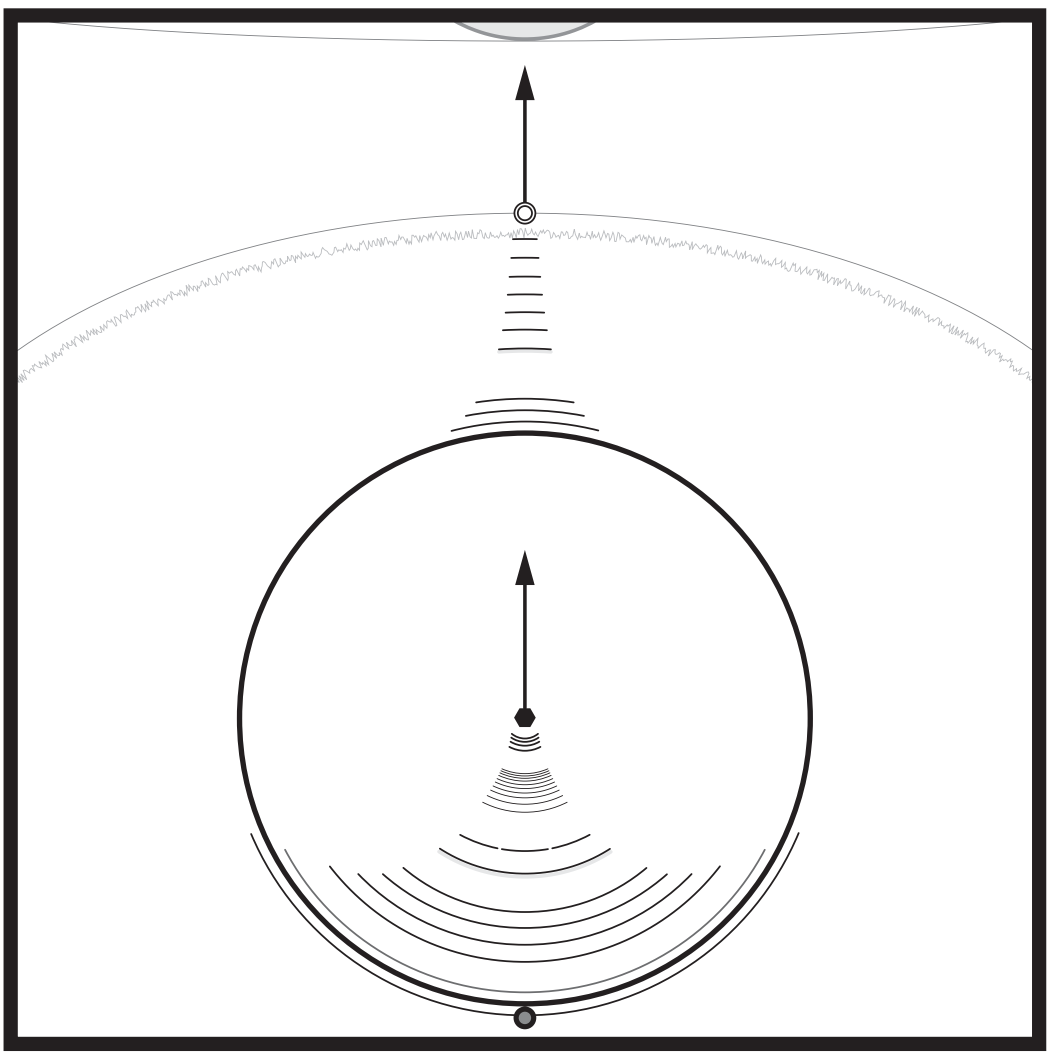
Chapter 3

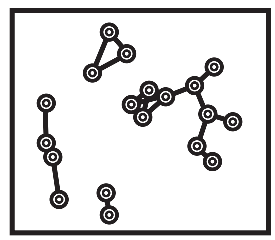
Euler Characteristic



The simplest and most elegant non-obvious topological invariant is the Euler characteristic, an integer-valued invariant of suitably nice spaces. This chapter is comprised of observations and applications of this invariant, meant to serve as motivation for the algebraic tools to follow that justify its remarkable properties.

3.1 Counting

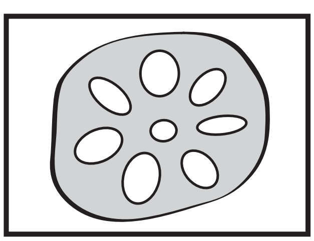
Euler characteristic is a generalization of counting. Given a finite set X, the Euler characteristic is its cardinality $\chi(X) = |X|$. Connect two points together by means of an edge (in a cellular/simplicial structure); as the resulting space has one fewer component, the Euler characteristic is decreased by one. Continuing inductively, the Euler characteristic counts vertices with weight $+1$ and edges with weight -1.



This intuition of counting connected components works at first; however, for certain examples, the addition of an edge does not change the count of connected components. Note that this occurs precisely when a cycle is formed. To fill in such a cycle in the figure with a 2-cell would return to the setting of counting connected components again, suggesting that 2-cells be weighted with $+1$. This intuition of counting with weights inspires the combinatorial definition of Euler characteristic. Given a space X and a partition thereof into a finite number of open cells $X = \sqcup_\alpha \sigma_\alpha$, where each k-cell σ_α is homeomorphic to $\mathbb{R}^k$, the **Euler characteristic**[1] of X is defined as

$$\chi(X) := \sum_\alpha (-1)^{\dim \sigma_\alpha}. \tag{3.1}$$

This quantity is well-defined for a reasonably large class of spaces (see §3.5) and is independent of the decomposition of X into cells: χ is a homeomorphism invariant. It is *not* a homotopy invariant for non-compact cell complexes, as, *e.g.*, it distinguishes $\chi((0,1)) = -1$ from $\chi([0,1]) = 1$. Among compact finite cell complexes, χ *is* a homotopy invariant, as will be shown in Chapter 5.



Euler characteristic can determine the homotopy type of a compact connected graph; *e.g.*, such is a tree (a contractible graph) if and only if $\chi = 1$. Euler characteristic is also sharp invariant among connected compact orientable 2-manifolds: $\chi = 2 - 2g$, where g equals the genus. Any compact convex subset of $\mathbb{R}^n$ has $\chi = 1$. Removing k disjoint convex open sets from such a convex superset results in a compact space with Euler characteristic $1 - k(-1)^n$.

Example 3.1 (Configuration spaces of graphs) ⊙

Recall from Example 2.11 that the discrete configuration space $\mathcal{D}^2(K_5)$ of two points on a complete connected graph of five vertices is a cubical 2-manifold. Being built from finitely many cells, it is clearly compact. It is also orientable and connected, as the reader may check. To determine the genus of this configuration space, one

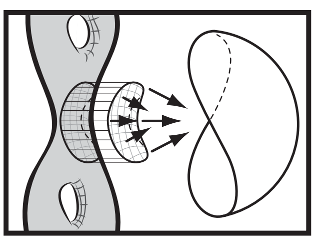
[1] This is sometimes called the *combinatorial*, *geometric*, or *motivic* Euler characteristic.

computes the Euler characteristic. Vertices of $\mathcal{D}^2(K_5)$ correspond to ordered pairs of distinct vertices of K_5, of which there are $(5)(4) = 20$. Edges correspond to pairs of one closed edge and one disjoint vertex: there are $(2)(10)(5-2) = 60$ such. Faces correspond to ordered pairs of closure-disjoint edges in K_5, of which there are $(10)(3) = 30$. Thus, $\chi(\mathcal{D}^2(K_5)) = 20 - 60 + 30 = -10$ and this surface has genus $g = 1 - \frac{1}{2}\chi = 6$. ⊙

The remainder of this chapter complements the enumerative interpretation of χ with geometric, dynamical, analytic, and probabilistic perspectives.

