Volume 1

1. [Lines, Planes] Give a parametrization of the the line passing through the point $(2, 1, 0)$ and *parallel* to the plane $2x+3y-z=1$. How many such lines are there?

2. [Vectors and More] $\vec{a} = (1, 2, 0)$, $\vec{b} = (1, 1, 1)$ and $\vec{c} = (-1, -2, 1)$. Which pair of vectors are closest to being parallel?

3. [Planes, Vectors] For what value of k are the planes

$$P_1 : (1+k)x - 3y + 6z = u$$
$$P_2 : x + (5+k)y + 3kz = v$$

orthogonal, if $(u, v) \in \mathbb{R}$?

4. [Cross Products] If vectors $\vec{a}, \vec{b}, \vec{c}$ are all mutually orthogonal, what can you say about the vector $\vec{V} = \vec{a} \times \vec{b} \times \vec{c}$?

5. [Sketch] Sketch the surface $z = 9 - x^2 - y^2$ for $z \geq 0$. Can you parametrize this surface using the parameters (s, t)?

6. [Lines, Planes, Vectors] Write the parametrization of the line perpendicular to the plane containing the points $(1, 2, 0)$, $(1, 0, 1)$, $(2, 1, 0)$.

7. [Planes and Vectors] At what angle do the planes $2x+y+z=4$ and $x-2y+z=1$ meet? Parametrize their intersection.

8. [Projections] In 5 dimensional space, a 5-cube (a cube with 32 vertices, 80 edges, 80 square faces, 40 cubic cells, and 10 tesseract 4-faces, if you're curious) is pushed in the direction given by $\boldsymbol{X} = (2, 3, -4, 1, 2)$. The force acting on the box is $\boldsymbol{F} = (1, 2, 0, 4, 5)$

   (a) What is the component of $\boldsymbol{F}$ acting in the direction of the displacement of the box?

   (b) What is the angle between $\boldsymbol{F}$ and $\boldsymbol{X}$?

9. [Parallelepiped] Find the volume of the parallelepiped spanned by the vectors $\vec{a} = (-2, 3, 1)$, $\vec{b} = (0, 4, 0)$, $\vec{c} = (-1, 3, 3)$.

10. [Vector Differentiation] Suppose $\vec{\gamma}(t)$ is the position of a particle in $\mathbb{R}^4$.

$$\vec{\gamma}(t) = \begin{pmatrix} t^3 \\ e^{2t} \\ 1-t \\ 5t^3 \end{pmatrix}.$$

Write (but do not evaluate) an expression for the arclength of the particle in the interval $t = 0...10$.

11. [Cross Products] Compute $3\hat{\imath} \times (1\hat{\imath} + 4\hat{\jmath} - 1\hat{k})$.

12. [Planes] Find the equation of the plane spanned by the vectors $\boldsymbol{u} = (1, 0, -1)$, $\boldsymbol{v} = (1, 1, 0)$ and containing the point $(1, 2, 2)$.

13. [Area] Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

14. [Volume] Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, $R($ and $S(2, 1, 3)$.

15. [Arc Length] What is the arc length element of the swirly slide modeled by the parametrization $x(t) = 3\sin(3t)$, $y(t) = \cos(3t)$, $z(t) = t + 1$?

16. [TNB] Find $\vec{T}$, $\vec{N}$, $\vec{B}$ for $\vec{r}(t) = \cos(2t)\hat{\imath} + \sin(2t)\hat{\jmath}$.

17. [Somewhat of a challenge] In $\mathbb{R}^3$ the unit normal vector $\vec{N}$ is orthogonal to the unit tangent vector $\vec{T}$. Prove this relation. Hint: $\frac{d}{dt}\left(\vec{T} \cdot \vec{T}\right) =?$

18. [Matrices] Write the size of the matrix resulting from the multiplication of matrices with sizes
    (a) $[3 \times 2][2 \times 2]$
    (b) $[2 \times 3][3 \times 9][9 \times 100]$
    (c) $[4 \times 8][8 \times 2][2 \times 4][4 \times 7]$

19. [Matrix Multiplication] Compute the following matrix multiplication:

$$\begin{pmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix}$$

20. [Matrix Multiplication] Compute the matrix multiplication below, but think about how you might be able to rearrange things here... This will become *very* important *very* soon!

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 4 & -2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

21. [Linear Equations] A system of three linear equations in $\mathbb{R}^3$ can have one solution, infinite solutions, or no solution at all. Sketch a picture of each case.

22. [Row Reduction] What are the 3 rules of row reduction? Why does row reduction even work?

23. [Solving Linear Equations] Solve (if possible) the system of linear equations using **row reduction**:

(a)	(b)	(c)
$4x - 3y + z = 2$	$x - 3y + z = 4$	$2x + y - 3z = 0$
$2x + y + 3z = 1$	$-x + 2y - 5z = 3$	$4x + 2y - 6z = 0$
$-x + 2y - 5z = 1$	$5x - 13y + 13z = 8$	$x - y + z = 0$

24. [Matrix Inverse] Compute the inverse of the product of the two matrices

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}.$$

25. [Hmm what do you think?] Compute the inverse of the function

$$\begin{pmatrix} u \\ v \end{pmatrix} = f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ x - y \end{pmatrix}.$$

26. [Linear Transformations] Derive the rotation matrix R_θ in $\mathbb{R}^2$ using trigonometry.

27. [Linear Transformations] Write a linear transformation in $\mathbb{R}^2$ that sends all vectors to the line $y = 3x$.

28. [Linear Transformations] Write the linear transformation in $\mathbb{R}^2$ that

    I. Sends the vertical to the line $y = 4x$

    II. Stretches the x-axis $2\times$

    III. Rotates the plane 30 degrees counterclockwise

    Is you answer unique?

