CONFIGURATION SPACES AND BRAID GROUPS ON GRAPHS IN ROBOTICS

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Dedicated to Joan Birman on her 70^{th} *birthday.*

Abstract

Configuration spaces of distinct labeled points on the plane are of practical relevance in designing safe control schemes for Automated Guided Vehicles (robots) in industrial settings. In this announcement, we consider the problem of the construction and classification of configuration spaces for graphs. Topological data associated to these spaces (*e.g.*, dimension, braid groups) provide an effective measure of the complexity of the control problem. The spaces are themselves topologically interesting objects. We show that they are $K(\pi_1, 1)$ spaces whose homological dimension is bounded by the number of essential vertices. Hence, the braid groups are torsion-free.

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1 Configuration Spaces in Manufacturing

1.1 Background

In several manufacturing and industrial settings, the following scenario arises: there is a collection of independent mobile Automated Guided Vehicles (*aka* AGVs) which traverse a factory floor replete with obstacles en route to a goal position (say, a loading dock or an assembly workstation), from whence the process iterates. For these applications, it is of the utmost importance to design a control scheme which insures that (1) the AGVs not collide with the obstacles; (2) the AGVs not collide with each other; (3) the AGVs complete the assigned task with a certain efficiency with respect to various work functionals.

In practice, control schemes are often effected by employing high factor-of-safety algorithms which guarantee safety but reduce efficiency. For example, one partitions the factory floor into "zones" and then all algorithms are written such that only one

AGV is allowed in a zone at any given time [Cas91, BS91]. Clearly, such practices are not optimally efficient.

Within the past decade, the idea of using abstract configuration spaces to model the workspace and then fabricating a control scheme on this topological space, has filtered through the robotics community [Lat91, KR90]. Surprisingly, topologists have been generally unaware of and uninvolved in this important development.

The technique is very straightforward: assume that the individual AGVs are represented as points on the workspace floor X. The set $\mathcal{O} \subset X$ represents those obstacles which are to be avoided. The configuration space $\mathcal{F}$ of N noncolliding labeled AGVs in the workspace is thus:

$$\mathcal{F} := [(X - \mathcal{O}) \times (X - \mathcal{O}) \cdots \times (X - \mathcal{O})] - \Delta, \qquad (1)$$

where Δ denotes the pairwise diagonal $\Delta := \{(x_1, x_2, \ldots, x_N : x_i = x_j \text{ for some } i \neq j\}$. A point in $\mathcal{F}$ thus represents a "safe" configuration of AGVs. One may also consider the unlabeled configuration space (the quotient of $\mathcal{F}^N$ by the action of the symmetric group S_N): the results of this announcement carry over to these spaces as well.

Let $g \in \mathcal{F}$ denote a desired goal configuration. In order to enact a control scheme which realizes this goal safely, it is sufficient to build a vector field X_g on $\mathcal{F}$ which (1) has g as a sink with a large basin of attraction; and (2) is transversally inward on the boundaries left from removing the diagonal and the obstacles. By evolving initial states with respect to this control field, even initial configurations which are near a collision are immediately repulsed onto a 'safe' pathway.

This topological/dynamical formulation has several advantages:

1. There are no *ad hoc* restrictions on how many AGVs can inhabit a subdomain of the workspace;

2. There is an analytical measure of *safety* — the distance to the boundary of $\mathcal{F}$ in the natural (product) metric inherited from the workspace — which allows for rigorous treatment of this issue; and

3. The control field can be designed to be *stable* in the sense that, when the basin of attraction of g is very large (it often can be a submanifold of full measure), arbitrarily large perturbations in the state of the system (*e.g.*, momentary failure of a steering component, slippage, the effect of debris on the floor, etc.) do not affect the attainment of the goal state.

The robotics community has effectively employed these ideas into real control schemes which work in certain industrial settings: see [Lat91] for an introduction to and description of various techniques.

It is by no means true, however, that this problem is completely solved. In the case where the configuration space is a manifold, the existence of a vector field which realizes the desired goal and is repulsive along the unsafe boundaries follows from a simple application of Morse theory (*cf.* [KR90]). The specification of such a vector field is of course a more difficult problem which requires detailed knowledge of the configuration space (at least on the level of charts), though general formulae are available for some specific instances [KR90].