3.2 Curvature

A blend of Euler characteristic and integration is prevalent in geometry. The classical result that initiated the subject is the **Gauss-Bonnet Theorem**. Let M be a smooth surface embedded in $\mathbb{R}^3$. The **Gauss map** is the map $\gamma\colon M \to \mathbb{S}^2$ that associates to each point of M the direction of the unit vector normal to M in $\mathbb{R}^3$. The **Gauss curvature** $\kappa = \det(D\gamma)$ is the determinant of the derivative of the Gauss map. Note that the curvature is a geometric quantity: rigid translations and rotations leave it invariant, but stretching and deformation of M change κ. This change is local, but not global, thanks to the classical:



Theorem 3.2 (Gauss-Bonnet Theorem). *For M a compact smooth oriented surface in $\mathbb{R}^3$, the integral of Gauss curvature with respect to area on M equals*

$$\int_M \kappa \, dA = 2\pi\chi(M). \tag{3.2}$$

That this is the beginning of a much larger story is evidenced by the following mild generalization. Let M be a compact oriented surface in $\mathbb{R}^3$, piecewise smooth over a cell structure, perhaps with piecewise-smooth boundary curve(s). Again, $2\pi\chi(M)$ equals the integral of a certain curvature over M, but this curvature measure, like M, is stratified.

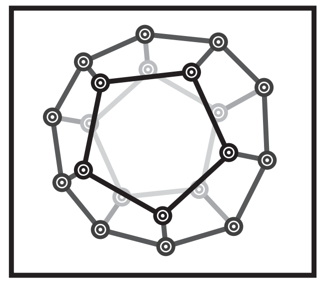
1. On 2-cells of M, $d\kappa$ means $\kappa \, dA$, Gauss curvature times the area element;
2. On 1-cells of M, $d\kappa$ means $k_g \, ds$, geodesic curvature times the length element;
3. On 0-cells of M, $d\kappa$ means the angle defect.

With this interpretation, the Gauss-Bonnet formula can be written as

$$\int_M d\kappa = \int_{M^{(0)}} d\kappa + \int_{M^{(1)}} d\kappa + \int_{M^{(2)}} d\kappa = 2\pi\chi(M), \tag{3.3}$$

i.e., the integral over the 2-cells of Gauss curvature plus the integral over 1-cells of geodesic curvature, plus the sum over 0-cells of angle defect equals $2\pi\chi(M)$. There are several corollaries of this result relevant to discrete and differential geometry:

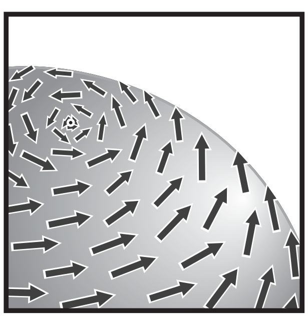
1. A smooth, closed surface has total (integrated) Gauss curvature constant, no matter how the surface is deformed.
2. For a geodesic triangle, $d\kappa$ vanishes along the geodesic edges and the sum of the angles of the triangle equals π plus the integral of Gauss curvature over the triangle face. This recovers classical notions of angle-sums for triangles on spheres and other curved surfaces.
3. For M the boundary of a compact convex polyhedron in $\mathbb{R}^3$, all faces and edges are flat, $\chi(M) = \chi(\mathbb{S}^2) = 2$, and the sum of the vertex angle defects (in this case, 2π minus the sum of the face angles) equals 4π.



The reader may rightly suspect whether an embedding in $\mathbb{R}^3$ is required. Recall, again, from Example 2.11, the cubical 2-manifold built from arranging six squares around each of 20 vertices. Placing a flat Euclidean metric on each square yields a space with an intrinsic geometry, locally flat except at the vertices. Since each vertex has angle defect $2\pi - 6\frac{\pi}{2} = -\pi$, Gauss-Bonnet implies that the surface has Euler characteristic $\chi = 20(-\pi)/2\pi = -10$: *cf.* Example 3.1.

3.3 Nonvanishing vector fields

Euler characteristic has many uses, one of which is as an **obstruction** to the existence of certain vector fields. Recall from §1.4 that a vector field is a continuous assignment of a tangent vector to each point in a manifold, and that a vector field vanishes at its zeros.

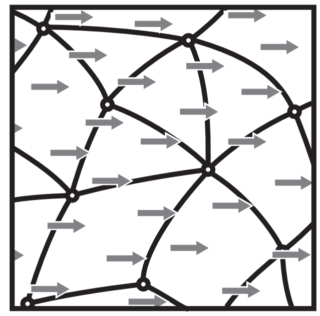


Theorem 3.3 (Hairy Ball Theorem). *A connected compact manifold M without boundary possesses a nonvanishing vector field if and only if $\chi(M) = 0$.*

This is not terribly useful in dimensions 1 and 3, since *all* compact manifolds in these dimensions have Euler characteristic zero. However, since $\chi(\mathbb{S}^2) = 2$, there are no fixed-point-free vector fields on a 2-sphere.

Proof. *(sketch)* The interesting direction (only if) reveals the obstructive nature of χ. The easiest proof without invoking advanced tools involves imposing a dense cellular mesh and working cell-by-cell, assuming smoothness when needed. Assume that M is a compact closed n-manifold that admits a smooth nonvanishing vector field V. Place a smooth cell structure on M with all cells represented in charts as convex polytopes of sufficiently small diameter. Invoking transversality and refining as needed, perturb the cell structure so that the vector field V is transverse to all cells. As the cells are sufficiently small and the vector field V is nonvanishing, V is approximately linear in a neighborhood of each cell. On each n-dimensional cell σ, consider the collection, $I(\sigma)$, of open faces of σ on which V points to the interior of σ.

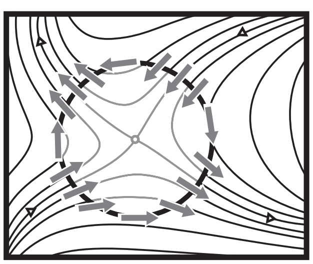
By transversality, $I(\sigma)$ is nonempty for each cell σ and contains both the interior cell of σ (homeomorphic to an open disc D^n) along with some boundary faces, the union of which (by linearity of V and convexity of the polyhedral cell structure) is homeomorphic to the open disc D^{n-1}. Thus, $\chi(I(\sigma)) = (-1)^n + (-1)^{n-1} = 0$ and, since the sets $I(\sigma)$ partition all cells of M, $\chi(M) = \sum_\sigma \chi(I(\sigma)) = 0$. ⊙



These invocations of transversality and convexity are admittedly glib. With better tools (from Chapters 4-5), it will be possible to drop the assumptions about smoothness, manifold, and cell structures: see Theorem 5.19. Note, however, how the proof constructed a type of local *index*, $I(\sigma)$, and built up a global inference by means of a sort of integration. These themes are recurrent in topology.

