Twistor Hecke eigensheaves in genus 2

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Abstract

Following the strategy outlined in [DP09, DP22] for bundles of rank 2 on a smooth projective curve of genus 2, we construct flat connections over the moduli of stable bundles, with singularities along the wobbly locus. We verify that the associated $\mathscr{D}$-modules are Hecke eigensheaves. The local systems are constructed by the nonabelian Hodge correspondence from Higgs bundles. The spectral varieties of the Higgs bundles are the Hitchin fibers corresponding to the Hecke eigenvalues.

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1 Introduction

1.1 Geometric Langlands

Suppose C is a smooth projective curve over the complex numbers. Let $\mathscr{X}$ be a moduli stack of vector bundles or principal G-bundles for a reductive group G. The open substack of semistable bundles has a good coarse moduli space X. The *geometric Langlands program* predicts the existence of certain perverse sheaves on $\mathscr{X}$. These are characterized by a property called the *Hecke eigensheaf property* with respect to a local system Λ on C, and it is predicted that to a given Λ there is a unique Hecke eigensheaf. In turn, a perverse sheaf on $\mathscr{X}$ leads to a local system on an open subset of X. So, we have a construction going from a local system Λ on the curve C to a local system on an open subset of X.

Our purpose in this paper, pursuing the program of the first two authors, is to look at this construction from the viewpoint of the Hitchin equations. These equations lead to what is variously known as the *nonabelian Hodge correspondence* or *Kobayashi-Hitchin correspondence* relating local systems to Hitchin pairs or *Higgs bundles*. These geometric objects showing up on the other side of the correspondence are in many respects more tangible than local systems, in particular they are associated to spectral data including a spectral cover which is typically a ramified covering of the base. We would like to understand the relationship between the spectral data of the Higgs bundle corresponding to the input local system Λ, and the spectral data corresponding to the resulting output local system over an open subset of X.

Before getting to a more detailed look at what we do, let's recall that the geometric Langlands program originated as a geometrization of the *Langlands program*, where automorphic functions are categorified to data of perverse or constructible sheaves. In positive characteristic for ℓ-adic sheaves, this categorification is based on the function-sheaf correspondence, where a constructible sheaf defines a function on the set of finite field-valued points by associating to each point the trace of its Frobenius action on the stalk of the sheaf. Due to several authors [Dri80, Dri83, Dri84, Lau87, Lau95, BD97, Gai15, Gai17], the geometric Langlands program was then carried from positive characteristic to characteristic zero, where we no longer have a function-sheaf correspondence, but the resulting statement at the sheaf level still makes sense. Somewhat surprisingly, over $\mathbb{C}$, or more generally over local fields of characteristic zero, there is an analytic function-theoretic version of the Langlands correspondence originally envisioned in [Lan71, Fre14] and further developed in [EFK21]. The recent works [EFDK22, EFK23, EFK22, BK22] have made exciting advances in understanding and proving instances of this analytic statement. The same works also make connections with the categorified geometric version of the Langlands conjecture and we expect them to ultimately have a direct relation with our Hodge theoretic approach. Exploring this analytic picture and the expected relation is a very interesting question which, unfortunately, is beyond the scope of the present paper.

A main player in both the function-theoretic and the categorified Langlands conjecture over $\mathbb{C}$ is the algebra of *Hecke correspondences*. These act on the moduli stack of principal G-bundles over a Riemann surface C. This is most easily understood for $G = GL_2$. Viewed as a multivalued function, the Hecke correspondence at a point $t \in C$ takes a rank 2 bundle E to the sum of its *Hecke transforms* at t: these are the elementary transforms E' fitting into exact sequences

$$0 \to E' \to E \to \mathbb{C}_t \to 0$$

where $\mathbb{C}_t$ is the skyscraper sheaf of length 1 at t. The set of Hecke transforms of E is parametrized by the set of rank 1 quotients $E_t \to \mathbb{C}_t$, which is a $\mathbb{P}^1$. Thus, a bundle E is sent to a formal sum of a $\mathbb{P}^1$'s-worth of new bundles E'. In the function world, one just takes the sum over the discrete set of points of $\mathbb{P}^1$. In the sheaf-theoretical viewpoint, the formal sum is replaced by the cohomology of a sheaf over $\mathbb{P}^1$.

The correspondence depends on the choice of point[1] $t \in C$. Thus, letting X denote the moduli space of bundles, we obtain the Hecke correspondence

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X \times C. \end{tikzcd}$$

If $\mathscr{F}_B$ is a perverse sheaf on X, its *Hecke transform* is the perverse sheaf $Rq_*(p^*\mathscr{F}_B)$ on $X \times C$. The value of the Hecke transform $\mathcal{H}(t)$ at a point t is the restriction of this on $X \times \{t\}$.

In a formal viewpoint, the Hecke operations at different points $t, t' \in C$ commute. The classical theory therefore views the whole algebra of Hecke operations as an algebra of commuting operators, and it becomes natural to look for a common diagonalization of these operators.

