Accumulations of infinite links

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Abstract

We consider smooth links in S^3 which consist of a countably infinite collection of tamely embedded circles having the property of being locally straight. We show that in general there are no restrictions on the types of accumulations that may appear in terms of the knot types: *e.g.*, trefoils may accumulate on figure-eights in a smooth manner. Our techniques rely on properties of "universal templates," or branched 2-manifolds. Our results apply to closed orbits of flows on S^3.

1 Introduction

Definition 1.1 *An* infinite link *in S^3 is a countably infinite collection of embeddings $\{K_i : S^1 \hookrightarrow S^3\}_0^\infty$ which are pairwise disjoint.*

Of course, we cannot embed the countable disjoint union of S^1's into S^3 all at once — by compactness, there must be accumulations of the components. It is the goal of this paper to examine such accumulations. Although infinite links have been approached in the past, the analysis consists primarily of looking at "direct limits" of finite links (*e.g.*, considering the direct limits of the braid groups) [12]. We wish to emphasize how the infinite link actually sits in S^3.

Definition 1.2 An infinite sequence of distinct knots $\{\gamma_i\}_{i=1}^\infty$ *accumulates* on the knot γ if there exists a point $x \in \gamma$ which is the accumulation point of a sequence of points $\{x_i \in \gamma_i\}_{i=1}^\infty$.

In §2, we consider a class of "regular" links which are locally straight — such links arise for example as links of closed orbits in flows. We begin our examination of accumulations in §3 by demonstrating the existence of strong restrictions on accumulations for infinite braids which have one-way crossings. Then, in §4 we prove our main theorem on the absence of restrictions on accumulations in S^3. In fact, we will show that a single infinite link realizes all such accumulations simultaneously:

Main Theorem: *There exists a smooth regular (*i.e.*, locally straight) infinite link $L_U \subset S^3$ which has the following property: given an arbitrary sequence of knot types $\{K_i\}_0^\infty$, there exists a sequence $\{\gamma_i\}_0^\infty$ of components of L_U having (respective) knot types K_i such that $\{\gamma_i\}_1^\infty$ accumulates on γ_0.*

Surprisingly, this "universal" link L_U appears as the set of periodic orbits in a flow on $\mathbb{R}^3$ induced by a third-order ordinary differential equation.

Given the connections we draw in Lemma 2.2 between locally straight links and the topology of flows, there are several works in foliation theory and dynamical systems theory which are related to the problem of infinite links. Note in particular the work of Gambaudo *et al.*, which considers *asymptotic linking* of accumulating knots in S^3 [6, 7]. We compliment this perspective by considering what might be termed the "asymptotic knotting" of accumulating knots.

For introductory material on knot and link theory, see [14, 4]. See also [1] for an introduction to braids, a related class of objects. A complete account of the template-theoretic methods used in this paper is to be found in [11].

2 Regular infinite links

Without any other restrictions, accumulations of knots can be immensely complicated, as demonstrated by some "wild" examples in Figure 1. It is necessary to restrict to a class of links which accumulate in a locally

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uniform manner. We therefore consider for the remainder of this work the following class of *regular* infinite links:

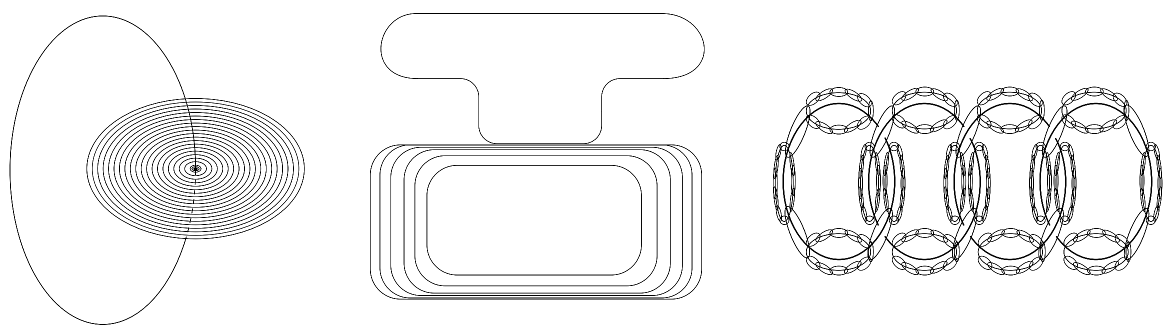

Figure 1: Examples of "wild" accumulations of link components.

Definition 2.1 Let L be an infinite link in S^3. Then L is *regular* if for every small neighborhood N of $p \in S^3$, there exists a diffeomorphism $h : N \to \mathbb{R}^3$ such that $h(L \cap N)$ is a (perhaps empty) set of vertical lines in $\mathbb{R}^3$.