29. [Linear Transformations] Write a linear transformation that transforms the vector $\vec{v} = (1, 2)^T$ to the new vector $\vec{p} = (3, 4)^T$. Is your answer unique?

30. [Change of Basis... Somewhat tricky] For a vector with components (x, y) in the basis $\{\hat{\imath}, \hat{\jmath}\}$, express the same vector with components (a, b) in the basis $\{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\hat{\imath} + 1\hat{\jmath},$$

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\hat{\imath} + 1\hat{\jmath}.$$

31. [Change of basis] Express the vector $\vec{v} = 2\hat{\imath} + 3\hat{\jmath}$ in the $\{\vec{u}, \vec{w}\}$ basis, where $\vec{u} = 1\hat{\imath} + 2\hat{\jmath}$, and $\vec{w} = -1\hat{\imath} + 1\hat{\jmath}$.

32. [Determinants] Compute the determinant of the 4×4 matrix

$$\begin{pmatrix} 0 & 2 & 4 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

33. [Determinants] Compute

$$\det\left[\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

34. [Determinants] Let $\boldsymbol{A}$ be a 200×200 matrix. If $\det(\boldsymbol{A}) = 17$, what is $\det(\boldsymbol{A}^T)$?

35. [Determinants] Use a determinant to compute the cross product $(3, 2, 1)^T \times (2, 0, 1)^T$.

36. [Volumes] Compute the volume of the parallelepiped spanned by the vectors $\vec{a} = (1, 0, 3)^T$. $\vec{b} = (2, 3, 1)^T$, $\vec{c} = (1, 1, 0)$.

37. [Determinants and Row Reduction] Use row reduction to compute

$$\det\begin{pmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{pmatrix}$$

38. [Determinants and... Row Reduction?] Compute the determinant of the following 5×5 matrix. Think first!

$$\begin{pmatrix} 1 & 2 & 8 & 9 & 81 \\ 3 & 2 & 4 & 4 & 36 \\ 213 & 342 & 2 & 5 & 45 \\ 32 & 245 & 42 & 1 & 9 \\ 344 & 53 & 23 & 2 & 18 \end{pmatrix}$$

39. [Determinants] Compute

$$\det\begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & 1 \end{pmatrix}.$$

What is the geometric meaning of this determinant?

40. [Determinants] Compute

$$\det\left[\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix}^{-1}\right].$$

- Suppose an equilateral triangle with area = 2 undergoes the three linear transformations inside the above determinant. What is the area of the resulting object after the 3 transformations?

41. [Determinants] Compute

$$\det \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Think first.

42. [Determinants] Compute the determinant of the matrix

$$\begin{pmatrix} 3 & 2 & 1 & 5 & 6 & 7 \\ 0 & 0 & 2 & 9 & 1 & 1 \\ 0 & 0 & 0 & -2 & 17 & 2 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

Beware of the bottom left!

43. [hmmm] For what value of c do the following vectors lie within a plane?

$$\vec{u} = \begin{pmatrix} 2c \\ 3 \\ -1 \end{pmatrix}, \vec{v} = \begin{pmatrix} c \\ 1 \\ 2 \end{pmatrix}, \vec{w} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

44. [Row Reduce Determinants] Compute using row reduction:

$$\det \begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & 1 \\ 3 & 2 & 3 \end{pmatrix}.$$

Volume 2

45. [Partial Derivatives] Compute all first order partial derivatives of $f(x, y, z) = xe^{(xyz)^2}$.

46. [Partial Derivatives] Compute

$$\frac{\partial}{\partial y}\left[xe^{xyz} + \arctan\left(x + z^2\right) - z^{e^{x^2}} + \cos^2\left(xze^{x^2z^2}\right)\right].$$

47. [Derivative Matrix] $[Df] = \begin{pmatrix} 2 & -10 & -1 & 0 \\ -2 & 8 & 1 & 5 \end{pmatrix}$

(a) How many inputs and outputs does f have?

(b) With respect to which input is the first output most sensitive to?

(c) If the rate of change of inputs is +1 (for each input), what is the rate of change of outputs?

48. [Derivative Matrix] Compute $[Df]$ for

$$f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\sin(xyz)\\ze^{xy}\\xz\cos(y)\\x+y-z\end{pmatrix}$$

49. [Rates of Change] Consider the function

$$g\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\sin(x)+\sin(y)+\sin(z)\\\sin(x-y-z)\end{pmatrix}$$

If the rate of change of inputs at the origin is given by $\dot{x}=1, \dot{y}=2, \dot{z}=1$, find the rate of change of outputs at the origin. Which variables(s) is the first output most sensitive to near the origin?

50. [The Derivative]

$$f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\sin(xy)\\3yz^2+x\end{pmatrix}.$$

At the input $(x, y, z) = (\pi, 1, 2)$, which input(s) is the first output most sensitive to? Second output?

51. [Rates of Change] Consider the function

$$h\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3xy\\\sin(xy)+x\end{pmatrix}$$

If the rate of change of outputs at $(1, 0)$ is $(9, 5)^T$, find the rate of change of inputs $\dot{x},\ \dot{y}$.

52. [Rates of Change] Given the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, if the derivative $[Df]$ satisfies

$$[Df]\begin{pmatrix}4\\-3\end{pmatrix}=\begin{pmatrix}1\\9\end{pmatrix}$$

at a particular input, what is the rate of change of outputs if the rate of change of inputs are 12 and -9 respectively (at the particular input)?