3.4 Fixed point index

Euler characteristic is the basis for numerous topological indices, the simplest of which applies to vector fields. Consider a vector field V on an oriented 2-manifold Σ. Let $p \in \mathrm{Fix}(V)$ be an isolated fixed point of V. Let B_p denote a sufficiently small ball about p with boundary $\gamma = \partial B_p$. The **index** of V at p, $\mathcal{I}_V(p)$, is defined to be the following line integral:



$$\mathcal{I}_V(p) := \frac{1}{2\pi} \oint_\gamma d\theta_V, \tag{3.4}$$

where θ_V is the angle made by V in local coordinates. More specifically, if in (x, y) coordinates based at p, the vector field is of the form $V = (v_x, v_y)$, then the integrand $d\theta_V$ is

$$d\theta_V = \frac{v_x\,dy - v_y\,dx}{v_x^2 + v_y^2},$$

so that $\mathcal{I}_V(p)$ represents the (signed integer) number of turns the vector makes in a small curve about p. This index is well-defined and independent of (a sufficiently small) B_p, thanks to Green's Theorem. Among the nondegenerate fixed points, sources and sinks have index $+1$; saddle points have index -1. It is clear that Equation (3.4) extends to define an index $\mathcal{I}_V(\gamma)$ to any closed curve γ which avoids $\mathrm{Fix}(V)$. One argues (in a manner not unlike that used in Green's Theorem or in contour integration) that index is additive. Let D be a disc whose boundary $\gamma = \partial D$ avoids $\mathrm{Fix}(V)$. Then,

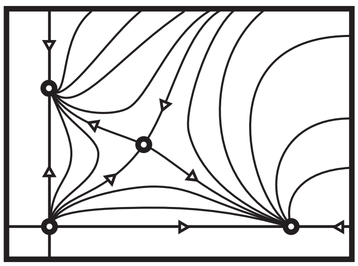
$$\mathcal{I}_V(\gamma) = \sum_{p \in D \cap \mathrm{Fix}(V)} \mathcal{I}_V(p). \tag{3.5}$$

This important result will be revisited and reinterpreted in §5.10 and §7.7.

Example 3.4 (Population dynamics) ⊙

Consider the following differential equation model of competing species with (normalized) population sizes $x(t)$, $y(t)$ as functions of time:

$$\frac{dx}{dt} = 3x - x^2 - 2xy \quad ; \quad \frac{dy}{dt} = 2y - xy - y^2. \tag{3.6}$$

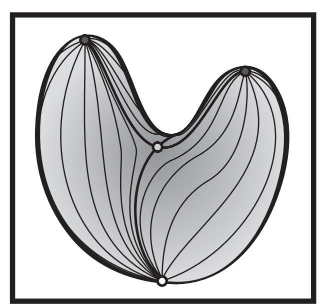


The equilibrium solutions consist of $(0,0)$ [mutual death], $(3,0)$ [species x survives], $(0,2)$ [species y survives], and $(1,1)$ [coexistence]. Linearization reveals that the coexistence solution is a saddle point; thus, $\mathcal{I}(1,1) = -1$. It is clear (from the equations and from the interpretation) that the x and y axes are invariant sets. Can there be a periodic orbit? *No.* The index of a periodic orbit γ must by Equation (3.4) equal $\mathcal{I}_V(\gamma) = +1$, since γ is everywhere tangent to the vector field. However, by Equation (3.5), γ must surround a collection of fixed points whose indices sum to $+1$. If $\gamma(t)$ is a nontrivial periodic solution, then it cannot intersect the fixed point set or the (invariant) x or y axes, and the only remaining enclosable fixed point has negative index. ⊙

The additivity present in Equation (3.5) is reminiscent of the Euler characteristic. The rationale for this equation (and the proper definition of the index of a vector field in all dimensions) will become clearer in Chapters 4-7: see Example 4.23, §5.10, and §7.7. The following classical theorem is a hint of these deeper connections:

Theorem 3.5 (Poincaré-Hopf Theorem). *For a continuous vector field V with isolated fixed points on a compact manifold M,*

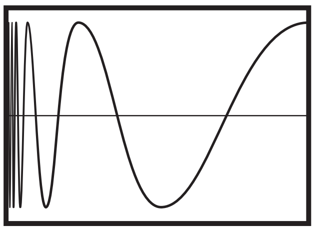
$$\sum_{\mathrm{Fix}(V)} \mathcal{I}_V(p) = \chi(M). \tag{3.7}$$



The proofs of Theorems 3.3 and 3.5 compute χ locally (on cells and fixed points respectively), then add up this local data to return the global χ. This *integrative* technique motivates an extension of Euler characteristic from sets to certain functions over sets: an integration theory. Its definition requires an excursion into exactly which sets have a well-defined Euler characteristic.

3.5 Tame topology

Euler characteristic is well-defined for spaces with a decomposition into a finite number of cells. Though an explicit cell structure is often present in, say, the simplicial setting, not all *"organic"* spaces come with a natural cell decomposition: *e.g.*, configuration space of points in a domain, level sets of a smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$, *etc.* Besides the lack of an explicit cell structure, worse things can occur, as with the graph of $\sin(1/x)$ for $0 < x < 1$. Though this is homeomorphic to an interval, it is a wild type of equivalence, as the closure of this set in $\mathbb{R}^2$ is *not* homeomorphic to a closed interval. In applications, one wants to avoid such oddities and focus on spaces (and mappings) that are for all intents and purposes *tame*.



Different mathematical communities focus on, *e.g.*: piecewise linear (PL) spaces, describable in terms of affine sets and matrix inequalities; **semialgebraic** sets, expressible in terms of a finite set of polynomial inequalities; or **subanalytic** sets, defined in terms of images of analytic mappings [277]. Logicians have created an axiomatic reduction of such classes of sets in the form of an **o-minimal structure**.[2]

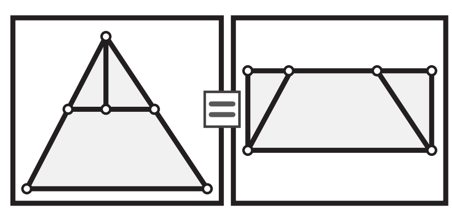
An o-minimal structure $\mathcal{O} = \{\mathcal{O}_n\}$ (over $\mathbb{R}$) is a sequence of Boolean algebras $\mathcal{O}_n$ of subsets of $\mathbb{R}^n$ (families of sets closed under the operations of intersection and complement) which satisfies certain axioms:

1. $\mathcal{O}$ is closed under cartesian products;
2. $\mathcal{O}$ is closed under axis-aligned projections $\mathbb{R}^n \to \mathbb{R}^{n-1}$;
3. $\mathcal{O}_n$ contains diagonals $\{(x_k)_1^n : x_i = x_j\}$ for each $i \neq j$;
4. $\mathcal{O}_2$ contains the subdiagonal $\{x_1 < x_2\}$; and
5. $\mathcal{O}_1$ consists of all finite unions of points and open intervals.

Elements of $\mathcal{O}$ are called **tame** or, more properly, **definable** sets. Canonical examples of o-minimal structures are semialgebraic sets and subanalytic sets. The finiteness of the final axiom is the crucial piece that drives the theory.