Lemma 2.2 *A link is regular if and only if there exists a smooth nonsingular flow on a neighborhood of L which has L as a subset of the closed orbits for the flow.*

Proof: If a locally nonsingular flow is given, then the Flowbox Theorem [13, p. 203] ensures that the link is regular. Conversely, if a link is locally diffeomorphic to a set of vertical lines in $\mathbb{R}^3$, we may pull back the flow in the vertical direction on $\mathbb{R}^3$ back via the local diffeomorphism. Since we are working in S^3, there is no obstruction to orienting the flow. □

Hence, regular links may be thought of as foliated subsets. Under this definition, regular links behave nicely, at least on a local level. For example, there must be a positive lower bound on the length of a component (fixing a metric on S^3). In addition, the irregular sorts of accumulations evident in Figure 1 are impossible for regular links.

Lemma 2.3 *If $x \in \gamma$ is a point of accumulation for the sequence of knots $\{\gamma_i\}$, then every point of γ is such an accumulation point (for a different sequence of points).*

Proof: Simply consider the accumulating sequence $\{x_i\} \to x$, and use the flow implicit in Lemma 2.2 to flow these points forwards in time. The continuity of the flow preserves accumulation points and the fact that γ has finite period implies that every point on γ is an accumulation point for the sequence. □

According to Lemma 2.3, the picture one should have in mind of a sequence of knots $\{\gamma_i\}$ accumulating onto γ is that the limiting subsequence of knots "wraps" around γ arbitrarily closely.

Corollary 2.4 *If $x \in \gamma$ is a point of accumulation for the sequence of knots $\{\gamma_i\}$, then, given N_γ a small tubular neighborhood of γ and i sufficiently large, either (1) $\gamma_i \subset N_\gamma$; or (2) an arbitrarily long segment of γ_i (under some fixed metric) is a subset of N_γ.*

Proof: This follows from the Implicit Function Theorem applied to the return map of a cross section to the flow implicit in Lemma 2.2: see [13, Thm. 8.1] for details. □

It is a subtle question to decide how the local structure implicit in Definition 2.1 implies global relations between the knot types of the components and the manner of accumulations for regular infinite links. We give an example to argue that this is a nontrivial issue:

Example 2.5 Let $X = \{0\} \cup \{1/n; n = 1, 2, 3 \ldots\} \subset [0, 1]$, and let L denote the infinite link $X \times S^1$ embedded in $\mathbb{R}^2 \subset \mathbb{R}^3$ as the set $\{(1+x)e^{2\pi i\theta}; x \in X, \theta \in [0, 1]\}$. This embedded set consists of concentric unknotted circles $\{\gamma_i\}_1^\infty$ accumulating onto a unit circle γ_0. Giving $L \subset \mathbb{R}^3$ the subspace topology, what happens when we reembed L? We may embed finitely many of the components $\gamma_1, \ldots, \gamma_N$ as any type of knot we wish, but upon embedding γ_0, all the γ_i for i large are forced to have the same knot type. Hence, the nature of accumulations in this infinite link dictates the topology of "most" of the components.

Hence, for any given regular link, the types of accumulations present convey relevant information on how the global embedding affects the embedding of the individual components.

3 Restrictions on positive links

We wish to answer the question of whether there are any restrictions on the types of accumulations that may occur in a regular link based on the embeddings of the accumulating sequence. We consider a class of links which may be thought of as "positive" in the sense that they have representatives as closed braids with crossings all in one direction. Gambaudo *et al.* have considered the linking properties of accumulations in a similar class of infinite links [7].

Proposition 3.1 *Let L be an infinite regular link in $D^2 \times S^1 \subset S^3$ (standard embedding) which is monotonic in both the S^1 direction and in the angular coordinate of D^2 in radial coordinates (that is, components of L are braids with all crossings of positive sign). Choose a sequence $\{\gamma_i\}_0^\infty$ of components such that $\{\gamma_i\}_1^\infty$ accumulates onto γ_0. Then,*

Case I: *If the lengths of all the γ_i (under some fixed metric) are bounded above, then for all but finitely many i, γ_i is a generalized cable of γ_0 (see below).*

Case II: *If the lengths of the γ_i are unbounded, then either the genera of the sequence is unbounded or γ_0 is an unknot.*

Proof:

Case I: Recall the definition of a *generalized cable* (a more restrictive version of a satellite). Given a closed braid $L \subset D^2 \times S^1$, and a knot $K \subset S^3$ with (closed) tubular neighborhood V_K, a generalized cable of L with respect to K is the image of L under a homeomorphism $h : D^2 \times S^1 \to V_K$.