53. [Rates of Change] Consider the volume and surface area of an open top cylindrical can. Denote the radius of the can as r and the height as h. If the radius of the can is 3 meters, and the height is 4 meters, which measurement (volume or surface area) is more sensitive to changes in radius? Which measurement is more sensitive to changes in height?

54. [Rates of Change] For the function f in the previous problem, if the rate of change of inputs at $(1, 1, 1)$ is $(\dot{x}, \dot{y}, \dot{z}) = (1, 2, 0)$, what are the rate of change of outputs? Which output is changing most at these inputs?

55. [Derivative Matrix] If the rate of change of outputs for a function are $(4, 1)^T$ and

$$[Df] = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix},$$

what are the rate of change of inputs?

56. [Derivative Matrix... This is a clever one] Given that the derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at a particular input satisfies

$$[Df] \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix},$$

how fast are the outputs changing at the particular input if the inputs are changing with rates of 6 and -9, respectively?

57. [Composition] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^p \to \mathbb{R}^k, h : \mathbb{R}^m \to \mathbb{R}^p$. Write all possible compositions of the functions f, g, and h.

58. [Chain Rule] Suppose

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(xyz) \\ e^{xy} + z^2 \\ x + y + z \end{pmatrix}, \text{and g} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^3 \\ u \cos v \\ uv \end{pmatrix}.$$

Compute $[D(f \circ g)]$ at $(u, v) = (0, 1)$.

59. [Chain Rule] If

$$[D(f \circ g)]_{\vec{a}} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \text{ and } [Dg]_{\vec{a}} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix},$$

what is $[Df]_{g(\vec{a})}$?

60. [Trivial Chain Rule] Use the chain rule to show that $(u + v)' = u' + v'$.

61. [Chain Rule] Use the chain rule/composition to show that the derivative of $f(u, v) = u^v$ is

$$vu^{v-1}u' + u^v \ln(u)v'.$$

62. [Chain Rule/Composition] A particle moves along the curve

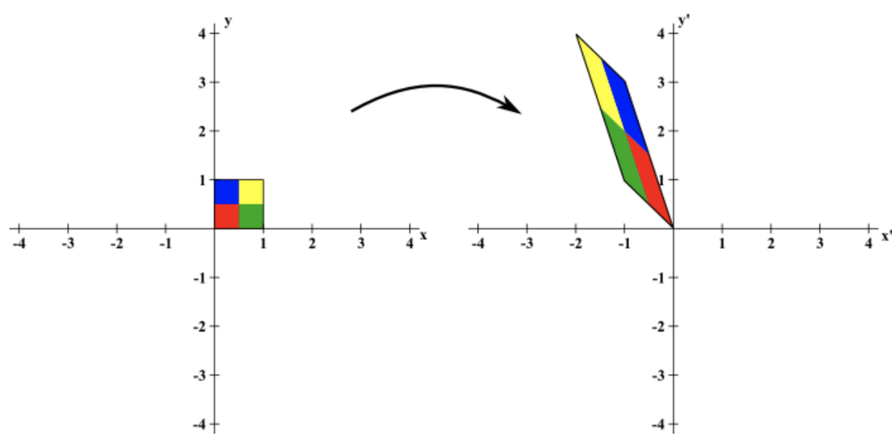
$$\vec{X}(t) = (\sin(t), \sin(t)\cos(t), \cos^2(t))^T$$

with the pressure field

$$\rho(x, y, z) = z + xy - yz.$$

Compute the rate of change of pressure with respect to time as the particle moves along its path. This is not as bad as it looks. Apply the chain rule using matrices.

63. [Linear Transformations] The diagram below (left) undergoes the linear transformation $\mathbf{A}$ in $\mathbb{R}^2$ (the result is shown on the right). Write the linear transformation.



64. Verify that $[Df]_{\vec{a}}^{-1} = [Df^{-1}]_{f(\vec{a})}$ with the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 2y \\ x + 3y \end{pmatrix}.$$

65. [IFT] Compute the derivative of the inverse of

$$w\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3uv + v^2 \\ u^2 v \end{pmatrix}$$

at $w(1, 2)$.

66. Write an expression that is satisfied by by the set of points for which the function w above is not invertible. What theorem are you using?

67. [Gradients/Directional Derivatives/Level Sets] An ant traverses its anthill. For unknown reasons, the temperature of the anthill at the location (x, y, z) is given by $T(x, y, z) = \sin(xy) + e^{yz}$.

    (a) At $(x, y, z) = (\sqrt{\pi}, \sqrt{\pi}, 0)$, in what unit vector direction must the ant move so that the temperature increases most rapidly?

    (b) At the same point, in what unit vector direction must the bug move so that its temperature *decreases* most rapidly?

    (c) At the same point, the bug moves in the direction of the vector $\vec{v} = (1, 0, -1)^T$. What is the rate of change of the temperature in this direction (at the same point)?

    (d) At the same point, in what unit vector direction must the bug immediately walk so that its temperature remains constant?

68. [Gradients/Level Sets] Find a vector orthogonal to the curve $y = x^2$ for any value of x.

69. [Level Sets] Draw level sets of the function $f(x, y) = x^2 + y^2$. Now sketch the corresponding surface $z = x^2 + y^2$ in $\mathbb{R}^3$. What type of surface is this?

70. Find an equation of the plane tangent to the surface $x^2 + yz + z = 3$ at $(x, y, z) = (1, 1, 1)$.

71. [Tangent Spaces - Parametric] Write a parametrization of the plane tangent to the surface $x(t_1, t_2) = t_1,\ y(t_1, t_2) = t_2,\ z(t_1, t_2) = t_1^2 + t_2^2 + 2t_1$ at $(t_1, t_2) = (0, 1)$. To avoid confusion, use parameters s_1 and s_2.

