A Topological Max-Flow-Min-Cut Theorem

(Invited Paper)

Robert Ghrist[(1,2)] and Sanjeevi Krishnan[(1)]
Departments of (1) Mathematics and (2) Electrical & Systems Engineering
University of Pennsylvania
Philadelphia, PA

***Abstract*—This note surveys a novel algebraic-topological version of the max-flow-min-cut (MFMC) theorem for directed networks with capacity constraints. Novel features include the encoding of capacity constraints as a sheaf of semimodules over the network and a realization of flow and cut values as a directed homology taking values in the sheaf. We survey the theorem and give applications to (1) multicommodity flows, (2) multi-source/multi-target flows, and (3) boolean-lattice-valued flows.**

***Index Terms*—topology, network flow, optimization, sheaves**

I. Introduction

Let G denote a finite directed and connected graph with distinct **source** and **target** vertices. The classical MFMC (Max-Flow Min-Cut Theorem) equates the maximal amount of flow on G subject to capacity constraints, and the minimal net-capacity weight of cuts separating the source from the target. For each edge $e \in E(G)$, there is a numerical capacity $k(e) \geq 0$ that encodes the maximal directed flow rate (of materials, signal, information, etc.) permitted along e. A **flow**

$$\Phi : E \to \mathbb{R}^+$$

is an assignment of flow rates to edges satisfying the following conditions:

Constraints: For each edge e, $0 \leq \Phi(e) \leq k(e)$.

Conservation Law: At each vertex v neither the source nor target,

$$\underbrace{\sum_{(x,v)\in E} \Phi(x,v)}_{\text{net flow in}} = \underbrace{\sum_{(v,y)\in E} \Phi(v,y)}_{\text{net flow out}}$$

The **value** of a flow is equal to the net flow rate out of (resp. in to) the source (resp. target). The *maxflow problem* is to determine the maximal value of a flow on a capacity-constrained network. A **cut** is a collection of edges $C \subset E(G)$ that disconnects source from target: specifically, there are no *directed* paths from source to target in $G \setminus C$. The **value** of a cut equals the sum over all edges $e \in C$ of $k(e)$. Note that cut values are untethered from flows — they are intrinsic to the network and its constraints.

The classical MFMC states that flow values and cut values are dual. Since its discovery in the 1950s (by Ford-Fulkerson and Elias-Feinstein-Shannon), it has been generalized and applied in countless settings, rightfully earning its place as a cornerstone of the theories of networks and optimization. Extensions to more complicated data types and constraints include an entire literature on multi-commodity flows (see, *e.g.*, [3], [8]). The most general type of capacity constraints in the literature to date are totally ordered commutative monoids [4]. The most direct proof of MFMC is, depending on taste, either the original algorithmic proof or as a direct consequence of LP duality.

II. Results

We survey a recent topological version of MFMC. This new proof exploits one of the canonical dualities in algebraic topology — Poincaré duality, which relates homology and cohomology in complementary dimensions. A straightforward application of Poincaré duality is not possible, however: classical Poincaré duality does not apply to (non-manifold) networks and seems unrelated to the capacity constraints. To address both complications, we turn to a deeper set of tools — the theory of sheaves. We overview these tools briefly in §III.

The flow/cut relationship to homology/cohomology is, on the surface, not a surprise and has been noticed by, *e.g.*, [2], who use an embedding of the graph into a surface to define cuts cohomologically. What is deep in the sheaf-theoretic MFMC is the relationship of the (co)homology to the constraints. Satisfaction of capacity constraints is an *intrinsic* feature of the topology of a capacity sheaf.

Krishnan [6], [7] has recently proved an extension of Poincaré duality for sheaves over directed spaces. Those results are tuned to sheaves taking coefficients in semimodules; that is, commutative monoids with an algebraic zero: see §IV. This duality is reviewed in §V-§VIII.

The algebraic and topological generality used by Krishnan is, of course, more than necessary to reprove the MFMC. The true reason for generalization is to point to deeper results. In §IX-§XI we provide three simple applications and generalizations of the topological MFMC to: (1) multicommodity flow problems; (2) multiple source/target network flows; and (3) flows of logical statements. We point to further capabilities in §XII.

III. Sheaves over graphs

The following definitions are not as general as possible and are restricted to the setting of cellular sheaves over a graph. We use the languages of algebraic topology [5] and category theory [1] freely but sparingly and with apologies to the reader.