From Lemma 2.2, there is a locally nonsingular flow which realizes L as a set of closed orbits. Yet, by Corollary 2.4, for i sufficiently large, γ_i must be arbitrarily close to γ_0 for arbitrarily long lengths. Hence, all but finitely many of the γ_i lie within a small tubular neighborhood of γ_0.

Case II: We will make use of the Birman-Williams genus formula for positive braids [2]. Given a nonseparable closed positive braid B of c crossings, n strands, and μ components, the genus of the link B is given by

$$g(B) = \frac{c - n + \mu}{2}. \tag{1}$$

Assume that the accumulation knot γ_0 is a nontrivial knot, so that $g_0 \equiv g(\gamma_0) \neq 0$. Let n_i, c_i, g_i denote the (respective) strand number, crossing number, and genus of the indexed components γ_i. Since the lengths of the γ_i are unbounded, the number of strands n_i is also unbounded in i.

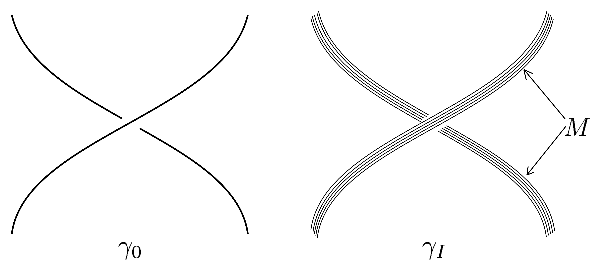

Figure 2: Near each crossing of γ_0, M strands of γ_I cross themselves, contributing M^2 to the crossing number c_I.

By Lemma 2.2 and Corollary 2.4, γ_i must run parallel to γ_0 for arbitrarily long lengths for sufficiently large i. Hence, given M arbitrarily large, there exists I large such that M strands of γ_I follow along γ_0 very closely. At each crossing of γ_0, γ_I appears locally as in Figure 2. Therefore, if $n_I = Mn_0 + N$ for some $N \geq 0$, the crossing number of γ_I is bounded as

$$c_I \geq M^2 c_0 + N, \tag{2}$$

since there are at least M^2 crossings at each crossing of γ_0 and every additional strand must induce at least one crossing. The genus of γ_I is thus bounded as

$$g_I \geq \frac{(M^2 c_0 + N) - (Mn_0 + N) + 1}{2} = \frac{M^2 c_0 - Mn_0 + 1}{2}, \tag{3}$$

which grows quadratically as $M \to \infty$ since n_0 is fixed and $c_0 \neq 0$. □

Remark 3.2 It is not hard to show that in Case II above there are also some restrictions on the local linking of knots accumulating on an unknot — any linking near the accumulation unknot will also force the genus to become unbounded.

The content of Proposition 3.1 is that flows which twist monotonically are very restricted in terms of which infinite links occur: *e.g.*, unknots cannot accumulate on a trefoil. One might expect similar restrictions to apply more generally — it seems counterintuitive (to this author, initially) that an infinite sequence of unknots can accumulate onto a genuinely knotted orbit in a locally straight manner. However, the case is quite the opposite.

4 A "universal" link

Theorem 4.1 *There exists a smooth regular (*i.e.*, locally straight) infinite link $L_U \subset S^3$ which has the following property: given an arbitrary sequence of knot types $\{K_i\}_0^\infty$, there exists a sequence $\{\gamma_i\}_0^\infty$ of components of L_U having (respective) knot types K_i such that $\{\gamma_i\}_1^\infty$ accumulates on γ_0.*

We prove this result by considering the limits of orbits on *templates*.

4.1 Templates and template theory

Definition 4.2 A *template* (a.k.a. knotholder) is a compact branched 2-manifold (with boundary) supporting a smooth semiflow which carries a decomposition into *joining* and *splitting charts*, given in Figure 3.

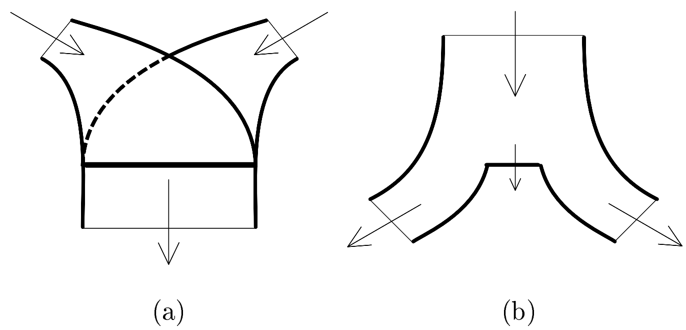

Figure 3: (a) Joining and (b) splitting charts form a template.