In the sheaf-theoretical viewpoint, it means that we are looking for *Hecke eigensheaves* $\mathscr{F}_B$ on X, corresponding to *Hecke eigenvalues* Λ_B that are perverse sheaves on C. The eigenvalue equation, saying in naive terms that the Hecke operation $\mathcal{H}(t)$ multiplies $\mathscr{F}_B$ by the eigenvalue that depends on t, is written as

$$Rq_*(p^*\mathscr{F}_B) \cong \mathscr{F}_B \boxtimes \Lambda_B.$$

One of the main tasks of the geometric Langlands program is to construct, for a given eigenvalue Λ_B (that's really the sheaf-theoretical version of the notion of "collection of eigenvalues one for each Hecke operation $\mathcal{H}(t)$"), a Hecke eigensheaf $\mathscr{F}_B$ corresponding to this eigenvalue. The "de Rham" version of the geometric Langlands conjecture a Hecke eigenvalue is

[1]This needs to be modified in case we look at bundles of fixed determinant as we do in the present paper—the parametrizing data is then a point in a covering $\overline{C}$ of C.

a flat bundle Λ_{dR} on C while a corresponding Hecke eigensheaf is a $\mathscr{D}$-module $\mathscr{F}_{dR}$ on X and these fit better with parametrizations for the next paragraph.

The translation of Langlands' conjecture from the automorphic world to the geometric context predicts that there will be a unique such eigensheaf for each eigenvalue Λ_{dR}. Furthermore, it states that as a function of Λ_{dR}, the eigensheaf varies in a coherent way. Namely, rather than starting with an individual flat bundle Λ_{dR} on C, we could start with a combination of these in the form of a coherent sheaf on the moduli space $Flat$ of Λ_{dR}'s. The (naive) geometric Langlands conjecture predicts that to such a coherent sheaf should be associated a unique $\mathscr{D}$-module $\mathscr{F}_{dR}$ on the moduli Bun bundles on C, and that this correspondence should set up a duality between complexes of coherent sheaves on $Flat$ and complexes of $\mathscr{D}$-modules on Bun.

The notion of *Langlands dual group* enters here: if we use a reductive group G to speak of the moduli stack $Bun(G)$ of principal G-bundles on C, then we need to take the Langlands dual group ${}^LG$ and look at coherent sheaves on the moduli $Flat({}^LG)$ of flat ${}^LG$ bundles on C.

Much important progress has been made on establishing the geometric Langlands correspondence. Drinfeld was the first to make a construction of Hecke eigensheaves for the group GL_2. His article [Dri83] changed over from the sheaf viewpoint to the function viewpoint somewhere in the middle, so it really constructs Hecke eigenfunctions. Laumon [Lau95, Lau87] formalized and generalized this construction, yielding a solution of the geometric Langlands problem for GL_2, and a conjectural framework for GL_n. Gaitsgory gave an alternative proof in his thesis [Gai97]. Then Lafforgue [Laf02] proved it for GL_n in the number-theoretical context, and Frenkel-Gaitsgory-Vilonen proved the geometric version for GL_n [FGV02]. In a first noncompact case, Arinkin [Ari01] treated the case of parabolic bundles on $\mathbb{P}^1$ with 4 singular points.

We will be looking in detail at the non-abelian Hodge theory approach to Drinfeld's original construction in Section 13, and will show how its Dolbeault version can be understandood via abelianization and the spectral cover construction. The constructions for higher rank can probably also be recast in terms that would be more familiar to geometers, although this is bound to contain a certain level of complication and we don't attempt it here.

It has remained, at least until fairly recently, elusive how one would attack the problem for general groups G. One may take note of recent progress such as [BC22, Ber21, Ber20, Ber19, Roz21, FR22, Fæ22] in the de Rham setting and [BZN18, AGK+22c, AGK+22b, AGK+22a] in the Betti setting.

It has also been understood, in the preceding period, that the simple description we have tried to approximate above is not adequate to describe a duality between the two sides of the geometric Langlands correspondence, and indeed that the two categories have different properties making it so that they couldn't be equivalent. Therefore, the categories need to be modified. The work of Gaitsgory and Rozenblyum on Ind-coherent sheaves [Gai11, GR14] and of Arinkin, Gaitsgory, and collaborators on nilpotent singular supports [AG15, AGK+22b, AGK+22a] aims to solve these problems. They mostly have to do with parts of the categories that are supported on or close to locations in the moduli spaces where various kinds of singularities occur. Therefore, in looking for a global understanding of at least some part of the correspondence, we will ignore these subtleties.

Recently, Gaitsgory and a group of co-authors have announced a full proof of the geometric Langlands conjecture for all groups, with first drafts available [GAB+24].