1.2 Graphs and Guidepath Networks

In this paper, we initiate the use of configuration space methods in an industrial setting which has heretofore resisted analysis.

The setting we describe above, in which the automated guided vehicles have a full two degree-of-freedom autonomy of planar motion, assumes a certain level of sophistication in the machinery. A more elementary (and hence, inexpensive and easier to install and maintain) system involves AGVs which are constrained to a network of guidepath wires, embedded either in the floor as tracks or suspended from the ceiling as wires [Cas91]. Such systems are currently in use in many industrial settings.

The problem of maneuvering AGV's on a graph is completely different from that of a full two-degree-of-freedom planar system. On the plane, two AGV's on a collision course may avoid disaster by "swerving" around each other at the last minute: not so on a 1-d graph. Here, the problem is much more global — the control scheme requires information about the global structure of the graph as a local perturbation is usually insufficient. Hence, it would appear that a topological approach could be of the greatest possible benefit.

We commence our investigation with the simplest nontrivial example: that of a pair of points on a 'Y-graph' — the tree Γ_Y having three edges $\{e_i\}_1^3$ meeting at a single 3-valent vertex v_0 as in Figure 1[left]. Let $\mathcal{F}^2(\Gamma_Y)$ denote the configuration space of two points on Γ_Y.

Lemma 1.1 *The configuration space $\mathcal{F}^2(\Gamma_Y)$ is homeomorphic to the embedded 2-complex illustrated in Figure 1[right].*

Proof: Let x and y denote the distinct points on Γ_y and let $D \subset \mathcal{F}^2(\Gamma_Y)$ denote

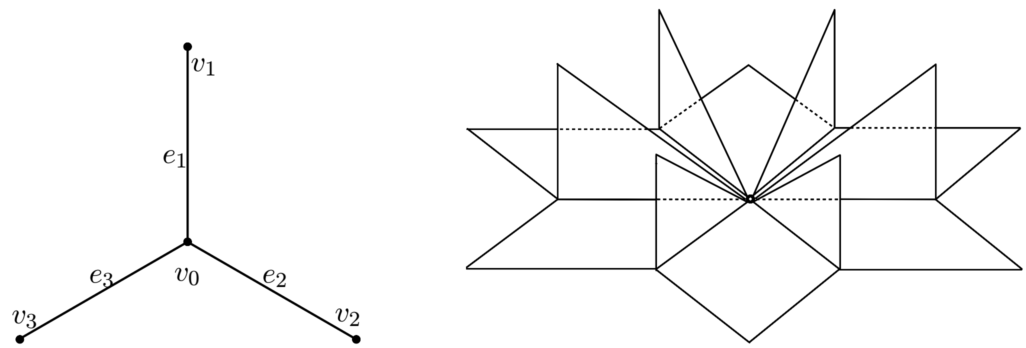

FIGURE 1: The Y-graph Γ_Y [left] has a configuration space $\mathcal{F}^2(\Gamma_Y)$ [right] which embeds in $\mathbb{R}^3$.

the region where x and y lie on distinct edges of Γ_Y. Then $\overline{D}$, the closure of D within $\mathcal{F}^2(\Gamma_Y)$ is easily seen to be a 2-manifold with boundary as follows: if x and y lie on distinct edges, the point (x, y) has a product neighborhood; if (say) x lies on the central vertex, then y lies within one of the three edges, and a neighborhood within $\overline{D}$ allows x to move onto the other two edges, these two edges together being homeomorphic to an interval. As there are two points and three intervals, $\overline{D}$ has a natural decomposition into six cells, each homeomorphic to $([0,1] \times [0,1]) \setminus \{(0,0)\}$ and joined along the edges $(0,1] \times \{0\}$ and $\{0\} \times (0,1]$ in a cyclic fashion. An Euler characteristic computation combined with keeping track of the boundary reveals that $\overline{D}$ is homeomorphic to a punctured disc. The complement of $\overline{D}$ in $\mathcal{F}^2(\Gamma_Y)$ consists of six "fins," each homeomorphic to $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y < 1\}$. These fins are attached to $\overline{D}$ along the edges $\{0\} \times (0,1]$ and $(0,1] \times \{0\}$ in each of the cells. ◇

Note that this space has a product decomposition as the Cartesian product of $(0,1]$ with the graph given by attaching six radial edges to a circle. The $(0,1]$ factor is a height function on $\mathcal{F}^2(\Gamma)$ which measures the distance between x and y.