Hence, a template is what one gets by gluing together a finite number of joining and splitting charts such that the directions of the semiflow line up. The term "semiflow" is used since the flow is not reversible at the branch lines. Templates were introduced in the seminal papers of Birman and Williams [2, 3]. Examples of templates appear in Figure 4.

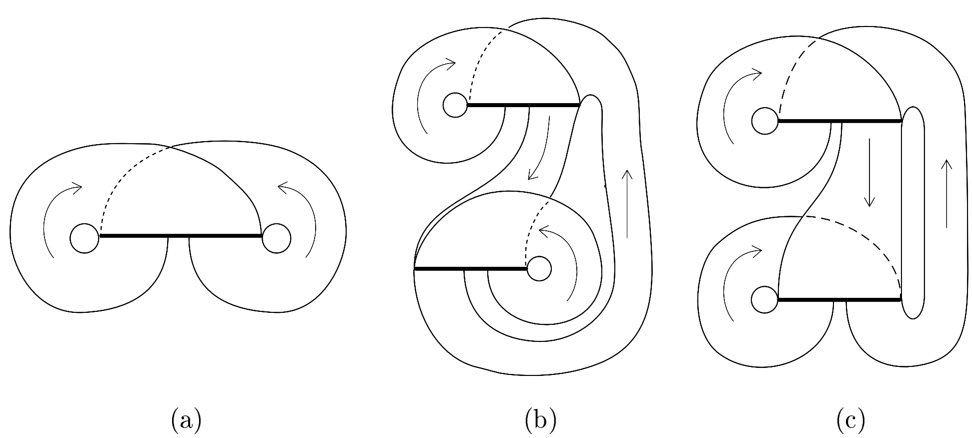

Figure 4: Examples of templates: (a) the Lorenz template; (b) the template $\mathcal{U}$; (c) the template $\mathcal{V}$.

We recall some elementary facts about templates (see [11] for a complete survey). Given an embedded template $\mathcal{T} \subset S^3$, there is a link of closed orbits for the semiflow, $L_{\mathcal{T}}$. This link is an infinite regular link which contains among its components an infinite set of knot types [5] (though not necessarily all [2]).

Templates come equipped with a symbolic dynamical (or *Markov*) structure which enables one to efficiently label the components of the link. Upon labeling the strips of a given template $\mathcal{T}$ with the symbols $\{x_1, x_2, \ldots, x_N\}$, there is a one-to-one correspondence between components of the link $L_{\mathcal{T}}$ and *itineraries*, or (periodically repeated) finite admissible words in the alphabet $\{x_1, \ldots, x_N\}$, up to cyclic permutations [3, 11]. A sequence of components of the link $L_{\mathcal{T}}$ accumulates onto some component γ_0 if and only if the corresponding sequences (or *itineraries*) agree for arbitrarily long lengths.

Example 4.3 In the example of Figure 5, the so-called *Lorenz template* [2] is shown along with a closed orbit of the semiflow. Under the labelling $\{x_1, x_2\}$ of the strips as shown, the closed orbit has itinerary $(x_1x_1x_2x_1x_2)^\infty$ (or any cyclic permutation thereof). In Figure 6, we give the Markov structures for the templates $\mathcal{U}$ and $\mathcal{V}$ of Figure 4.

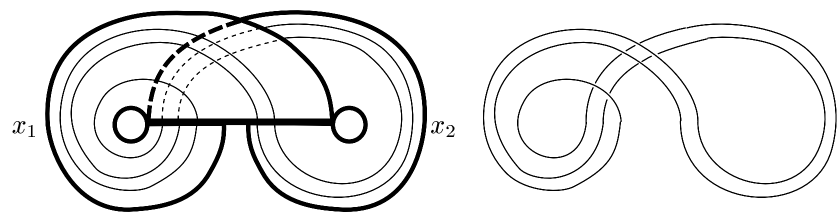

Figure 5: The Lorenz template with the closed orbit $\left(x_1^2x_2x_1x_2\right)^\infty$ (isotopic to a trefoil). Note this itinerary is equivalent to $(x_1x_2x_1x_2x_1)^\infty$.