Given a fixed o-minimal structure, one can work with tame sets with relative ease. Tame mappings are likewise easily defined: a (not necessarily continuous) function between tame spaces is *tame* (or *definable*) if its graph (in the product of domain and range) is a tame set. A **definable homeomorphism** is a tame bijection between tame sets. To repeat: *definable homeomorphisms are not necessarily continuous.* Such a convention makes the following theorem concise:

Theorem 3.6 (Triangulation Theorem). *Any definable set is definably homeomorphic to a finite disjoint union of open standard simplices. The intersection of the closures of any two of the simplices in this definable triangulation is either empty, or the closure of another open simplex in the triangulation.*



This result implies that tame sets always have a well-defined Euler characteristic and a well-defined dimension (the max of the dimensions of the simplices in a triangulation). The surprise is that these two quantities are not only topological invariants with respect to definable homeomorphism, they are *complete* invariants.

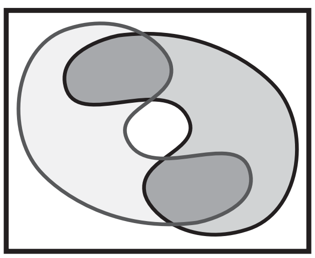
[2]The term derives from *order minimal*, in turn coming from model theory. The text of Van den Dries [293] is a beautifully clear reference.

Theorem 3.7 ([293]). *Two definable sets in an o-minimal structure are definably homeomorphic if and only if they have the same dimension and Euler characteristic.*

This result reinforces the idea of a definable homeomorphism as a **scissors equivalence**. One is permitted to cut and rearrange a space with abandon. Recalling the utility of such scissors-work in computing areas of planar sets, the reader will not be surprised to learn of a deep relationship between tame sets, the Euler characteristic, and integration.

3.6 Euler calculus

It is possible to build a topological calculus based on Euler characteristic. The integral in this calculus depends on the following **additivity**:



Lemma 3.8. *For A and B definable sets,*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{3.8}$$

This additivity has been foreshadowed in the imagery of χ as counting in §3.1, in the Gauss-Bonnet Theorem of §3.2, in the proof of Theorem 3.3, and in the additivity of the fixed point index of §3.4. The reader may prove (3.8) via triangulation, induction, and toil, until the more automatic homological tools of Chapters 4-5 are available. The similarity between Equation (3.8) and the definition of a measure is no coincidence. Following ideas that date back to Blaschke (at least), one constructs a measure[3] $d\chi$ over definable sets $A \subset X$ via:

$$\int_X \mathbb{1}_A \, d\chi := \chi(A). \tag{3.9}$$

Measurable functions in this integration theory are integer-valued and **constructible**, meaning that for $h\colon X \to \mathbb{Z}$, all level sets $h^{-1}(n) \subset X$ are tame. Denote by $\mathrm{CF}(X)$ the set of bounded compactly supported constructible functions on X. The **Euler integral** is defined to be the homomorphism $\int_X \colon \mathrm{CF}(X) \to \mathbb{Z}$ given by:

$$\int_X h \, d\chi = \sum_{s=-\infty}^{\infty} s\chi(\{h = s\}) = \sum_{s=0}^{\infty} \chi(\{h > s\}) - \chi(\{h < -s\}), \tag{3.10}$$

where the last equality is a manifestation of a discrete fundamental theorem of integral calculus.[4] Alternately, using the definition of tame sets, one may write $h \in \mathrm{CF}(X)$ as $h = \sum_\alpha c_\alpha \mathbb{1}_{\sigma_\alpha}$, where $c_\alpha \in \mathbb{Z}$ and $\{\sigma_\alpha\}$ is a decomposition of X into a disjoint union of open cells, then

$$\int_X h \, d\chi = \sum_\alpha c_\alpha \chi(\sigma_\alpha) = \sum_\alpha c_\alpha (-1)^{\dim \sigma_\alpha} \tag{3.11}$$

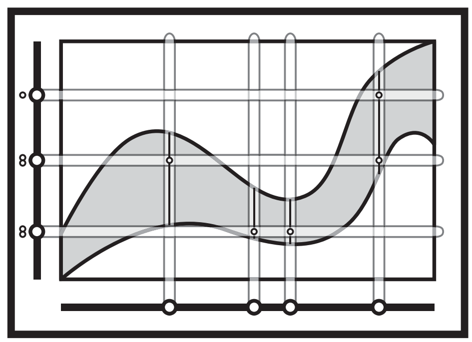
[3] The proper term is a **valuation**, not a measure; the abuse of terminology is to prompt the reader to think explicitly in terms of integration theory.

[4] That is, a telescoping sum.

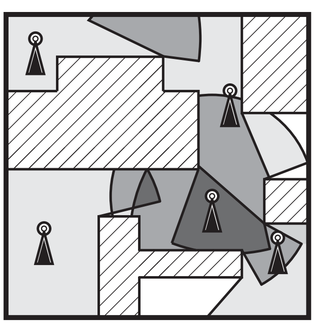
That this sum is invariant under the decomposition into definable cells is a consequence of Lemma 3.8 and Theorem 3.7.

3.7 Target enumeration

A simple application of Euler integration to data aggregation demonstrates the utility of this calculus. Consider a finite collection of **targets**, represented as discrete points in a space W. Assume a field of sensors, each of which observes some subset of W and counts the number of targets therein. The sensors will be assumed to be distributed over a region so densely as to be approximated by a topological space X.



There are many modes of sensing: infrared, acoustic, optical, magnetometric, and more are common. To best abstract the idea of sensing away from the engineering details, the following topological approach is used. In a particular system of sensors in X and targets in W, let the **sensing relation** be the relation $\mathcal{S} \subset W \times X$ where $(w, x) \in \mathcal{S}$ iff a sensor at $x \in X$ detects a target at $w \in W$. The horizontal and vertical **fibers** (inverse images of the projections of $\mathcal{S}$ to X and W respectively) have simple interpretations. The vertical fibers – **target supports** – are those sets of sensors which detect a given target in W. The horizontal fibers – **sensor supports** – are those target locations observable by a given sensor in X.



Assume that the sensors are additive but anonymizing: each sensor at $x \in X$ counts the number of targets in W detectable and returns a local count $h(x)$, but the identities of the sensed targets are unknown. This counting function $h\colon X \to \mathbb{Z}$ is, under the usual tameness assumptions, constructible. A natural problem in this context is to aggregate the redundant anonymous target counts: given h and some minimal information about the sensing relation $\mathcal{S}$, determine the total number of targets in W. This is not as easy as it sounds, and it is even less easy when X is not a continuum space, but rather a discretization thereof. It is therefore remarkable that a purely topological solution exists, independent of knowing a decomposition of the counting function h.

Proposition 3.9 ([24]). *If $h\colon X \to \mathbb{N}$ is a counting function for target supports U_α of uniform Euler characteristic $\chi(U_\alpha) = N \neq 0$ for all α, then*

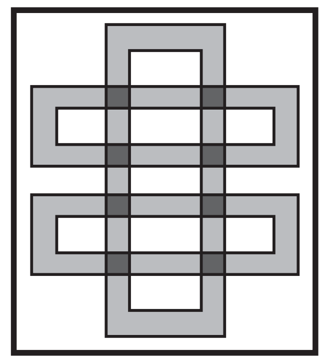
$$\#\alpha = \frac{1}{N} \int_X h \, d\chi.$$

Proof.

$$\int_X h\,d\chi = \int_X \left(\sum_\alpha \mathbb{1}_{U_\alpha}\right) d\chi = \sum_\alpha \int_X \mathbb{1}_{U_\alpha}\,d\chi = \sum_\alpha \chi(U_\alpha) = N\,\#\alpha.$$

⊙

For contractible supports (such as in the setting of beacons visible on star-convex domains), the target count is, simply, the Euler integral of the function. This solves a problem in the aggregation of redundant data, since many nearby sensors with the same reading are detecting the same targets; in the absence of target identification (an expensive signal processing task), it is nontrivial to aggregate the redundancy. Notice that the restriction $N \neq 0$ is nontrivial. If $h \in \mathrm{CF}(\mathbb{R}^2)$ is a finite sum of characteristic functions over annuli, it is not merely inconvenient that $\int_{\mathbb{R}^2} h\,d\chi = 0$, it is a fundamental obstruction to disambiguating sets. Given this, it is all the more remarkable that sets with $\chi \neq 0$ can be enumerated easily. Since the solution is in terms of an integral, local and distributed computations may be used in practice.