It is worthwhile to contemplate this example for hints as to what the general case of N points on a tree might reveal. Note that the braid group of this graph is especially simple ($\mathbb{Z}$) and that the space itself deformation retracts onto a graph. For an analysis of the dynamics and control of vector fields on this space, see [GK97, GK98].

After deriving our main results in the next sections, we conclude with several additional examples. The interested reader may wish to reference these as necessary for intuition.

2 Configuration spaces of trees

For the entirety of this section, Υ will denote a tree having distinguished vertex p. Let V denote the number of essential vertices in Υ — that is, the number of vertices of valency greater than two. The configuration space of N distinct points on Υ is denoted $\mathcal{F}^N(\Upsilon)$. This space can be analyzed by considering the codimension-one subset

$$\Sigma := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon) : x_n = p \text{ for some } n\}, \tag{2}$$

which splits into N disjoint (and disconnected) components

$$\Sigma_n := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon) : x_n = p\}. \tag{3}$$

Assume that at p there are $K > 2$ incident edges in Υ. Then

$$\Sigma \cong \coprod_{n=1}^{N} \left(\coprod_{|j|=N-1} \mathcal{F}^{j_1}(\Upsilon_1) \times \cdots \times \mathcal{F}^{j_K}(\Upsilon_K) \right), \tag{4}$$

where the Υ_i are the connected components of $\Upsilon - \{p\}$. The complement of Σ has the form

$$\mathcal{F}^N(\Upsilon) - \Sigma \cong \coprod \left(\coprod_{|j|=N} \mathcal{F}^{j_1}(\Upsilon_1) \times \cdots \times \mathcal{F}^{j_K}(\Upsilon_K) \right). \tag{5}$$

Using the subsets Σ_n to decompose $\mathcal{F}^N(\Upsilon)$ provides the basis for the induction arguments which fill the remainder of this announcement.

Consider the case where Υ is a tree having a vertex p on its boundary. Let e denote the unique edge connecting p to an essential vertex q of Υ and denote by Υ' the subtree given by $\Upsilon' := \overline{\Upsilon - \overline{e}}$. That is, Υ' contains the point q, but neither e nor p. Next define Σ'_n to be the set of configurations on Υ' which have the point x_n at q; hence, $\Sigma'_n := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon') : x_n = q\}$. Define also the "end", $\mathcal{E}_n := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon) : x_n = p\}$. Note that $\mathcal{E}_n$ is homeomorphic to $\mathcal{F}^{N-1}(\Upsilon)$ (after removing some inessential boundary points).

Lemma 2.1 *The space $\mathcal{F}^N(\Upsilon)$ is homeomorphic to*

$$\mathcal{F}^N(\Upsilon) \cong \mathcal{F}^N(\Upsilon') \bigcup_{\Sigma'_n}^{n=1..N} \{(\mathcal{E}_n \times (0,1]) \cup (\Sigma'_n \times \{0\})\}, \tag{6}$$

where the spaces $(\mathcal{E}_n \times (0,1]) \cup (\Sigma'_n \times \{0\})$ are glued to $\mathcal{F}^N(\Upsilon')$ along $\Sigma'_n \times \{0\}$.

Proof: The key observation is the following: one can decompose $\mathcal{F}^N(\Upsilon)$ into those configurations in which all points lie on Υ' and those configurations in which the point x_n lies on e and is the farthest such point from q: *i.e.*, no point lies on the interval of e from x_n to p. The set of configurations in which x_n is at a fixed point on e (and the farthest such point from q) is homeomorphic to a copy of $\mathcal{E}_n$, parameterized by the distance along e. This set $\mathcal{E}_n \times (0,1]$ is glued to $\mathcal{F}^N(\Upsilon')$ along the subset of $\Sigma'_n \times \{0\}$, since no other point may occupy the vertex q. ◇

Using this decomposition, we proceed with the following fundamental step.