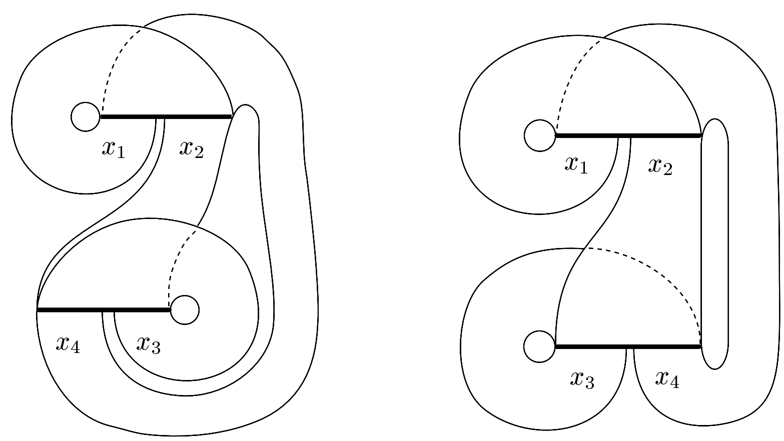

Figure 6: The Markov structures for the templates $\mathcal{U}$ (left) and $\mathcal{V}$ (right).

Several properties of certain embedded templates have been discovered by the author, following the work of M. Sullivan [15]:

Definition 4.4 A *universal template* $\mathcal{T} \subset S^3$ is a template which has *every* knot type represented among the components of $L_{\mathcal{T}}$.

Theorem 4.5 (Ghrist [8, 9, 11]) *The embedded templates $\mathcal{U}$ and $\mathcal{V}$ in Figure 4 are universal. Moreover, each has the following additional properties:*

1. *Given any finite* link*, there is an isotopic copy of this link among the closed orbits.*
2. *Given any (orientable) embedded template, there is an isotopic copy of this template embedded in $\mathcal{U}$ (or $\mathcal{V}$) as a* subtemplate *(see below). All of these subtemplates may be chosen to be disjoint and mutually unlinked.*

The proof of Theorem 4.5 is highly nontrivial and involves a combination of symbolic dynamics and topological arguments; in particular, the following two concepts are key.

Definition 4.6 Given a template $\mathcal{T}$, a *subtemplate* of $\mathcal{T}$, $\mathcal{T}' \subset \mathcal{T}$, is a subset of $\mathcal{T}$ which satisfies Definition 4.2 under the semiflow of $\mathcal{T}$ restricted to $\mathcal{T}'$.

Definition 4.7 A *template inflation* from a template $\mathcal{S}$ into a template $\mathcal{T}$ is an embedding $\mathfrak{R} : \mathcal{S} \hookrightarrow \mathcal{T}$ which respects the semiflow; hence, $\mathfrak{R}(\mathcal{S})$ is a subtemplate of $\mathcal{T}$. If $\mathcal{S}$ and $\mathcal{T}$ are embedded in S^3, we say $\mathfrak{R}$ is an *isotopic inflation* if $\mathfrak{R}(\mathcal{S})$ is isotopic in S^3 to $\mathcal{S}$.

Lemma 4.8 (Ghrist [8, 9]) *A template inflation induces a map on itineraries of $\mathcal{S}$ to itineraries of $\mathcal{T}$, the effect of which is to "inflate" each symbol x_i of the symbolic structure on $\mathcal{S}$ to a finite word of symbols $\mathfrak{R}(x_i)$ within the symbolic structure of $\mathcal{T}$.*

Example 4.9 A key step in the proof of Theorem 4.5 is showing that there is an isotopic inflation $\mathfrak{D} : \mathcal{V} \hookrightarrow \mathcal{V}$ acting on the template $\mathcal{V}$ which has an induced symbolic inflation

$$\mathfrak{D} : \mathcal{V} \hookrightarrow \mathcal{V} : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_1 x_2 \\ x_3 \mapsto x_3 \\ x_4 \mapsto x_3 x_4 \end{cases} . \tag{4}$$

Recall the symbolic structure placed on $\mathcal{V}$ from Figure 6. We show in Figure 7 that the image subtemplate $\mathfrak{D}(\mathcal{V}) \subset \mathcal{V}$ is isotopic to $\mathcal{V}$ (or, see [9, 11] for help).

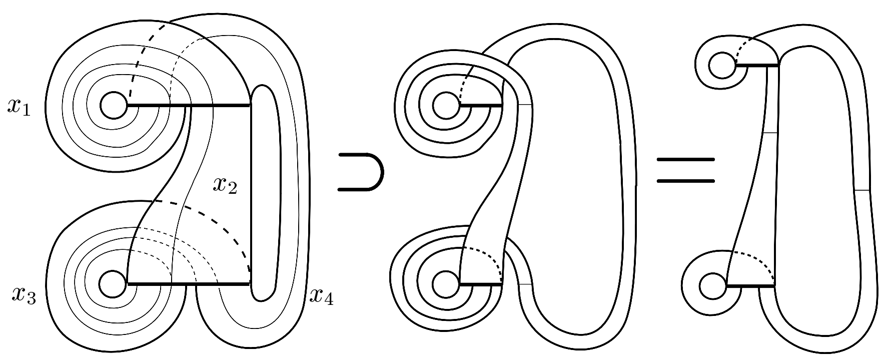

Figure 7: The image of the template inflation $\mathfrak{D} : \mathcal{V} \to \mathcal{V}$ is isotopic in S^3 to $\mathcal{V}$.