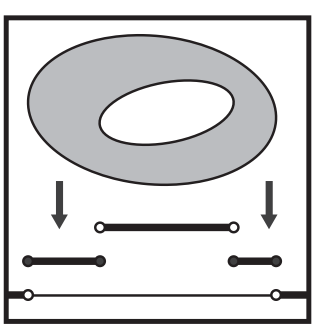


3.8 A Fubini Theorem

Euler characteristic is like a volume in another aspect: it is multiplicative under cartesian products.

Lemma 3.10. *For X and Y definable, $\chi(X \times Y) = \chi(X)\chi(Y)$.*

Proof. The product $X \times Y$ has a definable cell structure using products of cells from X and Y. For cells $\sigma \subset X$ and $\tau \subset Y$, the lemma holds via the exponent rule since $\dim(\sigma \times \tau) = \dim \sigma + \dim \tau$. Additivity of the integral completes the proof. ⊙



The assertion that $d\chi$ should be regarded as an honest topological measure is supported by this fact and its corollary: the Euler integration theory admits a Fubini Theorem. Calculus students know that:

$$\iint f(x,y)dx\,dy = \iint f(x,y)dy\,dx.$$

Real-analysis students learn to pay attention to finer assumptions on measurability (*cf.* tameness assumptions). This familiar result is the image of a deeper truth about integrations and projections. Given $F\colon X \to Y$, one can integrate over the **fibers**, or level sets, of F first, then integrate the resulting function over the projected **base** Y.

Theorem 3.11 (Fubini Theorem). *Let $F\colon X \to Y$ be a definable mapping. Then for all $h \in CF(X)$,*

$$\int_X h\,d\chi = \int_Y \left(\int_{F^{-1}(y)} h(x)\,d\chi(x) \right) d\chi(y). \tag{3.12}$$

Proof. If $X = U \times Y$ and F is projection to the second factor, the result follows from Lemma 3.10. The o-minimal Hardt Theorem [293] says that Y has a definable partition into tame sets Y_α such that $F^{-1}(Y_\alpha)$ is definably homeomorphic to $U_\alpha \times Y_\alpha$ for U_α definable, and that $F\colon U_\alpha \times Y_\alpha \to Y_\alpha$ acts via projection. Additivity of the integral completes the proof. ⊙

Example 3.12 (Enumerating vehicles) ⊚

The following example demonstrates an application of the Fubini Theorem to time-dependent targets. Consider a collection of vehicles, each of which moves along a smooth curve $\gamma_i\colon [0,T] \to \mathbb{R}^2$ in a plane filled with sensors that count the passage of a vehicle and increment an internal counter. Specifically, assume that each vehicle possesses a *footprint* – a support $U_i(t) \subset \mathbb{R}^2$ which is a compact contractible neighborhood of $\gamma_i(t)$ for each i, varying tamely in t. At the moment when a sensor $x \in \mathbb{R}^2$ detects when a vehicle comes within proximity range – when x crosses into $U_i(t)$ – that sensor increments its internal counter. Over the time interval $[0,T]$, the sensor field records a counting function $h \in \mathrm{CF}(\mathbb{R}^2)$, where $h(x)$ is the number of times x has entered a support. As before, the sensors do not identify vehicles; nor, in this case, do they record times, directions of approach, or any ancillary data.

Proposition 3.13 ([24]). *The number of vehicles is equal to $\int_{\mathbb{R}^2} h\,d\chi$.*

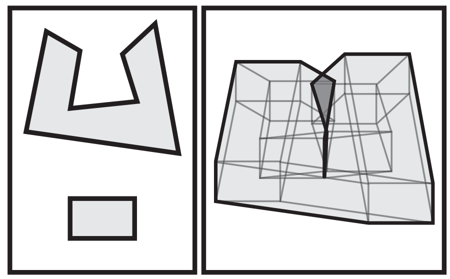
Proof. Each target traces out a compact tube in $\mathbb{R}^2 \times [0,T]$ given by the union of slices $(U_i(t), t)$ for $t \in [0,T]$. Each such tube has $\chi = 1$. The integral over $\mathbb{R}^2 \times [0,T]$ of the sum of the characteristic functions over all N tubes is, by Proposition 3.9, N, the number of targets. Consider the projection map $p\colon \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$. Since $p^{-1}(x)$ is $\{x\} \times [0,T]$, the integral over $p^{-1}(x)$ records the number of (necessarily compact) connected intervals in the intersection of $p^{-1}(x)$ with the tubes in $\mathbb{R}^2 \times [0,T]$. This number is precisely the sensor count $h(x)$ (the number of times a sensor detects a vehicle coming into range). By the Fubini Theorem, $\int_{\mathbb{R}^2} h\,d\chi$ must equal the integral over the full $\mathbb{R}^2 \times [0,T]$, which is N. ⊙⊚

3.9 Euler integral transforms

Euler integration admits a variety of operations which mimic the analytic tools so useful in signal processing, imaging, and inverse problems.

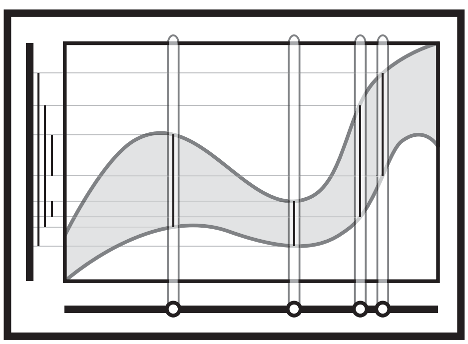
Example 3.14 (Convolution and Minkowski sum) ⊙

On a real vector space V, a **convolution** operation with respect to Euler characteristic is straightforward. Given $f, g \in \mathrm{CF}(V)$, one defines



$$(f * g)(x) := \int_V f(t)g(x-t)\, d\chi(t). \quad (3.13)$$

There is a close relationship between convolution and the **Minkowski sum**: for A and B convex, $\mathbb{1}_A * \mathbb{1}_B = \mathbb{1}_{A+B}$, where $A + B$ is the set of all vectors expressible as a sum of a vector in A and a vector in B [297, 269]. This, in turn, is useful in applications ranging from computer graphics to motion-planning for robots around obstacles [168]. For non-convex shapes, convolution of indicator functions can take on values larger than 1 in regions where the intersections of translations of A with B are disconnected. Unlike non-convex Minkowski sum, Euler-convolution is always invertible [27, 269]. ⊙