72. [Differentials] $f(x, y, z) = e^{xy}z + x^2yz^4$. Compute df.

73. [Linearization] A cylindrical solid can is designed to have radius r and height h. If measurements in h can vary by up to 10%, r up to 20% (yikes), by what percent can measurements of the volume of the can vary?

74. [Linearization] Suppose you are measuring a cylindrical tunnel in 3d (end open). If you measure the radius and length of the cylinder with 1% error, up to what percent is the error in the volume and surface area measurement? Which tends to be more accurate: measurement of volume or surface area?

75. [Linearization] Newton's law of universal gravitation states

$$F = \frac{Gm_1m_2}{r^2}.$$

Suppose m_1 is the mass of a distant star, and m_2 is the mass of a distant exoplanet orbiting the star. r is the distance between m_1 and m_2. For this problem, ignore all units. If the mass of the distant star (m_1) is measured with an error of up to 10%, and the distance (r) between the star and the exoplanet is measured with an error of up to 20%, what is the maximum error in the mass of the exoplanet, m_2? Assume that F is known ($F = F_0$).

76. [IFT] Compute the derivative of the inverse of the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2e^{x^2} - \cos(y^3) \\ y^4 + 4\sin(2x) \end{pmatrix}$$

at $f(0,0)$ if it exists.

77. [T/F] True or False: The inverse function theorem works for any function $f : \mathbb{R}^n \to \mathbb{R}^m$.

78. [hmmm] Determine whether or not the following system of equations can be solved for x and y as functions of u and v in the neighborhood of the origin:

$$\sin(xy) = u$$
$$\sin(x^2 + y) = v.$$

Write an expression that is satisfied by the set of all (x, y) such that the function $g(x, y) = (u, v)$ is *not* invertible. This question sounds frightening... don't let the language deceive you!

79. [Gradients/Directional Derivatives] Suppose $f(x, y, z) = \sin(xy) - \cos(yz) + xyz$.

    (a) What is the unit vector direction in $\mathbb{R}^3$ that leads to the greatest increase in f at the point $(\sqrt{\pi}, \sqrt{\pi}, 0)$?

    (b) Greatest decrease?

    (c) Find the rate of change of f in the direction of the vector $\vec{v} = (1, 1, 0)^T$ at the point $(\sqrt{\pi}, \sqrt{\pi}, 0)$.

    (d) What is the maximum rate of change of f at the point $(\sqrt{\pi}, \sqrt{\pi}, 0)$?

    (e) What is the minimum rate of change of f at the point $(\sqrt{\pi}, \sqrt{\pi}, 0)$?

80. [Level Sets] What is a level set?

81. [Level Sets] Sketch a level set of the function $f(x, y) = e^{x^2+y^2}$.

82. [Level Sets] Write an expression for the vector perpendicular to the curve $y = x^2$ in $\mathbb{R}^2$.

83. [Level Sets] Let $f(x, y) = x^2 - y^2 + 2\sin(x)$ and $g(x, y) = 2x^2 - y^3$. Suppose I tell you that at (x_0, y_0), the levels sets of f and g are orthogonal. How could you check whether or not this was actually true?

84. [Tangent Spaces] Write an equation for the plane tangent to the surface $-4x^2 + 3xy^3 + z^2 = 0$ at $(1, 1, 1)$. Find two vectors in the tangent plane at this point.

85. [Parametrized Tangent Spaces] Suppose a surface S is parametrized by $(x(u, v), y(u, v), z(u, v)) = (\sin(u^2 - v), u + v, v^2)$ in $\mathbb{R}^3$. Write a parametrization for the plane tangent to S at the point $(x, y, z) = (0, \pi, \pi^2)$.

86. [Level Sets] Let $f(x, y) = x^2 - y^2 + 2\sin(x)$ and $g(x, y) = 2x^2 - y^3$. Suppose I tell you that at (x_0, y_0), the levels sets of f and g are orthogonal. How could you check whether or not this was actually true?

87. [Tangent Spaces] Write an equation for the plane tangent to the surface $-4x^2 + 3xy^3 + z^2 = 0$ at $(1, 1, 1)$. Find two vectors in the tangent plane at this point.

88. [Taylor Series] Compute the Taylor series of $\ln(1+\sin(xy)) + e^{xy}$ about the origin up to (and including) order 4 terms. Describe/justify what the function looks like near the origin.

89. [Critical Points and Optimization] Compute and classify all critical points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$.

90. [Constrained Optimization] Suppose $f(x, y) = 2y^2 + x^2$ and that x and y must satisfy $x^2 + y^2 \leq 1$. Find the absolute maximum and minimum values of f in this region, and draw a rough sketch to justify your answer(s).

91. [Taylor Series - Challenge] Derive the second order Taylor expansion about $\vec{x} = \vec{0}$ for a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. Make sure your answer matches the result given by the formula covered in the video lectures. Hint: a Taylor series is just a power series representation of a function. Do this problem by representing f as a second order power series near the point of interest, $\vec{x} = \vec{0}$. In other words, approximate f by the power series series $T(x, y) = a + bx + cy + dx^2 + ey^2 + fxy$ in the neighborhood of the origin. Express a, b, c, d, e, f in terms of T.

92. [Critical Points and Optimization] Compute and classify all critical points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$.

93. [Constrained Optimization] Suppose $f(x, y) = 2y^2 + x^2$ and that x and y must satisfy $x^2 + y^2 \leq 1$. Find the absolute maximum and minimum values of f in this region, and draw a rough sketch to justify your answer(s).

94. [Lagrange] Maximize the volume of an open top cylindrical can of radius r, height h, and surface area α.