Lemma 2.2 *For any tree Υ and any vertex p, the inclusion of Σ into $\mathcal{F}^N(\Upsilon)$ is π_1-injective.*[1]

Proof: We induct on the number of essential vertices V, letting Σ denote the codimension-one subcomplex of configurations for which there is a point at the V^{th} vertex p. For $V = 1$, the inclusion map is π_1-injective since every connected component of Σ is contractible.

Let γ denote a representative of $\pi_1(\Sigma)$ which bounds a disc $D \subset \mathcal{F}^N(\Upsilon)$. By taking D in general position with respect to Σ and ordering the connected components of $\Sigma \cap D$ with respect to inclusion, we may assume without a loss of generality that the interior of D lies in a connected component of the complement of Σ. From Equation (5) it follows that D lies in the product of configuration spaces on graphs with strictly fewer essential vertices. However, since $D \cap \Sigma = \gamma$, the point x_n is located at p along γ, and no other points ever occupy Σ on D; hence, x_n moves within some subtree of $\Upsilon - \{p\}$. Thus, it suffices to consider the specialized case where Υ is a tree and p is a vertex on the boundary of the tree. Let Σ_n denote the set of configurations of N distinct points on Υ for which the point x_n is at p. Assume that γ is a representative of $\pi_1(\Sigma_n)$ which bounds a disc $D \subset \mathcal{F}^N(\Upsilon)$.

Let e denote the unique edge which connects an essential vertex $q \in \Upsilon$ to the boundary point p. Denote by Υ' the subgraph $\overline{\Upsilon - e}$. Then define the subset $\Sigma'_n := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon') : x_n = q\}$. From Lemma 2.1 we have

$$\mathcal{F}^N(\Upsilon) \cong \mathcal{F}^N(\Upsilon') \bigcup_{\Sigma'_n}^{n=1..N} \left\{(\Sigma_n \times (0,1]) \cup (\Sigma'_n \times \{0\})\right\}, \tag{7}$$

where the gluing is along the subset $\Sigma'_n \subset \Sigma_n \times \{0\}$ at which the n^{th} point is distance "zero" from q.

[1] Note, everything is basepoint dependent since Σ is not path connected. This theorem holds for arbitrary choice of basepoint.

We now have γ a loop in $\Sigma_n \times \{1\}$ which bounds a disc D within $\mathcal{F}^N(\Upsilon)$. Since γ is assumed nontrivial in π_1, D must intersect the gluing set Σ'_n in a nontrivial loop $\gamma' \subset \Sigma'_n$ which bounds a disc D' in $\mathcal{F}^N(\Upsilon)$. By the induction hypothesis, the inclusion $\iota' : \Sigma'_n \to \mathcal{F}^N(\Upsilon)$ is π_1-injective. Thus, γ is not contractible in $\mathcal{F}^N(\Upsilon)$. $\diamond$

Theorem 2.3 *Given the configuration space $\mathcal{F}^N(\Upsilon)$ of a tree Υ and a connected subset $K \subset \mathcal{F}^N(\Upsilon)$, if the homomorphism $\iota_* : \pi_1(K) \to \pi_1(\mathcal{F}^N(\Upsilon))$ induced by inclusion is trivial, then K is nullhomotopic in $\mathcal{F}^N(\Upsilon)$.*

Proof: Induct on the number of essential vertices V of Υ. The theorem is certainly true for $V = 0$. As before, let Σ denote the configurations which have a point at the V^{th} essential vertex. The complement of Σ in $\mathcal{F}^N(\Upsilon)$ is composed of products of configuration spaces of graphs with fewer numbers of essential vertices, *cf.* Equation (5); hence, if K lies within the complement of Σ, then K contracts by induction.

In the case where $K \cap \Sigma \neq \emptyset$, we know from Lemma 2.2 that $\pi_1(K \cap \Sigma) \mapsto 0$ under inclusion into $\pi_1(\Sigma)$. As Σ is composed of products of configuration spaces of graphs with smaller numbers of essential vertices (see Equation (4)), we have by induction that each connected component of $K \cap \Sigma$ contracts to a point within Σ. One may then "pinch" K off of Σ into a collection of connected sets K_i since Σ is a codimension one subcomplex. By induction, we may homotope each K_i to a point within the complement of Σ. By tracing out the path of the pinch-points under the homotopies, we have a homotopy of all of K to a graph, which must be nullhomotopic in $\mathcal{F}^N(\Upsilon)$. $\diamond$