A **cellular sheaf**, $\mathcal{F}$, over a graph, G, taking values in a category, **C**, of algebraic objects (vector spaces, groups, etc.)

and morphisms (linear transformations, homomorphisms, etc., resp.) is an assignment of $\mathbf{C}$-objects $\mathcal{F}(v)$ to vertices v and $\mathcal{F}(e)$ to edges e of G, along with an assignment of **restriction** morphisms $\mathcal{F}(v) \to \mathcal{F}(e)$ whenever e is incident to v.

The reader should think of a cellular sheaf over a graph as being a data structure. Each vertex and edge has data of a particular type, say, residing in a vector space. However, the dimension of the vector space can differ from place-to-place in the graph. Such vector spaces, unlike numerical edge weights, are not readily comparable. To compare data in one location to data in another, one can use the restriction maps.

A sheaf is also interpretable as a coefficient system that varies from point-to-point in the space. Recall that the homology, $H_\bullet(X;\mathbb{G})$, and cohomology, $H^\bullet(X;\mathbb{G})$, of a cell complex X with coefficients in an abelian group $\mathbb{G}$ are graded abelian groups which collate the number and types of holes in X, using basic homological algebra [5]. While most readers first learn of simplicial homology with integer coefficients (such coefficients denoting a multiplicity of simplices with orientation), there are very natural circumstances under which other coefficients are requisite. For example, Kirchhoff's Laws for electric circuits are really about the homology of the underlying graph with coefficients in the reals. One can think of a cellular sheaf on X with values in abelian groups as assigning a choice of coefficient group to each cell.

IV. Capacity sheaves

The first key idea of the sheaf-theoretic MFMC is this:

Capacity constraints on a network are encoded as a sheaf of semimodules.

The intuition is that the constraints, which vary from edge to edge, suggest a sheaf as the appropriate data structure. More specifically, the **capacity sheaf** $\mathcal{F}$ is a cellular sheaf over X, the topological space obtained from the geometric realization of the graph G by adding a **decoding edge**, e_*, directed from the target back to the source. Capacity sheaves can take very general, but not arbitrary, values: the most general category for capacities at present is sheaves taking values in semimodules.

A **semimodule** will mean a set $\mathbb{M}$ equipped with an associativity and binary operation (written as $+$) and containing both an identity element (written $0 \in \mathbb{M}$) and an **algebraic zero** (written $\infty \in \mathbb{M}$ and characterized by $\infty + x = \infty$ for all x). One thinks of ∞ as an algebraic *sink* or *failstate*.

For example, $\overline{\mathbb{N}}$ denotes the semimodule $\mathbb{N} \cup \{\infty\}$ under addition. Likewise, the extension of the non-negative reals to $\overline{\mathbb{R}^+} = [0,\infty] = [0,\infty) \sqcup \{\infty\}$ forms a semimodule under addition. One models the classical constraint on the edge of a graph with capacity $c > 0$ as the semimodule $\mathbb{M} = [0,\infty]/(c,\infty]$ with the quotient identifying all numbers greater than c with ∞. This yields an algebraic structure that resembles the closed interval $[0,c]$ under addition, with the proviso that exceeding c yields the failstate ∞. Our goal semimodules is to encode (as we show show in §IX) examples of nonlinear, nonconvex constraints.

Some of the results pertaining to the topological MFMC require an additional assumption. The semimodule is **semidivisible** if every pair of factorizations of an element not the algebraic zero admits a common refinement. The reason for this definition is to ensure that directed homology models flows on networks with possibly multiple incoming and outgoing edges at each edge.

Examples of interesting semidivisible semimodules include certain quotients of $[0,\infty]^n$ under vector addition, collections of sets under $\cup$ but closed under taking subsets, similarly certain lattices under join, and certain probability densities under convolution.

Motivated by the applications, a **capacity sheaf** on a graph G will mean a cellular sheaf on G assigning to each vertex and edge a quotient of a fixed semidivisible semimodule $\mathbb{M}$ by an ideal (subsemimodule closed under addition by elements in $\mathbb{M}$) and whose restriction morphisms are appropriate quotient maps. Such quotient semimodules are necessarily semidivisible by $\mathbb{M}$ semidivisible.

V. Poincaré duality

The kernel of the sheaf-theoretic MFMC is the following:

Flow-cut duality is a topological (Poincaré) duality on capacity sheaves.