One of the most general integral transforms is the **Radon transform** of Schapira [48, 67, 270]. Consider a locally closed definable relation $\mathcal{S} \subset W \times X$ (that may or may not come from sensing), and let π_W and π_X denote the projection maps of $W \times X$ to their factors. The Radon transform with kernel $\mathcal{S}$ is the map $\mathcal{R}_{\mathcal{S}} \colon \mathrm{CF}(W) \to \mathrm{CF}(X)$ given by lifting $h \in \mathrm{CF}(W)$ from W to $W \times X$, filtering with the kernel $\mathbb{1}_{\mathcal{S}}$, then integrating along the projection to X as follows:

$$(\mathcal{R}_{\mathcal{S}} h)(x) := \int_W h(w)\mathbb{1}_{\mathcal{S}}(x, w)\, d\chi(w). \quad (3.14)$$

Example 3.15 (Target enumeration) ⊙

Consider the sensor relation $\mathcal{S} \subset W \times X$, and a finite set of targets $T \subset W$ as defining an indicator function $\mathbb{1}_T \in \mathrm{CF}(W)$. Observe that the counting function which the sensor field on X returns is precisely the Radon transform $\mathcal{R}_{\mathcal{S}}\mathbb{1}_T$. In this language, Proposition 3.9 is equivalent to the following: assume that $\mathcal{S} \subset W \times X$ has vertical fibers $\pi_W^{-1}(w) \cap \mathcal{S}$ with constant Euler characteristic N. Then, $\mathcal{R}_{\mathcal{S}} \colon \mathrm{CF}(W) \to \mathrm{CF}(X)$ scales integration by a factor of N: $\int_X \circ \mathcal{R}_{\mathcal{S}} = N \int_W$. ⊙

A similar regularity in the Euler characteristics of fibers allows a general inversion formula for the Radon transform [270]. One must choose an 'inverse' relation $\mathcal{S}' \subset X \times W$.

Proposition 3.16 (Schapira inversion formula). *Assume that $\mathcal{S} \subset W \times X$ and $\mathcal{S}' \subset X \times W$ have fibers $\mathcal{S}_w$ and $\mathcal{S}'_w$ satisfying (1) $\chi(\mathcal{S}_w \cap \mathcal{S}'_w) = \mu$ for all $w \in W$; and (2) $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w'}) = \lambda$ for all $w' \neq w \in W$. Then for all $h \in \mathrm{CF}(W)$,*

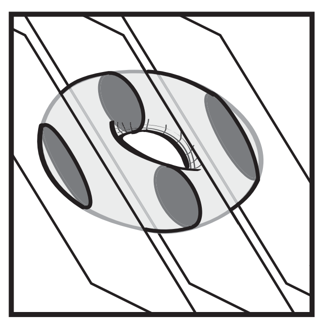
$$(\mathcal{R}_{\mathcal{S}'} \circ \mathcal{R}_{\mathcal{S}})h = (\mu - \lambda)h + \lambda \left(\int_W h \right) \mathbb{1}_W. \tag{3.15}$$

Proof. The conditions on the fibers of $\mathcal{S}$ and $\mathcal{S}'$ imply that $\int_X \mathcal{S}(w,x)\mathcal{S}'(x,w')d\chi = (\mu - \lambda)\delta_{w-w'} + \lambda$ for all $w, w' \in W$, where δ is the Dirac delta function. Thus, for any $w' \in W$,

$$\begin{aligned}
(\mathcal{R}_{\mathcal{S}'} \circ \mathcal{R}_{\mathcal{S}} h)(w') &= \int_X \left[\int_W h(w) \mathbb{1}_{\mathcal{S}}(w,x)\, d\chi \right] \mathbb{1}_{\mathcal{S}'}(x,w')\, d\chi \\
&= \int_W h(w) \left[\int_X \mathbb{1}_{\mathcal{S}}(w,x) \mathbb{1}_{\mathcal{S}'}(x,w') d\chi \right] d\chi \\
&= \int_W \left[(\mu - \lambda) h(w) \delta_{w-w'} + \lambda h(w) \right] d\chi \\
&= (\mu - \lambda) h(w') + \lambda \int_W h\, d\chi,
\end{aligned}$$

where the Fubini Theorem is used in the second equality. ⊙

Recall that the sensor counting field $h\colon X \to \mathbb{Z}$ is equal to $\mathcal{R}_S \mathbb{1}_T$, where $T \subset W$ is the set of targets. If the conditions of Proposition 3.16 are met *and* if $\lambda \neq \mu$, then the inverse Radon transform $\mathcal{R}_{S'} h = \mathcal{R}_{S'} \mathcal{R}_S \mathbb{1}_T$ is equal to a (nonzero) multiple of $\mathbb{1}_T$ plus a multiple of $\mathbb{1}_W$. Thus, one can localize the targets by performing the inverse transform. It is remarkable that enumerative data alone can yield not only target counts but target positions as well. By changing T from a discrete set to a collection of (say, contractible) compact sets, one notes that Radon inversion has the potential to not merely localize but recover shape. This *topological tomography* is the motivation for Schapira's incisive paper [270].



Example 3.17 (Topological tomography) ⊚

Assume that $W = \mathbb{R}^3$ and that one scans a compact subset $T \subset W$ by slicing $\mathbb{R}^3$ along all flat hyperplanes, recording simply the Euler characteristics of the slices of T. Since a compact subset of a plane has Euler characteristic the number of connected components minus the number of holes (which, in turn, equals the number of bounded connected components of the complement – see §6.5), it is feasible to compute an accurate Euler characteristic, even in the context of noisy readings.

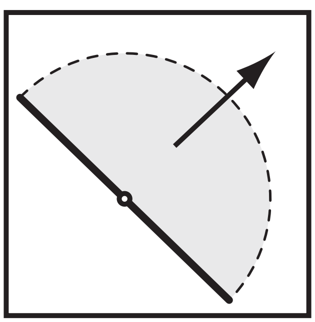
This yields a constructible function on the sensor space $X = \mathbb{AG}_2^3$ (the affine Grassmannian of all planes in $\mathbb{R}^3$) equal to the Radon transform of $\mathbb{1}_T$. Using the same sensor relation to define the inverse transform is effective. Since $\mathcal{S}_w \cong \mathbb{P}^2$ and $\mathcal{S}_w \cap \mathcal{S}_{w'} \cong \mathbb{P}^1$, one has $\mu = \chi(\mathbb{P}^2) = 1$, $\lambda = \chi(\mathbb{P}^1) = 0$, and the inverse Radon transform, by (3.15), yields $\mathbb{1}_T$ exactly: one can recover the shape of T based solely on connectivity data of black and white regions of slices. ⊚