The inflation $\mathfrak{D}$ of Equation 4 will be used in the proof of Theorem 4.1.

4.2 Proof of Main Theorem

Given an arbitrary knot K, we will construct a template containing a sequence of unknots which accumulate onto K. The manner in which this accumulation is constructed was inspired by a remark of Cameron Gordon. Then, we will use these unknots as anchors for a sequence of universal subtemplates, each of which can be chosen to contain an arbitrary knot within it. Finally, we use Theorem 4.5 to conclude that all these templates reside within the universal templates $\mathcal{U}$ and $\mathcal{V}$.

Step I: Choose a projection for a knot type K, and consider the template $\mathcal{T}_K$ given in Figure 8: take two parallel annuli and tie them into a pair of knotted annuli each of of type K with the blackboard framing. Then insert the "template plug" into the annuli as shown. Note that the template semiflow on $\mathcal{T}_K$ is oriented oppositely on each annulus. Under the symbolic description of the strips given as in Figure 8, the two boundary components with itineraries $(x_1)^\infty$ and $(x_3)^\infty$ are knots of type K.

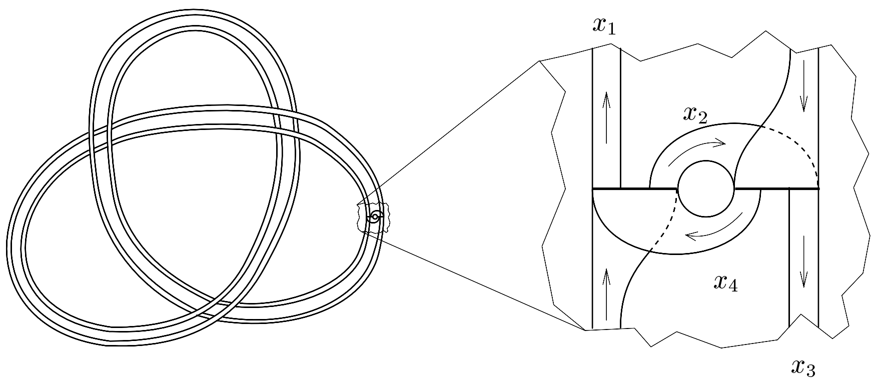

Figure 8: Two parallel annuli tied into a flat trefoil K form the template $\mathcal{T}_K$ upon inserting the "template plug."

Lemma 4.10 *If K is the unknot, then $\mathcal{T}_K$ is isotopic to the universal template $\mathcal{V}$.*

Proof: A simple exercise. □

Motivated by Lemma 4.10 and the discussion of the template inflation $\mathfrak{D} : \mathcal{V} \hookrightarrow \mathcal{V}$ in the prior subsection, we show the following:

Proposition 4.11 *The symbolic inflation $\mathfrak{D}$ given in Equation (4) defines an isotopic template inflation $\mathfrak{D} : \mathcal{T}_K \hookrightarrow \mathcal{T}_K$ for any knot type K.*

Proof: Any template may be considered as a sort of "thickened" graph — the strips correspond to edges and the branch lines to vertices. The graph associated to the template $\mathcal{T}_K$ consists of three cycles, represented by the orbits $(x_1)^\infty$, $(x_3)^\infty$, and $(x_2x_4)^\infty$, attached at two vertices (*i.e.*, two branch lines). The orbits $(x_1)^\infty$ and $(x_3)^\infty$ are separable knots of type K, and this is preserved under mapping via $\mathfrak{D}$ (since it leaves these itineraries invariant). The orbit $(x_2x_4)^\infty$ is an untwisted unknot, separable from the others; that the same is true of $\mathfrak{D}((x_2x_4)^\infty) = (x_1x_2x_3x_4)^\infty$ is clear from Figure 9. Here, a portion of a Seifert spanning surface for the orbit is shown. From the picture it is clear that the spanning surface is a flat disc; hence, its boundary is an untwisted unknot.