One of the subtleties lies in the directedness of the base graph. Krishnan [6], [7] has developed a homology $\overrightarrow{H}_\bullet$ for directed spaces that respects the directeness. For example, the directed first homology of the directed graph $\overrightarrow{S}^1$ having one vertex and one edge, with coefficients in the natural numbers $\mathbb{N}$, is $\overrightarrow{H}_1(\overrightarrow{S}^1) \cong \mathbb{N}$.

Classical Poincaré duality holds for oriented manifolds. For more general settings, one still needs a local sense of orientation. We encode this into a sheaf. The **orientation sheaf**, $\mathcal{O}$, over a directed graph G is defined as a local directed sheaf homology. On each vertex e, of G, $\mathcal{O}(e) = \overline{\mathbb{N}}$. On each vertex v, $\mathcal{O}(v)$ is the commutative submonoid of the free commutative monoid $\mathbb{N}[E_v]$ generated by the set E_v of edges incident to v, generated by all edges e from v to v and all sums $e_1 + e_2$ of edges e_1 to v and e_2 from v.

The following result from [6] is fundamental:

Theorem 1. *For X a smooth directed space of uniform local dimension n, $\mathcal{F}$ a sheaf of semimodules, and $\mathcal{O}$ the local orientation sheaf,*

$$\overrightarrow{H}_p(X;\mathcal{F}) \cong H^{n-p}(X;\mathcal{O}\otimes\mathcal{F}), \tag{1}$$

for all $p = 0,\ldots,n$, with the isomorphism natural.

In this paper, $n = p = 1$, the left side encodes feasible flows, and the right side implicates cut values. This isomorphism is the source of flow-cut duality. Here H^0 is a variant of the ordinary global sections functor appropriate for semimodules.

VI. Topological flows

Flows are directed homology classes of the capacity sheaf.

The cycle condition for homology enforces conservation of the flow. The key word above is *directed* — since the conservation property of flows states that the sum of incoming flow values equals the sum of outgoing flow values at each node, the directedness of the graph is essential and must be 'programmed' into the problem via directed homology $\overrightarrow{H}_\bullet(X;\mathcal{F})$, which takes into account extra directed structure on X, with coefficients in a capacity sheaf $\mathcal{F}$. Here, X denotes the augmentation of G by an extra feedback edge e_* from target to source (with infinite capacity), $\mathcal{F}$ is a capacity sheaf on X, and a **flow** is a class in $\overrightarrow{H}_1(X;\mathcal{F})$ that is not the algebraic zero. The capacity constraints are automatically satisfied since they are built into the capacity sheaf $\mathcal{F}$.

The **flow value** can be defined in terms of the restriction of the flow 1-cycle to the feedback edge. However, this leaves flow values and cut values incomparable. The delicate operation is to use Theorem 1 to compare flow and cut values.

VII. Topological cuts

Cut values are represented by cohomology classes.

Given a cut C and feedback edge e_* from target to source, the cut value equals the quotient of $\mathcal{F}(e_*)$ by the smallest congruence equating all values in $\mathcal{F}(e_*)$ whose every lift to a global section of $\mathcal{O}\otimes\mathcal{F}$ represents an algebraic zero in

$$H^0(C;\mathcal{F}\otimes\mathcal{O}) \tag{2}$$

For example, when $\mathcal{F}$ is the capacity sheaf encoding classical scalar constraints, such a quotient semimodule is of the form $[0,\infty]/(k_C,\infty]$ for k_C the value of a minimal cut C. Moroever, such a quotient can also be expressed as a quotient of the set $H^0(C;\mathcal{F}\otimes\mathcal{O})\setminus\infty$; thus cohomology classes lift cut values.

The computation of cut values is difficult in general. For certain classes of capacity sheaves (including the scalar case), the notion of classical cuts coincides and the computation of cut values is a straightforward summation of capacities.

VIII. Optimization

The final ingredient is the proper interpretation of max and min. These operations generalize in some settings to union and intersection, respectively. In the most general setting, max becomes a categorical colimit and min a categorical limit. In its most general form, the MFMC becomes:

colimit over flows of flow values	**equals**	**limit over cuts of cut values**

The sheaf-theoretic MFMC therefore describes the colimit (generalized union) of all flow values – the semimodule of feasible flow values – as the limit (generalized intersection) over all cuts of cut values minus their algebraic zeros. These semimodules of values need not be totally orderable; hence there is no well-defined "max" among flow values.