Corollary 2.4 *The configuration space $\mathcal{F}^N(\Upsilon)$ is an Eilenberg-MacLaine space of type $K(\pi_1, 1)$:* i.e., *$\pi_k(\mathcal{F}^N(\Upsilon)) = 0$ for all $k > 1$.*

Corollary 2.4 is significant in that π_1 determines the homotopy type of the configuration space. We thus consider the (pure) braid groups of trees. For Υ a planar graph, the inclusion $\iota : \Upsilon \hookrightarrow \mathbb{R}^2$ induces a map on the level of braid groups. However, note that this map is neither injective nor surjective. Some vestiges of the planar braid groups do however survive:

Theorem 2.5 *For any tree Υ and any $N > 0$, the fundamental group $\pi_1(\mathcal{F}^N(\Upsilon))$ is torsion-free.*

Proof: A theorem of P. Smith (see [Han91, p. 17]) implies that for a compact finite-dimensional Eilenberg-MacLane space X of type $(\pi_1(X), 1)$, the fundamental group is torsion-free. Hence, we may use Corollary 2.4, noting that one can deformation retract $\mathcal{F}^N(\Upsilon)$ to a compact complex by enlarging the diagonal slightly.

We give another proof that does not depend upon the information concerning higher homotopy groups. The space $\mathcal{F}^N(\Upsilon)$ is built from products of configuration spaces of subgraphs via gluing together along the sets Σ. These gluing regions are highly disconnected; however, one may perform the requisite gluings in a sequential order. By Van Kampen's theorem, the effect on π_1 of gluing two disjoint spaces together along a connected set S gives an amalgamated free product of the components over $\pi_1(S)$. Likewise, gluing a connected set along two disjoint copies of a connected subset S yields an HNN extension of the fundamental group over $\pi_1(S)$.

The theorem is trivially true when $N = 1$, as well as when there are no essential vertices. Hence, we may use Equation (6) for an induction argument as follows. By Lemma 2.2 and the induction hypothesis on N and V that fundamental groups of the pieces are torsion free, we have either an HNN extension or an amalgamated free product of torsion free groups over an injective subgroup. By standard results in geometric group theory (see [Ser80, p.6-8],[SW79]) the resulting fundamental group is torsion-free. ◇

Results about the homotopy type of the configuration spaces are of interest in understanding the topology of the spaces, but a different set of issues must be dealt with in order to have a practical solution to the control-theoretic problems mentioned in the Introduction. For example, the configuration space of N points on Υ is an N-dimensional complex. Any simplification of the space which reduces this dimension will more easily permit the construction of explicit control vector fields on the space. We offer as a partial solution to this dilemma a bound on the homological dimension of these spaces realizable through deformation retraction.

Theorem 2.6 *For any tree Υ and any $N > 0$, the configuration space $\mathcal{F}^N(\Upsilon)$ deformation retracts onto a V-dimensional subcomplex, where V is the number of essential vertices of Υ.*

Proof: Choose p an essential vertex which is outermost in the sense that it is one edge away from the boundary of Υ. We induct on the number of points N, the number of edges K incident to p, and the number of essential vertices V in Υ. In order to later apply this argument in the case where Υ is a general graph, the precise induction hypothesis will be that the configuration space deformation retracts as a pair $(\mathcal{F}^n(\Gamma), \Psi)$, where Ψ denotes the subcomplex

$$\Psi := \{\mathbf{x} \in \mathcal{F}^N(\Upsilon) : x_i \in \partial\Upsilon \text{ for some } i\}, \tag{8}$$

and $\partial\Upsilon$ denotes the boundary of Υ. In other words, the restriction of the deformation retraction to Ψ induces a deformation retraction of Ψ. If $V = 0$ and $N > 1$, or if $V > 0$ and $N = 1$, the configuration space pair deformation retracts to at most a V-dimensional subcomplex and thus satisfies the conclusion. Assume, then, that the result holds for all graphs with less than V essential vertices. If $K = 2$, then the vertex is not essential and the conclusion is true by induction on V.