Example 3.18 (Fourier transforms and curvature) ⊚

One relationship between Euler and Lebesgue measures is encoded in the Gauss-Bonnet theorem of §3.2. This, too, can be lifted from manifolds to definable sets and then to $\mathrm{CF}(\mathbb{R}^n)$. The mechanism is the **Fourier-Sato transform**. Given $h \in \mathrm{CF}(\mathbb{R}^n)$, a point $x \in \mathbb{R}^n$, and a unit *frequency* vector $\xi \in \mathbb{S}^{n-1}$, the Fourier-Sato transform is defined via:

$$(\mathcal{F}_{\mathbb{S}}\, h)_x(\xi) := \lim_{\epsilon \to 0^+} \int_{B_\epsilon(x)} \mathbb{1}_{\xi\cdot(y-x)\geq 0}\, h\, d\chi(y), \tag{3.16}$$

where B_ϵ denotes the *open* ball of radius ϵ. Like the classical Fourier transform, $\mathcal{F}_{\mathbb{S}}$ takes a *frequency* vector ξ and integrates over *isospectral sets* defined by a dot product. For Y a compact tame set, $\mathcal{F}_{\mathbb{S}}\mathbb{1}_Y(\xi)$ is the constructible function on Y that, when averaged over ξ, yields a curvature measure $d\kappa_Y$ on Y implicated in the Gauss-Bonnet Theorem (as shown by Bröcker and Kuppe [48]). For any $U \subset \mathbb{R}^n$ open, define the net curvature of Y on U to be:



$$\int_U d\kappa_Y := \frac{1}{\mathrm{Vol}\, \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_U (\mathcal{F}_{\mathbb{S}} \mathbb{1}_Y)_u(\xi)\, d\chi\, d\xi.$$

This curvature measure has support on Y and, up to a constant, agrees with the three notions of curvature mentioned in §3.2. The Euler-calculus interpretation of the Gauss-Bonnet Theorem says that this rescaled net curvature of Y is precisely its Euler characteristic.

$$\int_Y d\kappa_Y = \chi(Y).$$ ⊚

3.10 Intrinsic volumes

The humble combinatorial definition of χ has matured in this chapter to play the role of a measure. This has precedent in the subject of **integral geometry**. There is a family of *"measures"* on Euclidean $\mathbb{R}^n$ that mediate Euler and Lebesgue while entwining topological and geometric data. The k^{th} **intrinsic volume**[5] μ_k is characterized uniquely by the following: for all A and B tame subsets of $\mathbb{R}^n$,

1. **Additivity:** $\mu_k(A \cup B) = \mu_k(A) + \mu_k(B) - \mu_k(A \cap B)$;

[5]Intrinsic volumes are also known as **Hadwiger measures**, **quermassintegrale**, **Lipschitz-Killing curvatures**, **Minkowski functionals**, and, likely, a few more names unknown to the author.

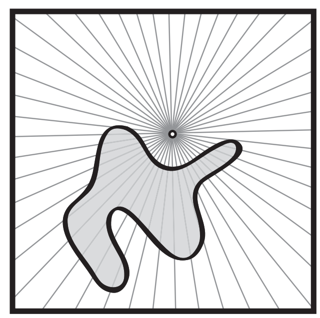
2. **Euclidean invariance:** μ_k is invariant under rigid motions of $\mathbb{R}^n$;
3. **Homogeneity under scaling:** $\mu_k(c \cdot A) = c^k(A)$ for all $c \geq 0$; and
4. **Normalization:** μ_k of a closed unit ball in $\mathbb{R}^n$ equals 1.

These measures (more properly, *valuations*) generalize Euclidean n-dimensional volume ($\mu_n = \mathrm{dvol}_n$) and Euler characteristic ($\mu_0 = d\chi$). There are several equivalent definitions, all revolving about the notion of an average Euler characteristic. One way to define the intrinsic volume $\mu_k(A)$ is in terms of the Euler characteristic of all slices of A along affine codimension-k planes:

$$\mu_k(A) = \int_{\mathbb{AG}^n_{n-k}} \chi(A \cap P)\, d\lambda(P) = \int_{\mathbb{AG}^n_{n-k}} \int_P \mathbb{1}_A \, d\chi \, d\lambda(P), \tag{3.17}$$

where $d\lambda$ is an appropriate measure[6] on $\mathbb{AG}^n_{n-k}$, the space of affine $(n-k)$-planes in $\mathbb{R}^n$. The details of this construction are not elementary [234]: the point here stressed is that all the intrinsic volumes are certain Lebesgue-averaged Euler integrals.

These intrinsic volumes are more than isolated examples. The classical theorem of Hadwiger [174] characterizes those Euclidean-invariant valuations which are continuous with respect to the Hausdorff metric on compact convex sets (the *compact-continuous* valuations):



Theorem 3.19 (Hadwiger Theorem). *The space of all compact-continuous Euclidean-invariant valuations on $\mathbb{R}^n$ is a vector space of dimension $n+1$ with basis $\{\mu_k\}_{k=0}^n$.*

These amalgamations of Euler and Lebesgue measure are the key to several interesting applications.

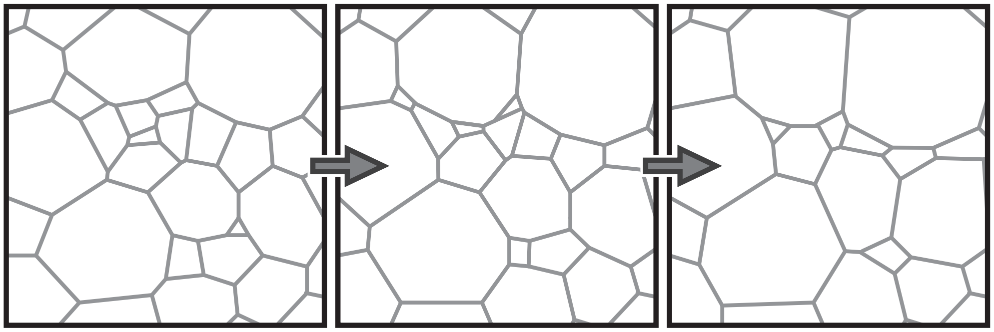
Example 3.20 (Microstructure coarsening) ⊚

Anyone who has observed foam in a pilsner glass knows that the *cell walls* of foam evolve over time so that some cells grow, while others shrink unto disappearance (neglecting pops). The same processes are ubiquitous in the microstructure coarsening of froth, metals, and some ceramics. It has long been known how idealized cells evolve and coarsen in the 2-dimensional setting. The **von Neumann - Mullins formula** states that the area $A(t)$ of a cell in an ideal dynamic 2-d microstructure evolves as:

$$\frac{dA}{dt} = -2\pi M\gamma \left(1 - \frac{|C_0|}{6}\right), \tag{3.18}$$

where M is a mobility constant, γ is a surface tension constant, and $|C_0|$ equals the number of corners (vertices) of the cell. Thus, cells with fewer than six corners shrink: those with more grow. The proper extension of this formula to three-dimensional cell structures was recently discovered by MacPherson and Srolovitz [214]. The key to the

[6] This measure is derived from the Haar measure on the Grassmannian $\mathbb{G}^n_{n-k}$ and Lebesgue measure on the orthogonal $\mathbb{R}^k$.



extension was to note that the "1" in Equation (3.18) is in fact the Euler characteristic of the (contractible, compact) cell. To lift from 2-d to 3-d requires lifting the relevant interior volumes from μ_0 $(= \chi)$ to μ_1. The MacPherson-Srolovitz formula for the volume $V(t)$ of a cell in a 3-d microstructure is:

$$\frac{dV}{dt} = -2\pi M\gamma \left(\mu_1 - \frac{|C_1|}{6}\right), \tag{3.19}$$

where μ_1 is the intrinsic 1-volume of the cell and $|C_1|$ is the total length of all the edges (1-dimensional boundary curves) of the cell. When contemplated in the light of intrinsic volumes, it is clear that Equation 3.19 is in fact an elegant relationship between μ_3 and μ_1. ⊙