The exchangeability of limits and colimits is highly sensitive to the particular algebraic conditions on the sheaf, crucial in avoiding phenomena described in the scalar setting as **duality gaps**. Certain variants of injective resolutions, adapted for sheaves of semimodules, can extend the above flow-cut duality for more general sheaves. We emphasize, however, that there are no duality gaps in topology – there are merely obstructions to computing/exchanging limits and colimits.

At this point, the reader may wish to contemplate some explicit examples.

IX. Multicommodity flows

Our first example generalizes MFMC to systems with multiple commodities. This requires merely a change of the capacity sheaf $\mathcal{F}$: instead of considering quotients of $[0,\infty]$, we consider quotients of $[0,\infty]^n$ for some fixed $n>1$.

We note that such quotients encompass certain nonlinear and nonconvex capacity constraints, as in Figure 1. Nevertheless, even with this type of non-linear and nonconvex constraint, MFMC duality holds, but with a 'max' flow that is not numerical, but rather the semimodule of feasible flow values. The literature often scalarizes the problem [8]. By dualizing first, and then scalarizing if needed, we recover a fuller picture of flow values. To use the MFMC requires that one can effectively compute the dual cut capacities. This turns out to be simple.

Theorem 2. *The value of a cut in a network with capacity sheaf taking values in quotient semimodules of $[0,\infty]^n$ is the Minkowski sum of the semimodules in $[0,\infty]^n$. The limit of the cut values consists of taking the intersection of the cut values over all cuts.*

In other words, the union over all possible flows of the flow values equals the intersection over all possible cuts of the Minkowski sums of the capacities. See Figure 1 for an example with two commodities and nonlinear/nonconvex constraints.

X. Multisource/target flows

Our second example is a modification of the first, adapted to a multi-source and multi-target system. Assume that G has sources $\{s_i\}$ and targets $\{t_i\}$ for $i=1,\ldots,N$. In the traditional form of the problem, all (directed) edges are given a numerical capacity, flows from s_i to t_i are additive, and the goal is to maximize the flows with respect to some preset demand ratios (imposing a scalarization) [8].

We can reformulate this problem using a multicommodity capacity sheaf. For an edge with net capacity c, let the capacity sheaf be the quotient of $[0,\infty]^N$ by the subset $\sum_i x_i > c$. Modify the graph G by adding two vertices: a global source s_* and a global target t_*. Add edges from s_* to each s_i and from each t_i to t_*, with a single feedback edge e_* from $t_*\to s_*$. Set the capacity sheaf to be $[0,\infty]^n$ on the feedback edge, and set the sheaf to be $[0,\infty]$ on all the other edges from s_* and into t_*, with the restriction maps being projection to the i^{th} factor. Thus, only the i^{th} commodity type can pass through s_i and t_i, but is otherwise additive on edges.

As with the previous example, there is no single notion of maximal flow without an explicit scalarization. More general settings are possible. One can encode constraints at the nodes (router capacities) or bias certain edges with respect to certain

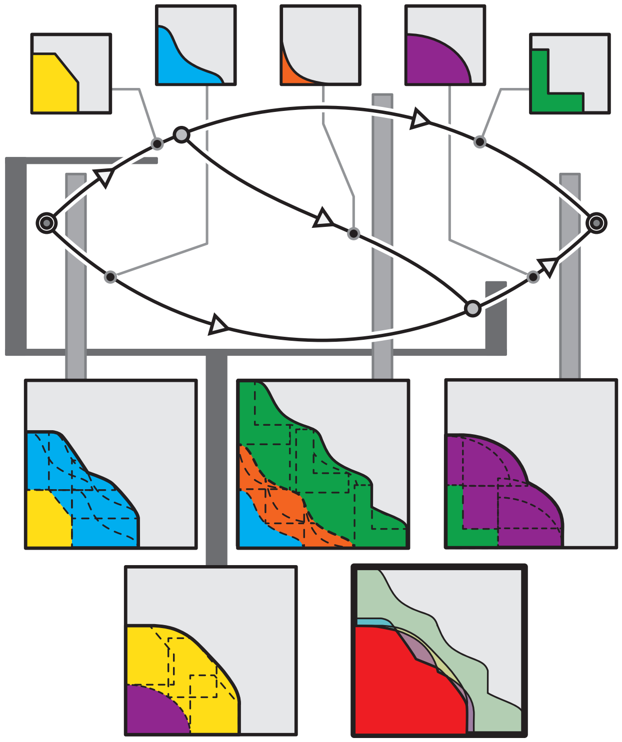

Fig. 1. A directed graph with 5 edges and 2-commodity capacity semimodules expressing nonlinear and nonconvex constraints [top]. Flows from the source (left) to target (right) admit 4 minimal edge cuts, each of whose cut value is the Minkowski sum of the edge capacity semimodules [middle, bottom, left]. The feasible ('min') flow values is the intersection of the four cut values [bottom, right].

commodities by programming the restriction maps to be something other than projections to factors.