Again denote by $\mathcal{E}_n$ the n^{th} "end" of the configuration space, homeomorphic to $\mathcal{F}^{N-1}(\Upsilon)$. Write $\mathcal{F}^N(\Upsilon)$ as the union of pieces (as in Equation (6))

$$\mathcal{F}^N(\Upsilon) \cong \mathcal{F}^N(\Upsilon') \bigcup_{\Sigma_n}^{n=1..N} ((\mathcal{E}_n \times (0,1]) \cup (\Sigma_n \times \{0\})). \tag{9}$$

Each $\mathcal{E}_n \times (0,1]$ product end is attached along $\Sigma_n \times \{0\}$; hence, this product deformation retracts to $\Sigma_n \times [0,1) \cup \mathcal{E}_n$ rel $\Sigma_n \cup \mathcal{E}_n$ as follows. For x not in a neighborhood of Σ_n, shrink the segment $\{x\} \times (0,1]$ to $\{x\} \times \{1\} \subset \mathcal{E}_n$, using a bump function with support on a neighborhood of Σ_n. Then use another bump function to collapse this to $\Sigma_n \times \{t\}$ for all $t \in (\epsilon, 1-\epsilon)$ for ϵ small. Rounding out the corners in the standard way finishes this stage of the deformation. By induction on V, the subsets Σ_n deformation retract to subcomplexes of dimension at most $V - 1$. Hence, we may deform each $\Sigma_n \times (0,1)$ to a subcomplex of dimension at most $(V-1)+1 = V$.

By induction on K, one may then deformation retract the base $\mathcal{F}^N(\Upsilon')$ to a subcomplex of dimension at most V without pushing Σ off of itself. An application of the Homotopy Extension Property yields a deformation retraction of the entire complex. As a final step, we may by induction on N deformation retract the ends $\mathcal{E}_n \times \{1\}$ within themselves and use the Homotopy Extension Property again to extend this to a global deformation. $\diamond$

This deformation retraction is simple enough that one may be able to specify a control vector field on the simplified configuration space, and then computationally invert the deformation retraction to lift this to a control field on the original space.

3 Configuration spaces of graphs

In order to extend the results of the previous section to configuration spaces of general graphs, some additional techniques and insights are requisite. Not all results carry over perfectly. For example, the configuration space of N points on a circle always deformation retracts to a circle, even though this graph has no essential vertices. Fortunately, this is the exception and not the rule.

In what follows we will denote by Γ a general (*i.e.*, not necessarily simply connected) graph, reserving Υ for trees. Choose $Q := \{q_i\}_1^M$ a collection of points in the edge set of Γ such that $\Gamma - Q$ is a connected open tree. Denote by Υ the graph obtained from $\Gamma - Q$ by adding distinct endpoints; hence, every point $q_i \in Q$ is split into two point q_i^+ and q_i^- in Υ. The configuration space $\mathcal{F}^N(\Gamma)$ decomposes as $\mathcal{F}^N(\Upsilon)$ with certain ends identified pairwise. More specifically, the graph Υ has $2M$ "ends" corresponding to the points $q_i^\pm$. Likewise, for each such end of Υ, the configuration space $\mathcal{F}^N(\Upsilon)$ has N "ends" which come from Equation (6). Hence, the configuration space $\mathcal{F}^N(\Upsilon)$ has $2MN$ product ends.

Theorem 3.1 *The configuration space $\mathcal{F}^N(\Gamma)$ is a $K(\pi_1, 1)$.*

Proof: Decompose the configuration space by splitting along the aforementioned set Q. Then, given a representative $f : S^k \to \mathcal{F}^N(\Gamma)$ of π_k, we know that f must intersect the clipped ends nontrivially via Corollary 2.4. However, the proof of Lemma 2.2 is presented in the context of contracting a loop in a product end of a tree; hence, the inclusion of the ends of $\mathcal{F}^N(\Gamma)$ into $\mathcal{F}^N(\Upsilon)$ is π_1-injective, and the proof of Theorem 2.3 applies. $\diamond$

Corollary 3.2 *The pure braid group of a graph is torsion-free.*

Theorem 3.3 *For any graph Γ not homeomorphic to a circle, the configuration space $\mathcal{F}^N(\Gamma)$ deformation retracts to a V-dimensional subcomplex, where V is the number of essential vertices in Γ.*

