221–233.
18. P. Schapira, "Operations on constructible functions," J. Pure Appl. Algebra 72, 1991, 83–93.
19. P. Schapira, "Tomography of constructible functions," in proceedings of 11th Intl. Symp. on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, 1995, 427–435.
20. J. Schürmann, Topology of Singular Spaces and Constructible Sheaves, Birkhäuser, 2003.
21. L. Van den Dries, Tame Topology and O-Minimal Structures, Cambridge University Press, 1998.
22. O. Viro, "Some integral calculus based on Euler characteristic," Lecture Notes in Math., vol. 1346, Springer-Verlag, 1988, 127–138.