Proofs that $\det(A^t) = \det A$.

Eric O. Korman

1 Proof 1

We consider two cases: $\det A = 0$ and $\det A \neq 0$. First assume that $\det A = 0$. Then by a theorem in the text, A is not invertible. This implies that A^t is not invertible since we have seen that a matrix is invertible if and only if its transpose is. Thus $\det A^t = 0$ so in this case we have $\det A^t = \det A$.

Now assume that $\det A \neq 0$. Then A is invertible and can therefore be written as a product $E_1 \cdots E_k$ of elementary matrices. We claim that $\det E^t = \det E$ for any elementary matrix. This is because if E is of the second or third type of elementary matrix then $E = E^t$ so that $\det E^t = \det E$. If E is of the first type then so is E^t. But from the text we know that $\det E = 1$ for all elementary matrices of the first type. This proves our claim. Using properties of the transpose and the multiplicative property of the determinant we have

$$\begin{aligned}
\det A^t &= \det((E_1 \cdots E_k)^t) \\
&= \det(E_k^t \cdots E_1^t) \\
&= \det(E_k^t) \cdots \det(E_1^t) \\
&= \det E_k \cdots \det E_1 \\
&= \det E_1 \cdots \det E_k \\
&= \det(E_1 \cdots E_k) \\
&= \det A.
\end{aligned}$$

2 Proof 2

We will prove that $\det A = \det A^t$ using the fact that the determinant can be computed by cofactor expansion along any row or column (this fact really shouldn't be assumed...)

We proceed by induction on n. For the base case $n = 1$ we have $A = A^t$ so that $\det A = \det A^t$, as desired. For the inductive step assume the result is true for $n = k - 1$ and let A be a $k \times k$ matrix. Write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

so that

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{kk} \end{pmatrix}.$$

Using cofactor expansion along the first column of A we have

$$\det A = a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{k+1} a_{k1} \det A_{k1}$$

where A_{ij} is the matrix obtained from A by removing the ith row and jth column. Using cofactor expansion along the first row of A^t we have

$$\det(A^t) = a_{11} \det(A^t)_{11} - a_{21} \det(A^t)_{12} + \cdots + (-1)^{k+1} a_{k1} \det(A^t)_{1k}.$$

We see that $(A^t)_{ij} = (A_{ji})^t$. Since A_{ji} is a $(k-1) \times (k-1)$ matrix we can use the inductive hypothesis to see that

$$\det(A^t)_{ij} = \det((A_{ji})^t) = \det A_{ji}.$$

Making this substitution into the above formula for $\det(A^t)$ gives $\det(A^t) = \det A$.