Proof: In the deformation retraction of Theorem 2.6, the deformation can always be accomplished rel the ends (except when the tree is the trivial line segment with one point on it). Hence, we may deformation retract that portion of $\mathcal{F}^N(\Gamma)$ which corresponds to $\mathcal{F}^N(\Gamma - Q)$ down to a V-dimensional subcomplex. Then, the remaining portions of the configuration space may be likewise deformation retracted by induction on the number of points as in the previous theorem. $\diamond$

We close our treatment without a specific classification of the configuration spaces of graphs. By Corollary 3.1, the isomorphism class of the pure braid groups on graphs determines the homotopy type of the configuration space. Given the examples from the next section, we conjecture that the configuration spaces are all homotopic to copies of tori of various dimensions, glued together along incompressible subtori: a reasonable refinement of this would be the following.

Conjecture 3.4 *The group $\pi_1(\mathcal{F}^N(\Upsilon))$ is an* Artin right angle group*: the group has a presentation in which all of the relations are commutators of the generators.*

4 Examples

EXAMPLE 1: Let $\mathcal{F}_K^N$ denote the configuration space of N points on a K-pronged radial tree (*i.e.*, having K edges attached to a single central vertex). According to Theorem 2.6, $\mathcal{F}_K^N$ deformation retracts to a one-dimensional graph. Since the homotopy type of a graph is determined by its Euler characteristic, we can derive the following:

Proposition 4.1 *The braid group $\pi_1(\mathcal{F}_K^N)$ is isomorphic to a free group on Q generators, where Q equals*

$$Q := 1 + (NK - 2N - K + 1)\,\frac{(N+K-2)!}{(K-1)!} \tag{10}$$

Proof: Using Equation (6), one derives a recursion relation for the Euler characteristic:

$$\chi(\mathcal{F}_K^N) = \chi(\mathcal{F}_{K-1}^N) + N\left[\chi(\mathcal{F}_K^{N-1}) - E\right], \tag{11}$$

where E denotes the number of connected components of $\mathcal{E} - i(\Sigma)$. Each such component contributes one edge in the deformation retracted space and hence contributes a -1 to the value of $\chi(\mathcal{F}_K^N)$. A simple combinatorial argument shows that

$$E = \prod_{i=1}^{K-1}(N+i-2) = \frac{(N+K-3)!}{(K-2)!}. \tag{12}$$

The seed for the recursion relation is the fact that $\mathcal{F}_K^1$ is homeomorphic to the underlying tree which is homotopically trivial. Solving (11) yields

$$\chi(\mathcal{F}_K^N) = -\,(NK - 2N - K + 1)\,\frac{(N+K-2)!}{(K-1)!}, \tag{13}$$

which, in turn, implies that the configuration space is homotopic to a wedge of $1-\chi$ circles. ◇

EXAMPLE 2: The spaces $\mathcal{F}_K^N$ of Example 1 are, by Proposition 4.1, homotopic to a wedge of circles. However, the deformation retraction of Theorem 2.6 does not compress the space to this extreme, but rather leaves some structure. In the simple case of $\mathcal{F}_3^2$, the deformation retraction yields a 1-d graph which resembles a 'benzene ring': homeomorphic to S^1 with six radial edges attached. This reduction of the configuration space has the advantage that each vertex corresponds to a necessary passage through the central vertex.