Something that has not, in our view, been sufficiently emphasized in previous works in this area is the fact that the geometric Langlands program predicts something very specific about the topology and geometry of moduli spaces of vector bundles on curves. This question was however raised in Sawin's MathOverFlow post [Saw16]. For one thing, the moduli stack is a pretty wild beast, being only locally of finite type. However, it contains an open substack that is the moduli of semistable bundles. This substack is in turn close to being a projective variety, in that its coarse moduli space is the projective moduli space X of S-equivalence classes of semistable bundles, whose points are in $1:1$ correspondence with the polystable bundles.

Therefore, for a perverse sheaf on $Bun(G)$ we will have an open subset X^{vs} on which the perverse sheaf is a locally constant sheaf, i.e. a representation of $\pi_1(X^{\mathrm{vs}})$. It was known early on by Laumon that if the perverse sheaf in question is a Hecke eigensheaf, the corresponding open subset X^{vs} has an explicit description as the moduli space of *very stable* bundles: a very stable bundle is one that does not admit a nonzero nilpotent Higgs field.

So, and in spite of the oversimplifications in the above presentation, the abstract geometric Langlands correspondence predicts something concrete and easily understandable: that to a perverse sheaf Λ, let us say itself a local system on C, there should be naturally associated a local system over X^{vs}. It is this construction that we would like to study in the present paper.

There are several motivations for the viewpoint we adapt here. The first was a small detail immediately noticed by the first author, in the Manin volume of the *Duke Mathematical Journal*. Hitchin's article on the moduli space of Higgs bundles [Hit87b] and Laumon's article on the geometric Langlands correspondence [Lau87] were both in this volume. In Remarque 5.5.2, Laumon states that Deligne, in unpublished communication, calculated the multiplicity of the zero-section in the characteristic cycle of the $\mathscr{D}$-module: this is the same as the rank of the local system over X^{vs}. The answer (for SL_2-bundles on a curve of genus g) was 2^{3g-3}. This number was the same as the degree of the map from a general fiber of the Hitchin system to the moduli space of bundles.

The general fiber of the Hitchin system is a subvariety in what is, basically, the cotangent bundle of X (the necessary modifications to that statement will be the subject of discussion later).

In light of the nonabelian Hodge correspondence, started in Hitchin's paper [Hit87a] and developed in the highest level of generality by the third author [Sim92] and Mochizuki [Moc06, Moc09] some time later, it would look natural to think of the general fiber of the Hitchin system, a subvariety of $T^{\vee}X$ of degree 2^{3g-3} over the base X, as a good candidate for being the spectral variety of the geometric Langlands local system of rank 2^{3g-3}. This idea became the conjecture of the first two authors [DP09], and is the essence of what we will be trying to do here.

Some early ideas in this direction were contained in a letter from Hausel to Hitchin (unpublished), and in Faltings' talk at Deligne's 61th birthday conference.

A next element of motivation explains more precisely how this should be organized. This is known as *electric-magnetic duality* in the work of Kapustin and Witten [KW07] or the *classical limit of geometric Langlands* [DP12, DP09]. Hausel and Thaddeus [HT03, Hau21] view it as a form of mirror symmetry.

The Hitchin moduli spaces of principal Higgs bundles for the groups G and ${}^LG$ fit into a diagram

$$\begin{tikzcd}[column sep=small] \mathsf{Higgs}(G) \arrow[dr, "\boldsymbol{h}"'] & & \mathsf{Higgs}({}^LG) \arrow[dl, "{}^L\boldsymbol{h}"] \\ & \mathcal{B} & \end{tikzcd}$$

where:

Theorem 1.1. *The base affine spaces of the Hitchin fibration* $\mathbf{B}$ *for* G *and* ${}^LG$ *are naturally isomorphic. Furthermore, the two Hitchin maps* $\boldsymbol{h}$ *and* ${}^L\boldsymbol{h}$ *are generically dual SYZ-type torus fibrations.*

We refer to [HT03, KW07, DP12] for the proof.

The prediction of the de Rham version of the geometric Langlands correspondence may be stated as in the following several paragraphs. Let X_G be the coarse moduli space of semistable G-bundles on C. Then given an ${}^LG$-flat bundle Λ_{dR} on C, there should be a $\mathscr{D}_X$-module $\mathscr{F}_{dR}$ on X_G that is (reasonably approximately[2]) a Hecke eigensheaf for Λ_{dR}.

Through the non-abelian Hodge theorem the eigenvalue ${}^LG$-flat bundle Λ_{dR} corresponds to a point Λ_{Dol} of $\mathsf{Higgs}({}^LG)$, in the fiber ${}^L\boldsymbol{h}^{-1}(\mathsf{b})$ over a point $\mathsf{b} \in \mathcal{B}$ in the Hitchin base that we will suppose to be general.