95. [Lagrange] Find all critical points of the function $f(x, y, z) = 2x^3 + \frac{4}{3}y^3$ constrained to a circle of radius 2. Use Lagrange multipliers. **For a challenge**, see if you can work out the same results using only constrained optimization. You might need a calculator.

96. [Lagrange] Find the least distance between the origin and the plane $x+3y-2z=4$.

97. [Lagrange] Find the point on the line $ax+by=1$ closest to the point (x_0, y_0) in the plane. Assume a and b are nonzero constants. Use Lagrange multipliers, and don't worry about simplifying your answer.

Volume 3

98. [Double Integrals] Evaluate

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y} dydx.$$

99. [Double Integrals] Compute the volume below the surface $z = xy\sin(x^2y)$ for $x \in [0, \pi]$ and $y \in [0, \pi/2]$.

100. [Double Integrals... a bit of a challenge] Assuming $a > 0$, reverse te order of integration for the double integral

$$\int_{y=0}^{a}\int_{x=a}^{a+\sqrt{a^2-y^2}} dxdy.$$

101. [Triple Integrals] Reverse the order for the triple integral below, integrating with respect to x, y, z (in that order)

$$\int_0^1\int_0^1\int_{x^2}^1 dydxdz.$$

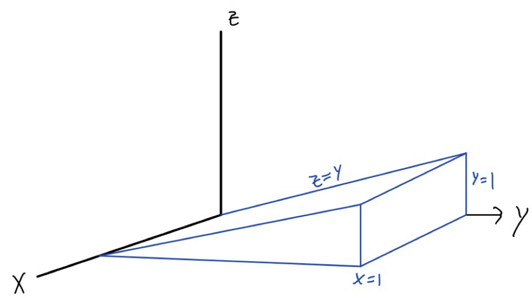
After reversing the order of integration, compute the integral.

102. [Triple Integrals] Compute the volume of the region $(x, y, z) \geq 0$ between the planes $x+y+z=1$ and $x+y+4z=1$.

103. [Centroids] Find the centroid of the region defined by $0 \leq (x, y, z) \leq 1$, $x+y \geq 1$.

104. [Center of Mass] Compute the y coordinate of the center of mass for the solid object shown (above) with density $\rho(x, y, z) = x + y$. What would be the x coordinate of the center of mass if $\rho(x, y, z) =$ constant? The solid is bounded below by the xy plane.

105. [Triple Integrals] Suppose the density of a solid object is $\rho(x, y, z) = xz$. The object occupies the tetrahedron with vertices (0,0,0), (0,1,0), (1,1,0), and (0,1,1). Set up the integral(s) to find the mass of the object.



106. [Triple Integrals] Set up the integral(s) to find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$. What are the x and y centroids for the solid object contained in this region?

107. [Inertia] Consider the rotation of a 2 dimensional object about the z-axis. The object is bounded by the lines $y = x$, $x = L$, and $y = 0$. Assume $L > 0$. The object has uniform density α.

    (a) What is the mass M of the object?

    (b) What is the moment of inertia element, dI?

    (c) Calculate the moment of inertia about the z-axis. Express your answer in terms of the mass M.

    (d) (Bonus) If the center of mass of the object is $(\bar{x}, \bar{y}) = (\frac{2}{3}L, \frac{1}{3}L)$, compute the moment of inertia about the center of mass without doing another integral.

108. [Inertia Matrix] Suppose the inertia matrix for a solid object in a particular coordinate system is given by

$$\begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the moment of inertia of the solid about the axis in the direction of the vector $\vec{V} = (1, 1, 1)^T$

109. [Probability] The probability density for the position of a particle in a 2-dimensional box is proportional to x^2y, where x, y is the position coordinate of the particle inside the box. Suppose the physical dimensions of the box permits a certain range of values for x, y, namely $0 \leq (x, y) \leq 1$.

    (a) Find the exact expression for the PDF $\rho(x, y)$.

    (b) Suppose the box is split into two regions by the line $y = x$. What is the probability that the particle will be found in the upper region? The lower region?

(c) Now, suppose the box is split into four equal area quadrants. Without computing any integrals, in which quadrant is the particle most likely to be found? Why?

110. [Cylindrical/Spherical] Compute the volume of the sphere $x^2 + y^2 + z^2 = 9$ for $(x, y, z) \geq 0$. This is an answer you should be able to check!

111. [Cylindrical/Spherical] Find the volume of the cylinder $x^2 + y^2 = 1$ bounded above by the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 0$.

112. [Cylindrical/Spherical] Suppose the temperature at a position (x, y, z) is

$$T(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

(a) Find the average temperature inside the unit sphere centered at the origin.

(b) Setup the integral(s) to find the average temperature *inside* the cone $z = \sqrt{x^2 + y^2}$ for $z \in [0, 4]$.

113. [Cylindrical/Spherical] Find the moment of inertia for a cylinder of radius R, height h and mass M about its central axis.

114. [Change of variables] Use change of variables to show that the polar area element is $rdrd\theta$. This is a nice check to see if you understand how to do a change of variables problem.

115. [Cylindrical/Spherical] Compute the volume of the region outside $z = \sqrt{3(x^2 + y^2)}$ and inside $-x^2 - y^2 - z^2 + 25 = 0$. Attempt this problem using both cylindrical and spherical coordinates for some good practice.

116. [Change of Variables] Compute the double integral

$$\iint\limits_D \left(12x^5y^5 + 4xy^9\right) dxdy$$

where D is the region bound by the inequalities

$$\begin{array}{c} 3 \leq x^4 - y^4 \leq 6 \\ 2 \leq xy^3 \leq 4 \end{array}.$$

A reasonable change of variables given the domain is $u = x^4 - y^4$ and $v = xy^3$.