XI. Logical flows

Our final example is to flows of logical statements, where one encodes operations on statements by means of an algebraic lattice. Consider, for simplicity, a finite alphabet $\mathcal{A}$ of variables and the Boolean algebra generated by $\mathcal{A}$ under the operations $\{\cup, \cap\}$. Then, compatible flows implicate $\cap$ and cut values are found by applying $\cup$ to the edge capacities (*cf.* the Minkowski sum). The flow-cut duality becomes a $\cup\cap - \cap\cup$ duality.

For an explicit example, consider the graph of Figure 2 with edges labeled by subsets of $\mathcal{A} = \{A, B, C, D\}$. The capacity sheaf over such an edge is the semimodule of the powerset $\mathcal{P}$ of the edge labels under the operation of union. This semimodule encodes which elements are *legal* to travel across the edge via a logical *or*. The reader can see that there are three possible flow paths from source to target. For each such flow path, the feasible flow value is the intersection of the edge capacities as sets (encoding a logical *and*). Given a cut, the cut value is the union of the edge capacities over the cut. Flow-cut duality becomes: *the set of feasible flows equals the intersection over all cuts of the union of edge capacities over the cut*.

In this example, it is obvious that the net feasible flow

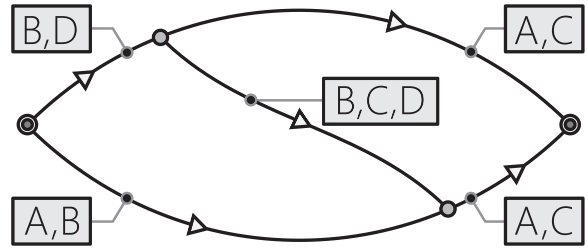

Fig. 2. The feasible flow of a lattice-valued capacity sheaf is A, whereas the 'smallest' cut is generated by A and C. It is the *intersection* of the cut values that determines the feasible flow values A.

consists of A and $0 = \emptyset$. The reader may be puzzled by this, since there is no single 'minimal' cut whose cut value equals A; indeed, the smallest cut has value $\mathcal{P}(A, C)$. However, the proper topological lifting (or *categorification*) of max/min is to union/intersection. Duality gaps, here and elsewhere, are failures to decategorify properly.

XII. Concluding remarks

This introduction to the sheaf theoretic MFMC [7] is necessarily brief. We note the following:

1) The problem of efficient computation of cut values and limits thereof is important and untouched. Our goal in this note is simply to explain the duality theorem.
2) There are other commutative operations besides vector addition and union. Dualities based on signals or probability densities under convolution and lattices under $\vee$ are potentially very useful.
3) Sheaves can encode linear transformations at nodes, accommodating network coding, reaction balancing (stoichiometry), and other constraints expressible as homomorphisms.
4) Non-directed networks can be handled with sufficient modification to the graph.
5) The most exciting open directions involve higher-dimensional base spaces and a purely topological approach to LP duality.

Acknowledgement

This work is supported by federal contracts FA9550-12-1-0416 and FA9550-09-1-0643.

References

[1] S. Awodey, *Category Theory*, 2nd ed., Oxford Press, 2010.
[2] E. Chambers, J. Erickson, and A. Nayyeri, "Homology flows, cohomology cuts." *SIAM J. Computing*, 41(6), 2012, 1605–1634.
[3] S. Even, A. Itai, and A. Shamir, "On the Complexity of Timetable and Multicommodity Flow Problems". *SIAM J. Comput.* 5(4), 1976, 691–703.
[4] A. Frieze, "Algebraic flows." *Algebraic and Combinatorial Methods in Operations Research*, 95, 1984, 135–146.
[5] A. Hatcher, *Algebraic Topology*. Cambridge University Press, 2001.
[6] S. Krishnan, "Directed Poincaré Duality", preprint, Aug 2013.
[7] S. Krishnan, "Max-flow-min-cut for sheaves on graphs", preprint, Aug 2013.
[8] T. Leighton and S. Rao, "Multicommodity Max-Flow Min-Cut Theorems and Their Use in Designing Approximation Algorithms", *J. ACM*, 46(6), 1999, 787–832.