The fiber $\mathcal{P} := \boldsymbol{h}^{-1}(\mathsf{b}) \subset \mathsf{Higgs}$ is the dual torus of ${}^L\boldsymbol{h}^{-1}(\mathsf{b})$, so the Fourier-Mukai transform for the dual torus fibrations says that the point Λ_{Dol} (and hence the point Λ_{dR}) corresponds to a line bundle $\mathfrak{L}$ over $\mathcal{P}$.

The moduli stack of G-Higgs bundles is isomorphic to the cotangent stack of $Bun(G)$. For moduli spaces, this is no longer true globally, but it remains true birationally. The set of very stable points X^{vs} of the coarse moduli space of G-bundles is smooth, and its cotangent space $T^\vee(X^{\mathrm{vs}})$ is the subspace $\mathsf{Higgs}^{\mathrm{vs}}$ of G-Higgs bundles whose underlying bundle is very stable.

The first two authors propose to use $\mathcal{P}$, viewed in a birational sense, after blowing up, as a subvariety Y of the (logarithmic) cotangent bundle of X, to be a spectral cover of X. And to use $\mathfrak{L}$ as input spectral datum to construct a logarithmic parabolic Higgs sheaf $\mathscr{F}_{Dol}$ on X that, under the non-abelian Hodge correspondenc [Moc06, Moc09], should correspond to the Hecke eigensheaf $\mathscr{D}_X$-module $\mathscr{F}_{dR}$.

The divisor of singularities, complement of Laumon's open set X^{vs} of very stable points, was termed the *wobbly divisor* Wob $\subset X$ in [DP09]. Points of Wob are semistable bundles such that $H^0(\mathrm{ad}(E) \otimes \omega_C)$ has a nonzero nilpotent element.[3]

Under this birational transformation thhe degree 0 line bundle $\mathfrak{L}$ is modified to a line bundle $\mathscr{L}$ on Y, and parabolic structure is supposed to be added, along divisors lying over the wobbly divisor.

Here is a first, albeit incomplete, formulation of the conjecture of the first two authors.

Conjecture 1.2 ([DP09, DP22]). *Suppose Λ_{dR} is an ${}^LG$ local system on C corresponding to*

[2] The statement for the coarse moduli space X_G is not exactly the same as the stack-theoretical statement, because the extension from a local system on X_G^{vs} to a perverse sheaf on X_G is a different object than the extension to a perverse sheaf on Bun_G; we can expect some modifications to the Hecke eigensheaf property. Such a distinction does not seem to intervene at the level discussed in the present paper.

[3] It turns out, looking into the study of the classical *quadric line complex* which is our moduli space X_1, that the wobbly locus in that case was known as the union of what are called *special lines* [GH94, page 792].

a point Λ_{Dol} in the Hitchin fiber of $\mathsf{Higgs}(^L G)$ *over a general point $b \in \mathcal{B}$ in the Hitchin base, and let $\mathfrak{L}$ be the dual line bundle on the Hitchin fiber $\mathcal{P}$ of* $\mathsf{Higgs}(G)$ *over the same point b. Then the Hecke eigensheaf $\mathscr{F}_{dR}$ for eigenvalue Λ_{dR} corresponds, via Mochizuki's Kobayashi-Hitchin correspondence, to a parabolic logarithmic Higgs bundle $\mathscr{F}_{Dol,\bullet}$ on* (X_G, Wob). *The spectral data over the open subset* $X_G^{\mathrm{vs}} = X_G - \mathrm{Wob}$ *consists of the pullback open subset $\mathcal{P}^{\mathrm{vs}}$ considered as a subvariety of $T^\vee(X_G^{\mathrm{vs}})$, with the restriction of $\mathfrak{L}$ as spectral line bundle.*

The nonabelian Hodge correspondence is fundamentally global, involving the solution of Hermite-Yang-Mills-Higgs equations minimizing a Yang-Mills functional over the manifold. In particular, the specification of the Higgs bundle by its spectral data over an open subset X_G^{vs} does not uniquely specify how it might be extended to (X_G, Wob). Such an extension involves choosing an extension of the bundle, and possibly a parabolic structure.

On the smooth parts of the wobbly divisor $\mathrm{Wob} = X_G - X_G^{\mathrm{vs}}$, one might be able to formulate a more precise conjecture spelling out how the parabolic structure is supposed to look. This was done for the case of the root stack of $\mathbb{P}^1$ with five singular points in [DP22].

One of the objectives of the study we do in the present paper is to continue the investigation of what type of behavior to expect for this structure.

The wobbly divisor will, in general, have singularities that are more complicated than normal crossings. Therefore, in order to apply the general theory [Moc06], we need to blow up to resolve those singularities. Because of the Bogomolov-Gieseker inequality, the way to extend the parabolic structure—that will have been defined in codimension 1—is unique. It is determined by the condition of minimizing c_2^{par} (or equivalently maximizing $\mathrm{ch}_2^{\mathrm{par}}$) subject to the constraint $c_1^{\mathrm{par}} = 0$. However, no recipe is currently known for doing this. In the present paper, we are faced with two different situations of this type, and we adopt two different strategies for attaining the minimum, and showing that it is $c_2^{\mathrm{par}} = 0$ that leads to a harmonic bundle and hence a local system.