117. [Change of Variables] Evaluate

$$\int_0^2 \int_{(x-2)/2}^{(2-x)/2} (3x + 2y)\, dydx$$

using the change of variables $u = x + 2y$ and $v = x - 2y$.

118. [One of my favorites...] Use change of variables to compute the volume of the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
What's a good choice for the change of variables here? I'll leave that up to you. *Hint: This is a 3d problem so you will need 3 variables – (u,v,w). Choose each to make the equation for the ellipsoid something simpler/more familiar. This problem is shockingly easy with the right change of variables!*

119. [Surface Integrals] Using a surface integral, find the area of the plane $x+y+z=1$ in the first octant.

120. [Surface Integrals] Find the surface area of the cone $x^2 + y^2 = z^2$ for $z \in [0, 2]$.

121. [Surface Integrals] Use a surface integral to show that the surface area of the sphere $x^2 + y^2 + z^2 = R^2$ is $4\pi R^2$.

122. [Triple Integrals] Compute the volume of the region outside $z = \sqrt{3(x^2 + y^2)}$, inside $-x^2 - y^2 - z^2 + 25 = 0$. Try this problem using both spherical coordinates and cylindrical coordinates.

123. [Triple Integrals] Find the volume of the solid enclosed by the surface $a^2z^2 = h^2x^2 + h^2y^2$ for $0 \le z \le h$.

124. [Probability] The probability density for the position of a particle in a 2 dimensional box is directly proportional to x^2y, where the box is the region with $0 \le (x, y), \le 1$, and x, y is the position coordinate of the particle inside the box.
    (a) Find the exact expression for the PDF $\rho(x, y)$.
    (b) What is the probability that the particle will be detected in the upper half of the box?
    (c) Suppose the box is split into 4 quadrants by the lines $x = 0.5$ and $y = 0.5$. Without computing the integrals, in which quadrant is the particle most likely to be found?

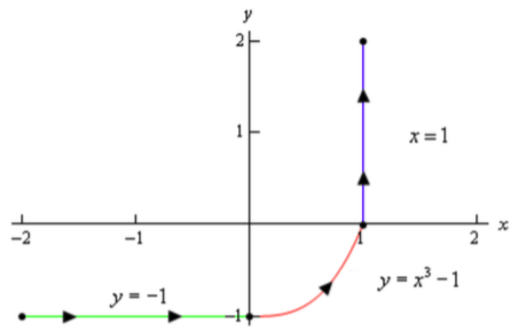
125. [Change of Variables] Use change of variables to compute the volume of the ellipsoid in $\mathbb{R}^3$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
where (a, b, c) are positive constants.

126. [Change of Variables] Evaluate $\iint_R 2x dA$, where R is the triangle with vertices $(0, 3),\ (4, 1),\ (2, 6)$. Use the change of variables $x = \frac{1}{2}(u-v),\ y = \frac{1}{4}(3u+v+12)$.

Volume 4

127. [Vector Fields] Sketch the vector field $\mathbf{F}(x, y) = y\hat{\imath} - x\hat{\jmath}$.

128. [Scalar Path Integrals] Evaluate $\int_c x dl$, where c is the curve shown below. *Hint:* $\int_0^1 t\sqrt{1+9t^4}dt = 0.94$.



The next two questions are an attempt to help you understand 1-forms intuitively. These questions are simple, but try to think about what's going on here deeply!

129. [Dot Products?] Suppose $\mathbf{F}(x, y) = 2y\hat{\imath} - 3x^2\hat{\jmath}$ and $\mathbf{v} = -1\hat{\imath} + 4\hat{\jmath}$. Compute $\mathbf{F}(2, -1) \cdot \mathbf{v}$. Interpret the result.

130. [1-Forms... or dot products?] Suppose $\alpha = 2ydx - 3x^2dy$. Evaluate the 1-form α at the point $x = 2, y = -1$ acting on the vector $\mathbf{v} = -1\hat{\imath} + 4\hat{\jmath}$.

131. [1-Forms] $\mathbf{F} = x\hat{\imath} + y^2\hat{\jmath}$. Evaluate the one form of $\mathbf{F}$, $\alpha_{\mathbf{F}}$, at $(x = -1, y = 3)$ acting on the vector $\mathbf{v} = 2\hat{\imath} - 2\hat{\jmath}$.

    This question is an attempt to help you understand what a gradient 1-form is:

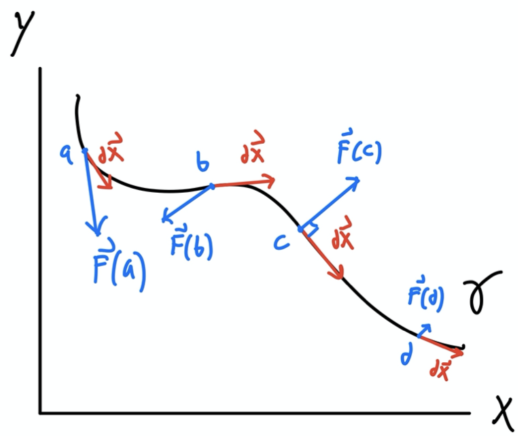
132. [Gradient 1-Forms] As you hopefully figured out in the previous problems, $\alpha_{(a,b)}(\mathbf{v}) = \mathbf{F}(a, b) \cdot \mathbf{v}$. More generally, $\alpha_{\mathbf{F}} = \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}$. Suppose α is a gradient 1-form. What must be true about the vector field $\mathbf{F}$?