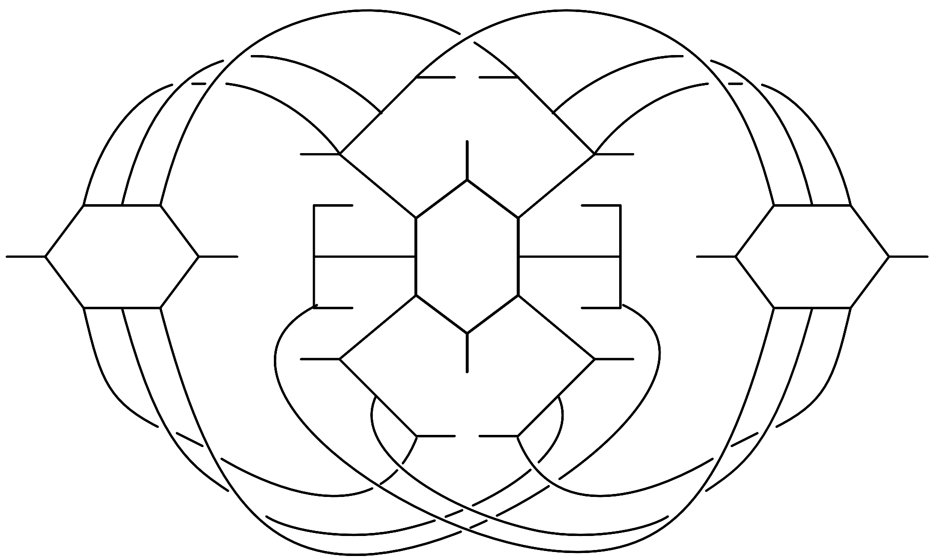

FIGURE 2: The space $\mathcal{F}_3^3$ deformation retracts to a graph.

In Figure 2, we illustrate this semi-minimal reduction of the space $\mathcal{F}_3^3$. According to Proposition 4.1, $\mathcal{F}_3^3$ deformation retracts to the wedge of 13 circles. However, under the piecewise deformation given by the proof of Theorem 2.6, one obtains a graph that is non-planar and clearly has vestigial copies of the 'benzene' graph of $\mathcal{F}_3^2$. It is interesting to note that this is the configuration space associated to a type of "Towers of Hanoi" problem [Hat]: the condition that one may only move one ring at a time corresponds to the condition that the central vertex may be occupied by at most one point. The diameter of the graph in Figure 2 corresponds to the minimal number of steps required to reverse the order of three points initially on the same edge.

EXAMPLE 3: Consider the graph Γ_H which is homeomorphic to the letter "H": two essential vertices. According to Theorem 2.6, the configuration space $\mathcal{F}^N(\Gamma_H)$ deformation retract to a 2-complex. However, a more careful analysis of individual cases yields more insightful results. In what follows, we have executed the proof of Theorem 2.6 step-by-step, applying the results of Example 1 at each stage. We denote by F_p the free group on p generators.

EXAMPLE 3A: $\pi_1(\mathcal{F}^2(\Gamma_H)) \cong F_3$.

EXAMPLE 3B: $\pi_1(\mathcal{F}^3(\Gamma_H)) \cong F_{25}$.

EXAMPLE 3C: $\pi_1(\mathcal{F}^4(\Gamma_H)) \cong F_{195} * \left[*_1^6 (\mathbb{Z} \times \mathbb{Z})\right]$.

There are exactly six 2-tori in a homotopically minimal representative of this space. Each torus corresponds to a configuration where pairs of points trace out loops in a neighborhood of the individual vertices.

It is entirely possible that Example 3c illustrates the canonical way in which non-free components of braid groups on graphs can arise: *cf.* Conjecture 3.4.

EXAMPLE 4: The simplest non-tree graph, Γ_Q, is homeomorphic to a circle with one edge attached. As this graph is the identification of two edges of the Y-graph Γ_Y, one may obtain the configuration space (up to homeomorphism) via the proper identifications. We display the result in Figure 3. Note that this space deformation retracts onto a one-dimensional graph homotopic to the wedge of three circles (the point on the left of the diagram is a puncture).

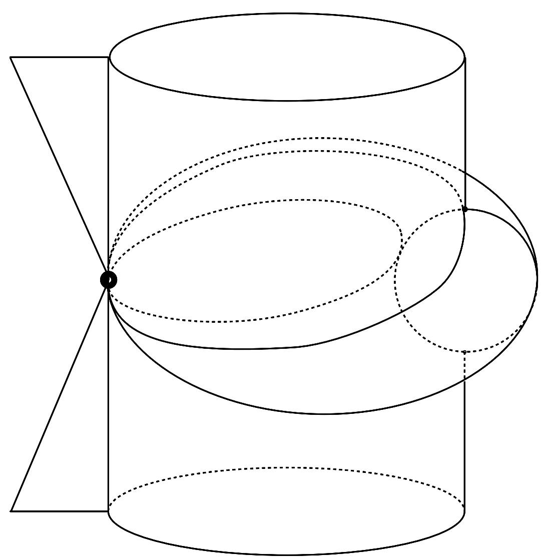

FIGURE 3: The space $\mathcal{F}^2(\Gamma_Q)$ embeds in $\mathbb{R}^3$.

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