The long-term hope is that we could understand these processes in a uniform way for all groups G, obtaining a uniform construction of Hecke eigensheaves at least for generic initial eigenvalue data Λ.

1.2 The case of genus 2 and rank 2

In this paper, we take a much less lofty goal: to understand how this works in the case where G is PSL_2 and the curve C has genus 2. At the end of the paper, we will prove a comparison with Drinfeld's construction. This proof in Chapter 13 will actually prove

Conjecture 1.2 for SL_2 on curves of arbitrary genus, but it does not immediately give the further information about parabolic structures. That seems attainable and could be the topic of future work.

Let us now look more carefully at what is to be done. The reader may refer to Section 1.3 below for concise statements of the main theorems.

Here are a few introductory notations. The curve C is a smooth projective curve of genus $g = 2$. It is hyperelliptic, with degree 2 map $\mathsf{h} : C \to \mathbb{P}^1$ and hyperelliptic involution $\boldsymbol{\iota}_C$. Out of the 6 ramification or ***Weierstrass*** points, fix one of them denoted $\mathbf{p}$ to be used as a basepoint (e.g. for Abel-Jacobi maps) throughout the paper.

One of the first main inputs is the beautiful classical theory of moduli spaces of semistable bundles of rank 2 and fixed determinant.

These were investigated extensively by Tyurin [Tyu64], Narasimhan and Ramanan [NR69], Desale and Ramanan [DR76], Newstead [New68], and then further by Previato-van Geemen [vGP96], Beauville [Bea06], Heu-Loray [Heu09, HL19, HL17], Pal-Pauly [PP21a], etc.

Let X be a moduli space of polystable vector bundles of rank 2 on C with a fixed determinant line bundle. Up to isomorphism given by tensoring with a line bundle, there are two possibilities depending on the parity of the degree. The following explicit descriptions are provided by Narasimhan and Ramanan [NR69]:

- X_0 is the moduli space of bundles of degree 0 with determinant $\mathcal{O}_C$, and $X_0 \cong \mathbb{P}^3$;
- X_1 is the moduli space of bundles of degree 1 with determinant $\mathcal{O}_C(\mathbf{p})$, and $X_1 = Q_1 \cap Q_2 \subset \mathbb{P}^5$ is the intersection of two quadrics.

The explicit descriptions were generalized to bundles of rank 2 over higher genus hyperelliptic curves by Desale-Ramanan [DR76].

In even degree X_0 is only a coarse moduli space: it is known that a universal family does not exist even over any Zariski open set. For bundles of odd degree, a universal family exists, and heuristically it seems to be significantly easier explicitly to describe points of X_1.

Let Higgs_0 resp. Higgs_1 denote the moduli space of Higgs bundles of the same rank, degree and determinant. These were studied early on by Previato and van Geemen [vGP96].

For comments that apply to both cases, and to some extent for other curves and other groups, we will just write X and Higgs for either of the spaces. For example we can say in that $\dim(X) = 3$ and $\dim(\mathsf{Higgs}) = 6$.

The ***Hitchin fibration*** is the map $\mathsf{Higgs} \xrightarrow{\boldsymbol{h}} \mathcal{B} = \mathbb{A}^3$ sending a Higgs bundle to the moduli point of its associated spectral curve $\widetilde{C} \subset T^\vee C$. The ***nilpotent cone*** has X as its principal component:

$$\boldsymbol{h}^{-1}(0) = X \cup \text{ (other components)}.$$

The other components usually correspond to the components of the wobbly locus Wob [PP21b, PN20, HH22], in the sense that the other components of $\boldsymbol{h}^{-1}(0)$ intersect X along components of Wob. Our case of Higgs_0 is an exception: there, Wob_0 has an extra component that does not correpond to an additional component of the nilpotent cone.

Indeed, the wobbly locus of X_0 decomposes as

$$\mathrm{Wob}_0 = \mathsf{Kum} \cup \left[\cup_{\kappa \in \mathsf{Spin}(C)} \mathsf{Trope}_\kappa\right],$$

where $\mathsf{Kum} \subset \mathbb{P}^3$ is the Kummer surface associated to C, and Trope_κ are the sixteen ***trope planes***, which are naturally labeled by the set $\mathsf{Spin}(C)$ of theta characteristics κ of C. The trope planes do correspond to the extra components of the nilpotent cone. But the Kummer surface does not, rather it consists entirely of singular points in Higgs_0 that could be viewed as some kind of infinitesimally nearby components of $\boldsymbol{h}^{-1}(0)$.

The Kummer surface itself has 16 nodes and these form, with the 16 trope planes, the famous ***Kummer*** 16_6 ***configuration*** [GH94, Bea96, Keu97, Dol20]. Each plane passes through 6 points and each point is contained in 6 planes.