133. [Gradient 1-Forms] Suppose $f(x, y) = xy^2 - x + y - 2$. Compute the gradient 1-form of f.

    The next question is an attempt to help you understand what it means to integrate a 1-form:

134. [Integrating 1-Forms... Or dot products?] Suppose you have a force field $\mathbf{F}(x, y)$ and you wish to find the work done by $\mathbf{F}$ over the path γ. Recall the formula for work: Work $= \int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma \alpha_{\mathbf{F}}$. At the four points shown, (a, b, c, d), rank

the contributions to the integral for the total work over the path from greatest to least.



______ ______ ______ ______

Greatest *Least*

135. [Integrating 1-Forms] Compute the path integral $\int_\gamma y^2dx - xdy$, where γ is the straight line path from $(1, 0)$ to $(2, 1)$.

A quick note on the above problem: Given the definitions of work and flux from chapter 5 of volume 4, I could have phrased this question in two additional ways...

(a) Compute the work done by the force field $\mathbf{F} = y^2\hat{\imath} - x\hat{\jmath}$ along the path γ. This one is hopefully obvious.

(b) Compute the flux of the vector field $\mathbf{F} = -x\hat{\imath} - y^2\hat{\jmath}$ through the path γ. This one is less obvious. See if you can work out why the resulting integral here is the same as in the original question.

136. [Integrating 1-Forms] Suppose $\mathbf{F} = -1\hat{\imath} - 2x^2\hat{\jmath}$. Compute the flux of the vector field $\mathbf{F}$ out of the semicircular arc for $y \geq 0$, oriented in the counterclockwise direction from $x = 1$ to $x = -1$. What if you oriented the arc in the clockwise direction (from $x = -1$ to $x = 1$)? Does your answer change? How would you interpret this result?

137. [Integrating 1-Forms] Setup the integral to compute $\int_\gamma y^2dx - x^2dy$, where γ is the path along $y = 4 - x^2$ oriented clockwise for $x \in [-2, 2]$.

138. [Shortcuts?] Compute $\int_\gamma \nabla\left(e^x \cos(y) + xyz + \frac{1}{2}z^2\right) \cdot d\mathbf{x}$, where γ is the straight line path from $(0, 0, 0)$ to $(0, 0, 1)$.

139. [Gradient 1-Forms] Suppose $\mathbf{F} = \nabla f$. If γ is a path going from point $\mathbf{a}$ to $\mathbf{b}$, what must be true about $\int_\gamma \alpha_{\mathbf{F}}$?

140. [Hmmm] Compute the counterclockwise circulation of the vector field $\mathbf{F}(x, y, z) = (y\cos(xy) + yze^{xyz} + 2x, x\cos(xy) + xze^{xyz}, xye^{xyz})$ around the intersection of the plane $2x + 3y - z = 4$ with the cylinder $x^2 + y^2 = 1$ (the intersection is an ellipse).

141. [hmm] Suppose we live in an n-dimensional universe where the force applied to some hyper-box is given by the vector field $\vec{F} = (x_1, x_2, ..., x_n)$. Compute the work done in moving the box from the origin to the point $(1, 1, ..., 1)$ in $\mathbb{R}^n$. Can you think of two ways of doing this problem?

142. [Green's Theorem] What does Green's theorem say for circulation? For flux?

143. [Green's Theorem] Compute the counterclockwise circulation in the plane for the vector field $\mathbf{F} = (-2y, 5x)$ along the region bounded by $x \geq y^2$ and $y \geq x^2$.

144. [Green's Theorem] Compute the flux of the vector field $\mathbf{F} = (3, 5)$ through the unit circle in the xy plane with positive flux going outwards. What do you expect the answer to be? Check!

145. [Green's Theorem] Compute the *clockwise* circulation of the vector field $\vec{F} = x^4\hat{\imath} + xy\hat{\jmath}$ around the triangle in the plane with vertices $(0, 0),\ (1, 0),\ (0, 1)$.

146. [Hmmm] Compute $\int_\gamma -xy^2dx + ydy$ over the circle of radius R in the pl

147. [Grad-Curl-Div] Which of the following area legal for a vector field $\mathbf{F}$?

    (a) $(\nabla \times \mathbf{F}) \cdot (\nabla \cdot \mathbf{F})$
    (b) $\nabla \times (\nabla \cdot \mathbf{F})$
    (c) $\nabla \times (\nabla \times \mathbf{F})$
    (d) $\nabla \cdot (\nabla \cdot (\nabla \times \mathbf{F}))$
    (e) $\nabla(\nabla \times \mathbf{F})$
    (f) $\nabla(\nabla \cdot ((\nabla \times \mathbf{F}) \times \mathbf{F}))$

148. [Grad-Curl-Div] Suppose $\mathbf{F} = xy^2\hat{\imath} + \sin(xy)\hat{\jmath} + z\hat{k}$. Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ at $(x, y, z) = (1, 0, 2)$.

149. [2-Forms] What is the area of the parallelogram spanned by the vectors $\mathbf{u} = (2, 1, 2)^T$ and $\mathbf{v} = (1, 2, 3)^T$ onto the yz plane?

150. [2-Forms] What are the basis 2-forms?

151. [2-Forms] Expand $(5dx - dy) \wedge (dx \wedge dy - 2dx \wedge dz)$.

152. [Differentiating Forms] If $\alpha = 2xydx - y^4dz$, compute $d\alpha$.

153. [Flux 2-Form] The constant vector field $\mathbf{F}(x, y, z) = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ pierces the parallelogram spanned by the vectors $\mathbf{u} = (1, 2, 2)^T$ and $\mathbf{v} = (1, 0, -1)^T$. Compute the flux through the parallelogram, where the direction of positive flux is given by the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$.