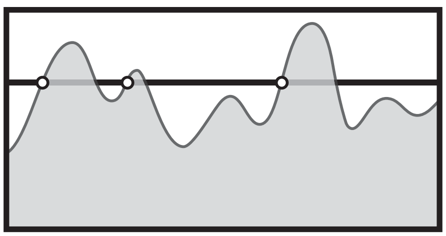
3.11 Gaussian random fields

Most readers will be comfortable with the utility (if not the details) of stochastic processes. A real-valued **stochastic process** is, in brief, a collection $\{h(x) \colon x \in X\}$ of random variables $h(x)$ parameterized by points in a space X. A **random field** is a stochastic process whose arguments vary continuously over the parameter space X. For example, a random field may be used to model the height of water on a noisy sea or the magnitude of noise in a sea of cell-phone towers, both examples having as parameter space a geometric domain of dimension two (or three, with the addition of time). As the simplest random variables are **Gaussian** (having well-defined finite mean and variance with normal distribution), so are the simplest random fields. A **Gaussian random field** with parameter space $X \subset \mathbb{R}^n$ is a random field h over X such that, for each $k \in \mathbb{N}$ and each k-tuple $\{x_i\}_0^k \subset X$, the collection $\{h(x_i)\}_0^k$ of random variables has a multivariate Gaussian distribution. For simplicity, attention will be restricted to Gaussian distributions with the following features:

1. **Centered:** all $h(x)$ have zero mean, $\mathbb{E}\{h(x)\} = 0$;
2. **Common variance:** all $h(x)$ have fixed variance $\sigma^2 > 0$;
3. **Stationary-isotropic:** the covariance $\mathbb{E}\{h(x)h(y)\}$ is a function of distance $|x - y|$.

These last two assumptions imply a well-defined constant λ, the **second spectral moment**, which, roughly, measures the variance of directional derivatives of h with respect to x.

Adler [3], Taylor [5], Worsley [302, 303], and others have led in the exploration of Gaussian random fields from a geometric perspective. There is implicit in this work no small amount of topology, with Euler characteristic featuring prominently. Let $h(x)$, $x \in X \subset \mathbb{R}^n$, be a stationary random Gaussian field. Knowing as much as possible about the tomography of the field h is important in medical imaging, astronomy, and a host of other applications. For example, one might wish to know the expected number of peaks in a field, or some other qualitative properties associated to the (upper) **excursion sets** $\{h \geq s\}$. One of the principal results in this area is that, although the fine details of the expected field excursion sets cannot be computed, the expected Euler characteristics are not merely computable, but computed and commensurate with experimental data.



The following classical result of Rice from the 1940s [247] helped initiate the subject: for $h(x)$, $x \in [0, T]$, a stationary, zero mean, C^1 Gaussian process with finite variance σ^2 and second spectral moment λ, the expected number of up-crossings at $h = s$ (locations where h increases past s) is given by

$$\mathbb{E}\#\{h \nearrow s\} = e^{-\frac{1}{2}\left(\frac{s}{\sigma}\right)^2} \frac{\sqrt{\lambda}}{2\pi\sigma} T. \tag{3.20}$$

The observation that (up to a boundary correction term) $\#\{h \nearrow s\}$, the number of up-crossings, is in fact the Euler characteristic $\chi\{h \geq s\}$ of the upper excursion set presages the deeper work of [3] for fields over higher-dimensional domains. It appears very difficult to understand the expected shape or even topological type of these excursion sets; however, the simplification that χ introduces and its commensurability with integral techniques yields to computational effort. The following result is a highly illustrative example:

Theorem 3.21 ([3, 5]). *Assume that $h(x)$, $x \in \mathbb{R}^n$, is a centered, stationary-isotropic Gaussian random field with common variance σ^2, second spectral moment λ, and sufficient regularity.*[7] *The expected Euler characteristic of the upper excursion set $\{h \geq s\}$ over a compact definable subset $X \subset \mathbb{R}^n$ is*

$$\mathbb{E}(\chi\{h \geq s\}) = e^{-\frac{1}{2}\left(\frac{s}{\sigma}\right)^2} \sum_{k=0}^{\dim X} (2\pi)^{-\frac{1}{2}(k+1)} \left(\frac{\sqrt{\lambda}}{\sigma}\right)^k \mathrm{He}_{k-1}\left(\frac{s}{\sigma}\right) \mu_k(X), \tag{3.21}$$

where He_k is the degree k Hermite polynomial in one variable and $\mu_k(X)$ is the k^{th} intrinsic volume of X.

Equation (3.21) is a remarkable formula in that it ties together so many ideas from integral geometry, topology, and stochastics in a package that permits honest applications to data. The ability to predict expected Euler characteristic from stochastic

[7] This consists of nondegeneracy assumptions on the joint distributions of the first and second derivatives of f and regularity assumptions on the covariance functions of the second derivatives.

data has had enormous impact in imaging, in everything from medical data to astronomy. Additional work [5] allows one to relax the assumptions of the field being stationary, isotropic, and Gaussian. In addition, expectations for intrinsic volumes μ_k of excursion sets are likewise derivable.

Notes

1. The scissors equivalence implicit in Theorem 3.7 is the hint of deeper structures. Euler characteristic and integration over CF is an elementary version of **motivic integration**, of great current interest in algebraic geometry. The papers of Cluckers, Denef, and Loeser [68, 92] detail this somewhat; the exposition of Hales [175] is a good starting point for this theory.
2. Since the Euler measure is the *universal motivic measure* over topological spaces, any other scissors-invariant group-valued measure on definable topological spaces must factor through χ. One way around this depressing result is to extend χ to a polynomial-valued measure of sequences of spaces. The work of Gal [137] encodes the Euler characteristics of $\mathcal{UC}^n(X)$ as coefficients of a series in a formal variable t. For X tame, the resulting series is the Taylor series of a rational function. Similar algebraic manipulations for Euler characteristics of sequences of spaces appears in work on ζ-functions from algebraic geometry [172]. It is to be suspected that a full theory of integration with respect to $d\chi[t]$ over (say) filtered spaces awaits development.
3. Euler integration is here, as in [262, 268], presented as a combinatorial theory. Higher perspectives are more illuminating. Chapter 5 will unfold the connection to homology; Chapter 7 to Morse theory; and Chapter 9 to sheaf theory. The literature on **normal cycles** [134, 234] allows, thanks primarily to results of Kashiwara [191], an approach in terms of **conormal cycles** and related geometric measure theory: see §6.11.
4. The issue of numerical Euler integration – how to approximate an integral with respect to $d\chi$ based on sampled data – is extremely interesting, technical, and relevant to applications. It appears to have received little treatment from the Mathematics community. See §6.5 for a first step.
5. It is possible and profitable to perform Radon transforms (and inversion thereof) with weighted kernels [27].
6. Intrinsic volumes $d\mu_k$ are well-defined on all of $\mathrm{CF}(\mathbb{R}^n)$ – not merely compact definable sets – thanks to the close relationship with $\int d\chi$. Continuity of these measures is possible, but requires a more delicate function space topology, relying on currents (see §6.11 and [28]).
7. Research in geometric random fields has been, sadly, largely ignored by topologists. It may help matters to employ (1) the language of Euler integration, as it removes much of the mystery as to the natural appearance of χ in this work, and (2) the o-minimal framework, as it would likely subsume many of the nondegeneracy assumptions. The author has taken the liberty of invoking definable sets in the statement of Theorem 3.21: this is not *quite* what Adler proves. Hopefully it is a true statement.