In X_1, the wobbly locus is a singular surface whose normalization is

$$\mathbb{P}^1 \times \overline{C} \to \mathrm{Wob}_1.$$

This map comes about in the following way: the curve $\overline{C}$, a 16-sheeted etale cover of C, sits inside Higgs_1 as the second fixed point locus of the $\mathbb{C}^\times$ action; the downward direction of the $\mathbb{C}^\times$ action is forms a projectively trivial bundle over $\overline{C}$ and the limits of downward orbits are the points of Wob_1. In particular, Wob_1 does equal the locus where the single extra component of the nilpotent cone meets X_1.

Several authors have recently investigated more closely the structure of the structure of the nilpotent cone and the wobbly locus in general situations: Bozec [Boz22], Gothen-Zúñiga-Rojas [GnR22], Pal-Pauly [PP21b], Peón-Nieto [PN20, FGOPN23], Zelaci [Zel20], Hellmann [Hel21], Hausel-Hitchin [HH22].

The fiber of the Hitchin fibration over a general point $\mathsf{b} \in \mathbb{A}^3$ of the Hitchin base is a Prym variety $\mathcal{P} := \boldsymbol{h}^{-1}(\mathsf{b})$. The point b corresponds to a spectral curve

$$\pi : \widetilde{C} \to C,$$

which is a $2:1$ covering ramified over 4 points. The abelian variety $\mathcal{P}$ is the Prym variety of line bundles on $\widetilde{C}$ whose norm down to C is the line bundle $\mathcal{O}_C(2\mathsf{p})$ for X_0 or $\mathcal{O}_C(3\mathsf{p})$ for X_1. The shift by $\mathcal{O}_C(2\mathsf{p}) \cong \omega_C$ is due to the fact that $\pi_*(\mathcal{O}_{\widetilde{C}}) \cong \mathcal{O}_C \oplus \omega_C^{-1}$ so the determinant line bundle of the rank 2 vector bundle is ω_C^{-1} times the norm line bundle.

Over the open subset of very stable points,

$$\begin{tikzcd} \mathsf{Higgs}^{\mathrm{vs}} \cong T^*X^{\mathrm{vs}} \arrow[d, "\mathfrak{f}"] \\ X^{\mathrm{vs}} \end{tikzcd}$$

The identification between $\mathsf{Higgs}^{\mathrm{vs}}$ and $T^\vee(X^{\mathrm{vs}})$ comes from Serre duality wherein the cotangent space

$$T^\vee(X^{\mathrm{vs}})_E = H^1(\mathrm{End}^0(E))^* \cong H^0(\mathrm{End}^0(E) \otimes \omega_C)$$

identifies with the space of Higgs fields on a given bundle $E \in X^{\mathrm{vs}}$.

On $\mathcal{P}^{\mathrm{vs}} := \mathcal{P} \cap \mathsf{Higgs}^{\mathrm{vs}}$ we obtain a map $f^{\mathrm{vs}} : \mathcal{P}^{\mathrm{vs}} \to X^{\mathrm{vs}}$ which, in our case, is a finite $8:1$ ramified covering.

In order to go from the given spectral data over the open subset X^{vs} to completed objects over X, the first task is to blow up $\mathcal{P}$ in order to resolve the rational map $\mathcal{P} \dashrightarrow X$, given by f^{vs} on $\mathcal{P}^{\mathrm{vs}}$, into a morphism.

This might be complicated in the general setting. In rough terms, the projection from our hoped-for spectral variety $\mathcal{P}^{\mathrm{vs}} \to X^{\mathrm{vs}}$ should extend to a finite map $Y \xrightarrow{f} X$ where $Y \to \mathcal{P}$ is some kind of a blowing-up of $\mathcal{P}$ along the locus $\mathfrak{Q} \cap \mathcal{P}$ for the complement $\mathfrak{Q} := \mathsf{Higgs} - \mathsf{Higgs}^{\mathrm{s}}$ of the set of Higgs bundles whose underlying bundle is stable (that's a little bigger than $\mathsf{Higgs}^{\mathrm{vs}}$). The subvariety $\mathfrak{Q}$ is the union of the collection of incoming directions to the higher level fixed point sets of the $\mathbb{C}^\times$ action.

In general, we don't know how to describe the appropriate blow-up. It will probably contain a sequence of blow-ups along various subvarieties within $\mathfrak{Q}$. Luckily, in the case of rank 2 bundles on a genus 2 curve, it suffices to blow up once and there are explicit descriptions:

- In $\mathcal{P}_3 = \{L \in \mathrm{Jac}^3(\widetilde{C}) \,|\, \mathsf{Nm}_\pi(L) = \mathcal{O}_C(3\mathsf{p})\}$ we have a smooth curve $\widehat{C} := \widetilde{C} \times_C \overline{C}$ and Y_1 is the blow-up of $\mathcal{P}_3$ along this curve. Let $\boldsymbol{E}_1 \subset Y_1$ denote the exceptional divisor, mapping to Wob_1. We will denote the blow-up maps by $\boldsymbol{\varepsilon}$ with indices if necessary.