154. [Integrating 2-Forms] Compute the flux of the vector field $\mathbf{F} = 3y\hat{\imath} - 2x\hat{\jmath} + 2z\hat{k}$ out of the triangular planar region with vertices $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$. Use the direction away from the origin perpendicular to the surface as the direction of positive flux.

155. [Integrating 2-Forms... Tricky] Compute the flux of the vector field $\mathbf{F} = y\hat{\jmath} - z\hat{k}$ into the paraboloid $y = x^2 + z^2$ for $y \in [0, 1]$ capped off by the disk $x^2 + z^2 \leq 1$ positioned at $y = 1$. Use the direction *into* the surface as the direction of positive flux. *Hint:* $\int\limits_0^{2\pi} \left(\cos^2\theta + 3\sin^2\theta\right) d\theta = 4\pi$.

156. [Integrating 2-Forms] Compute the flux of $\mathbf{F} = \frac{1}{9}(x, y, z)^T$ out of the cylinder $x^2 + y^2 = 9$ for $z \in [0, 5]$.

157. [Integrating 2-Forms the clever way] Integrate the flux 2-form $\phi = (x + z)dz \wedge dx$ along the paraboloid $y = x^2 + z^2$ for $y \in [0, 1]$. The positive direction of flux is inward (along the $+x$ axis at the origin).

Some of the problems below are from previous weeks, but now we have more tools to answer these questions... Let's use them!

158. [Stokes] Stoke's theorem states that $\int_{\partial D} \alpha = \int_D d\alpha$. Verify that the two integrals (LHS vs. RHS) are indeed equivalent by finding the flux of the curl of $\mathbf{F} = y\hat{\imath} + 4z\hat{\jmath} - 6x\hat{k}$ out of the surface $z = 16 - x^2 - y^2$ for $z \geq 0$. *Hint:* $\int_0^{2\pi} \sin^2 x dx = \pi$.

159. [Stokes/Greens] Compute the *clockwise* circulation of the vector field $\vec{F} = x^4\hat{\imath} + xy\hat{\jmath}$ around the triangle in the plane with vertices $(0, 0),\ (1, 0),\ (0, 1)$.

160. [Divergence/Greens] Compute the flux of the vector field $\vec{F} = 3\hat{\imath} + 5\hat{\jmath}$ through the circle $x^2 + y^2 = 1$ with positive flux in the direction of the outward unit normal.

161. [Divergence] Compute the flux of the vector field $\vec{F} = (x + z)\hat{\jmath}$ out of the paraboloid $y = x^2 + z^2$ for $0 \leq y \leq 1$, capped of by a disk of radius 1 at $y = 1$ centered at $x = z = 0$. Use the positive direction of flux as inward (along the +y axis at the origin and along the -y axis at y=1).

162. [hmm] Consider the vector field $\vec{F} = \frac{1}{9}(x, y, z)$

    (a) Compute the flux of the vector field $\vec{F}$ out of the cylinder of radius 3: $x^2 + y^2 = 9$ for z between 0 and 5. Assume (for part a) that the cyliunder is capped off at both ends by disks of radius 3. In other words, the cylinder is a *closed surface*.

(b) Now suppose you remove the caps and you are left behind with only the cylindrical body $x^2+y^2=9$ for z between 0 and 5. How could you compute the flux out of this surface by using your result in part (a)?

out of the cylinder $x^2+y^2=9$ for $0 \leq z \leq 5$.

163. [hmm] Compute the flux of the curl of the vector field

$$\vec{F} = e^{z\sin(xy)}\hat{\imath} + (z - xy^3)\hat{\jmath} + (z^3\cos(e^{xy-z}))\hat{k}$$

out of the sphere of radius R centered at the point $(2, 3, 1)$.

164. [hmm] Compute $\int_\gamma -xy^2dx + ydy$ over the circle of radius R in the plane, traversed counterclockwise.

165. [Gauss/Stokes] What is the flux of the vector field $\mathbf{F} = xy^2\hat{\imath} + yz^2\hat{\jmath} + x^2x\hat{k}$ out of the sphere of radius 3 centered at the origin?

166. [Gauss/Stokes] Compute the flux of the vector field

$$\mathbf{F} = \left(xy + \tanh\left(z - e^{-y}\right)\right)\hat{\imath} + \left(e^{x\cos(xz/2)} - 72\right)\hat{\jmath} + \left(\sinh^2(1-x) + x - yz\right)\hat{k}$$

into the cone $z = \sqrt{x^2+y^2}$ for $z \in [0,1]$, capped off by a disk of radius 1 positioned at $z=1$.

167. [Stokes/Gauss] You are trying to compute the amount of light exiting a light bulb for R & D purposes. You are supplied the rendering below, and determine that the information encoding the direction and intensity of each light ray emanating from the source can be modeled by the vector field $\nabla \times \mathbf{F}$, where $\mathbf{F} = \left(e^{z^2-2z}x, \sin(xyz) + y + 1, e^{z^2}\sin(z^2)\right)$. You also know that the base of the light bulb can be modeled by the unit circle $x^2+y^2=1$. What is the total amount of light leaving the bulb (i.e. the flux out of the surface), and what recommendation might you make to the supplier of the filament responsible for powering the bulb? Try to keep things professional, "you're a bunch of con artists" might not be appropriate.

168. [Stokes/Gauss] Suppose that the flux of the curl of some vector field $\mathbf{F}$ out of the hemisphere (left) with its base centered at the origin is given by H.

(a) What is the flux of the curl out of the ice cream cone like object (middle)?

(b) Do you have enough information to determine the flux of the curl out of the cone (right)?