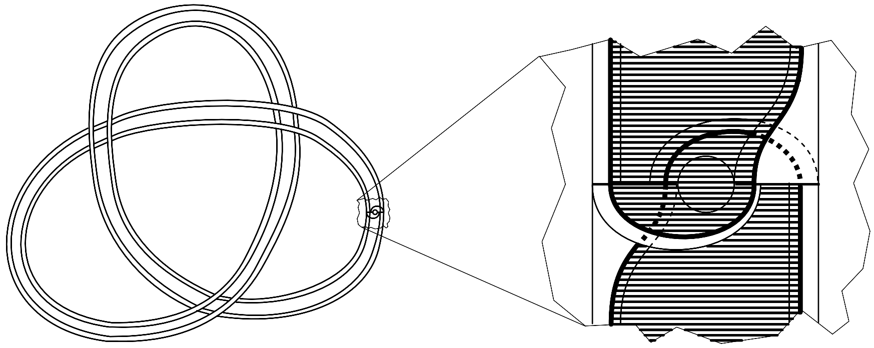

Figure 9: The orbit $\mathfrak{D}((x_2x_4)^\infty) = (x_1x_2x_3x_4)^\infty$ is an untwisted unknot.

Hence, as graphs, $\mathcal{T}_K$ and $\mathfrak{D}(\mathcal{T}_K)$ are isotopic. It remains to show that thickening the graphs is done in the same manner for each. Denote by ℓ (resp. ℓ') the branch line of $\mathcal{T}_K$ intersecting the orbit x_1 (resp. x_3). At ℓ, the strip containing the orbit x_1 crosses under the other incoming strip. At ℓ', the opposite is true: the strip containing x_3 crosses over the other incoming strip. That this is preserved under $\mathfrak{D}$ is shown in Figure 10.

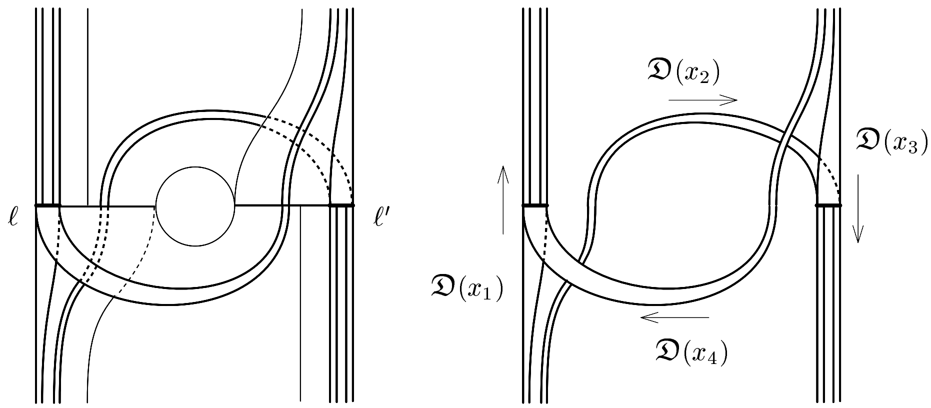

Figure 10: The inflation $\mathfrak{D}$ preserves the over/under crossings of strips.

Thus, as $\mathfrak{D}(\mathcal{T}_K)$ is a template with four strips embedded in the same manner as those of $\mathcal{T}_K$ and connected in the same manner, the templates are isotopic in S^3. $\square$

Corollary 4.12 *The set of orbits on $\mathcal{T}_K$ with itineraries $\left(x_1^i x_2 x_3^i x_4\right)^\infty$ is a collection of separable untwisted unknots which accumulate onto the boundary knots $(x_1)^\infty$ and $(x_3)^\infty$ (of knot type K).*

Proof: The orbit $(x_2x_4)^\infty$ is an untwisted unknot separable from all other orbits on $\mathcal{T}_K$. Iterating $\mathfrak{D}$ on this orbit generates a set of orbits which have the desired properties via Proposition 4.11. Since the itineraries of these orbits contain the subwords x_1^i and x_3^i for large i, these orbits accumulate onto the orbits $(x_1)^\infty$ and $(x_3)^\infty$. $\square$

Step II: The above template $\mathcal{T}_K$ has the property that there is an infinite sequence of isotopic subtemplates,

$$\cdots \subset \mathfrak{D}^i(\mathcal{T}_K) \subset \cdots \subset \mathfrak{D}^2(\mathcal{T}_K) \subset \mathfrak{D}(\mathcal{T}_K) \subset \mathcal{T}_K,$$

which accumulates on the boundary curves $(x_1)^\infty$ and $(x_3)^\infty$ of knot type K. The next step is to show that any sequence of predetermined knot types may be chosen to accumulate onto these boundary curves:

Lemma 4.13 *The template $\mathcal{T}_K$ is universal.*

Proof: We find an isotopic copy of the universal template $\mathcal{U}$ as a subtemplate of $\mathcal{T}_K$ (recall the Markov strucure for $\mathcal{U}$ given in Figure 6). According to Corollary 4.12, the orbits on $\mathcal{T}_K$ with itinerary $\left(x_1^i x_2 x_3^i x_4\right)^\infty$ are separable unknots of zero twist. As in the proof of Proposition 4.11, we choose three of these to use as "spines" for the graph of the template $\mathcal{U}$ corresponding to the image of the orbits $(x_1)^\infty$, $(x_3)^\infty$, and $(x_2x_4)^\infty$ on $\mathcal{U}$. The inflation

$$\mathcal{U} \hookrightarrow \mathcal{T}_K : \begin{cases} x_1 \mapsto x_4x_2 \\ x_2 \mapsto x_4 \\ x_3 \mapsto x_1x_2x_3^2x_4x_1 \\ x_4 \mapsto x_1x_2x_3 \end{cases}, \tag{5}$$

maps $(x_1)^\infty$, $(x_3)^\infty$, and $(x_2x_4)^\infty$ to orbits on $\mathcal{T}_K$ of the form $\left(x_1^i x_2 x_3^i x_4\right)^\infty$ for some i (or some cyclic permutation thereof), and so these are separable untwisted unknots. In Figure 11, it is shown that the over/under-crossings at the branch lines are in the proper configuration for the image subtemplate to be isotopic to $\mathcal{U}$. □

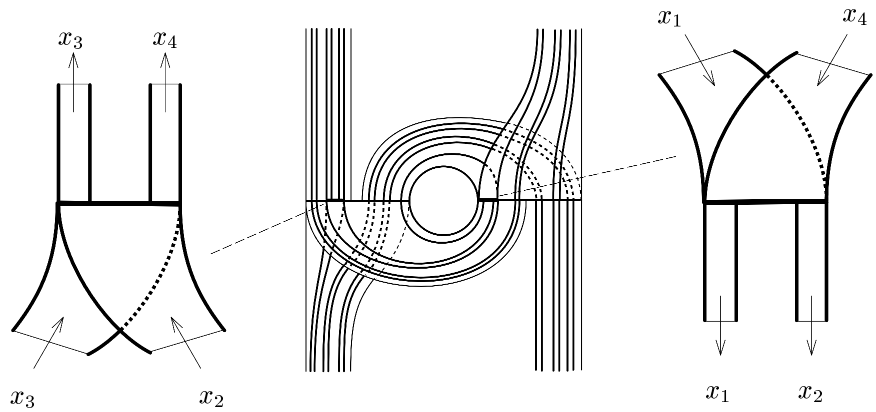

Figure 11: The template $\mathcal{U}$ is a subtemplate of $\mathcal{T}_K$.

Corollary 4.14 *Given any sequence of knot types $\{K_i\}$, there is a sequence of orbits γ_i on $\mathcal{T}_K$ each with knot type K_i, which accumulates on the boundary orbits (of knot type K).*

Proof: By Lemma 4.13, choose a knot of type K_i on $\mathcal{T}_K$ and inflate this orbit on $\mathcal{T}_K$ via $\mathfrak{D}^i$. For i large, the orbits in the image of $\mathfrak{D}^i(\mathcal{T}_K)$ accumulate on the boundary orbits of type K. □

Proof of Theorem 4.1: Given a knot type K, we have a template $\mathcal{T}_K$ which contains all types of accumulation sequences onto a knot of type K. Note that this template is orientable (in the sense of templates) since it is "flat" — it contains no twists in the given presentation. Thus, by property (2) of Theorem 4.5, both of the universal templates $\mathcal{U}$ and $\mathcal{V}$ contains an isotopic copy of $\mathcal{T}_K$ as a subtemplate for every knot type K. The isotopy of course preserves the types of accumulations. Hence, the link of $\mathcal{V}$ (or $\mathcal{U}$ or $\mathcal{T}_K$ for that matter) contains every possible type of accumulating sequence. □

Denote by L_U the *universal link* from the proof of Theorem 4.1.

Corollary 4.15 *The universal link L_U appears as the set of periodic orbits for a third-order ordinary differential equation on $\mathbb{R}^3$.*

Proof: In [10] it is shown that a certain class of flows arising from ODEs on $\mathbb{R}^3$ contains hyperbolic invariant sets which, upon collapsing out a contracting foliation, yield universal templates. Since this collapsing procedure does not alter the periodic orbits, the link of closed orbits on the template coincides with the link of closed orbits in the flow. See also [11] for details of this collapsing procedure. □

Hence, ostensibly simple infinite links may support a rich structure in the accumulations which is not present within the class of "positively twisted" links.

Question: In each instance in which a sequence of unknots accumulates onto a knot, it does so by accumulating onto a pair of knots of opposite orientation; *i.e.*, this accumulation pair "cancels." Is it possible for a sequence of unknots to accumulate onto a single knotted orbit?

Acknowledgments

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