- In $\mathcal{P}_2 = \{L \in \mathrm{Jac}^3(\widetilde{C}) \,|\, \mathsf{Nm}_\pi(L) = \mathcal{O}_C(2\mathbf{p})\}$ we should blow up 16 points; the exceptional locus $\boldsymbol{E}_0 = \sqcup_{\kappa\in\mathsf{Spin}(C)}\boldsymbol{E}_{0,\kappa}$ is a union of 16 planes in Y_0 labeled by theta characteristics of C and mapping to the corresponding trope planes in X_0.

The inclusion $\mathcal{P}^{\mathrm{vs}} \hookrightarrow T^\vee X^{\mathrm{vs}}$ extends to an inclusion

$$Y \hookrightarrow T^\vee X(\log \mathrm{Wob})$$

well-defined at least up to and including codimension 1. The essential reason for this is that the birational map from Y to $T^\vee X$ is given by a global 1-form $\boldsymbol{\alpha}$ on Y, termed the ***spectral* 1*-form*** that is in fact pulled back from a 1-form on $\mathcal{P}$ associated to the tautological 1-form of the original spectral covering $\widetilde{C}/C$. The section over Y of the pullback of $T^\vee X$ can have poles at points where Y/X is ramified and $\boldsymbol{\alpha}$ does not vanish on the vertical directions of the ramification, but in this case the poles are logarithmic.

The ***spectral correspondence*** [BNR89, Don95, DM96] says that from a spectral covering contained in the cotangent bundle, together with a line bundle, we obtain a Higgs sheaf on the base. For parabolic Higgs sheaves, the spectral covering should be inside the logarithmic cotangent bundle with logarithmic poles along Wob. In our situation, the inclusion of Y into the logarithmic cotangent bundle insures that the logarithmic property holds along the smooth points of Wob. Understanding what happens at singular points of Wob is one of the main technical difficulties.

The image of $Y \hookrightarrow T^\vee X(\log \mathrm{Wob})$ is the spectral covering predicted by the program of the first two authors [DP09, DP12, DP22], as stated in Conjecture 1.2 above. The proposal for realizing this subvariety as a spectral covering associated to a local system, is to apply the nonabelian Hodge correspondence initiated by Hitchin [Hit87a], and taken to the case of open varieties of higher dimension by Mochizuki [Moc07a, Moc07a, Moc06, Moc09], in order to get a harmonic bundle and therefore a corresponding local system on X^{vs}.

This requires resolving the singularities of Wob to get a normal crossings divisor, choosing a line bundle over Y and then over the resolution of singularities, and putting an appropriate parabolic structure on the resolution in order to get a logarithmic Higgs field with respect to a normal crossings divisor.

Once these are done, one needs vanishing of the parabolic Chern classes c_1^{par} and c_2^{par}. The first two authors conjecture that placing a well-adjusted parabolic structure over the wobbly locus should insure this vanishing. This was done for the case of $\mathbb{P}^1$ with 5 marked points in [DP22]. That paper involves a nontrivial interaction between parabolic structures on the curve C and these parabolic structures over (X, Wob).

In the present paper devoted to the same the program for bundles of rank 2 on smooth projective curves of genus 2, the parabolic weight structure is much simpler. Indeed, we find that a parabolic structure, with weight $\alpha = 1/2$, is needed for X_1; and that for X_0 no parabolic structure is needed over the trope planes or the Kummer surface.

The Chern class calculations will prescribe the numerical equivalence class of the line bundle $\mathscr{L}$ necessary to obtain a parabolic Higgs sheaf $\mathscr{F} := f_*\mathscr{L}$ on X with logarithmic structure along the smooth locus of Wob such that the appropriately calculated c_1^{par} and c_2^{par} vanish.

Over X_1, the parabolic structure is localized upstairs near the exceptional divisor $\boldsymbol{E}_1$. The map f has ramification index of 2 at a general point of $\boldsymbol{E}_1$, and that gives us the sub-bundle of the direct image that we will use to put the parabolic structure.

The standard restriction theorems for the Kobayashi-Hitchin correspondence [Moc06] say that it suffices to work down to codimension 2 in X. This convention will be adopted throughout the paper. An understanding of the codimension 2 pieces, namely the singular locus in codimension 1 of Wob, is needed in order to verify the vanishing of the second Chern class. In our case, up to codimension two in X, Wob_1 has normal crossings and curves along which it is cuspidal, whereas Wob_0 has normal crossings and curves along which there is a tacnode (the trope planes meet the Kummer surface in double conics).

Some special strategies are employed in Chapters 4 and 5 to obtain the calculation of parabolic Chern classes at these singularities, helped by the fact that the parabolic weights were limited to $0, 1/2$. This promises to be a stumbling block in the general situation.

The conclusion of the Chern class calculations is that the line bundle over Y_0 should be of the form

$$\mathscr{L}_0 = \boldsymbol{\varepsilon}_0^*(\mathfrak{L}_0) \otimes f_0^*\mathcal{O}_{X_0}(2) \otimes \mathcal{O}_{Y_0}(\boldsymbol{E}_0)$$

where $\mathfrak{L}_0$ is a degree 0 line bundle on the Prym variety $\mathcal{P}_2$, and $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ is the blow-up map.

Similarly, over Y_1 it should have the form

$$\mathscr{L}_1 = \boldsymbol{\varepsilon}_1^*(\mathfrak{L}_1) \otimes f_1^*\mathcal{O}_{X_1}(1)$$

where $\mathfrak{L}_1$ is a degree 0 line bundle on the Prym variety $\mathcal{P}_3$ and again $\boldsymbol{\varepsilon}_1$ is the blow-up map.

These degree 0 line bundles correspond to the line bundle $\boldsymbol{N}$ over $\widetilde{C}$ that is input in the geometric Langlands picture, i.e. the spectral line bundle defining the eigenvalue Higgs bundle Λ_{Dol} on C. Showing that these are the appropriate adjustments in view of the Hecke eigensheaf condition is a significant part of our work.

The modular spectral coverings Y_0 and Y_1 are irreducible so the constructed Higgs bundles $\mathscr{F}_{Dol,0}$ and $\mathscr{F}_{Dol,1}$ are automatically stable. Applying Mochizuki's Kobayashi-Hitchin correspondence [Moc06] yields a corresponding flat bundles $\mathscr{F}_{dR,0}$ and $\mathscr{F}_{dR,1}$ on X_0^{vs} and X_1^{vs}.

From this construction we derive immediately some basic properties of these flat bundles:

- For (X_1, Wob_1) the monodromy transformation in $\mathscr{F}_{dR,1}$ around a general point of Wob_1 is of order 2 with two eigenvalues of -1 and six eigenvalues of 1.
- For (X_0, Wob_0) the monodromy transformations in $\mathscr{F}_{dR,0}$ around general points of the trope planes are transvections, whereas the monodromy around a general point of Kum is a direct sum of four transvections.

These flat bundles are supposed to be the ones given by the geometric Langlands correspondence. In particular, we would like to show that they satisfy the Hecke eigensheaf property. We do not address the general problem of uniqueness of Hecke eigensheaves directly, however it will be shown that the ones produced by our construction agree with the ones constructed by Drinfeld-Laumon.

The Hecke property requires computing the higher direct image of a harmonic bundle in the Dolbeault framework. For this we use the formalism and results from [DPS16]. Some further work is needed here to generalize and adapt the computational machinery of [DPS16] to the specific setup given by the Hecke correspondences. This is done in Chapter 12, and for the reader's convenience we provide a summary of the end results that will be applied to the Hecke computations, in Section 3.11.

The general setting for the pushforward calculation is a map $H \to S$ from a surface to a curve, but let us look at how it comes about in our application. As we are dealing with bundles having a fixed determinant, the Hecke correspondence is parametrized by the $16:1$ etale covering $\overline{C} \to C$ parametrizing points $t \in C$ plus a square-root of $\mathcal{O}_C(t - \mathbf{p})$. Going from X_i $(i = 0, 1)$ to the other space X_j $(j = 1, 0)$ this is

$$
\begin{tikzcd}
 & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_i & & X_j \times \overline{C}.
\end{tikzcd}
$$

Choosing a point $a \in \overline{C}$ and restricting to a line $S := \ell \subset X_j \times \{a\}$, let $H = H_\ell$ be the pullback of ℓ in the Hecke correspondence $\overline{\mathcal{H}}$. There is a parabolic Higgs bundle $(\mathscr{E}, \varphi)$ on H coming by pullback from our constructed Higgs bundle $\mathscr{F}_{Dol,i}$ on X_i. One defines [DPS16] the H/S relative ***Dolbeault complex*** of $(\mathscr{E}, \varphi)$ by setting

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(H/S, \mathscr{E}, \varphi) := \left[W_0 \mathscr{E} \xrightarrow{\varphi_{H/S}} W_{-2}\mathscr{E} \otimes \Omega^1_{H/S} \right]$$

over H. Here, $W_k\mathscr{E}$ is the subsheaf of sections of $\mathscr{E}$ whose restriction to the horizontal part of the parabolic divisor in H lies in the k-th piece of the monodromy weight filtration of $\mathrm{res}(\varphi)$ on the parabolic weight 0 part.

To avoid cluttering the notation we will continue to write $q : H \to S$ for the restriction of the map $q : \overline{\mathcal{H}} \to X_j \times \overline{C}$. The Dolbeault higher direct image vector bundle on S is

$$(\mathscr{E}, \varphi) \mapsto \mathcal{F} := \mathbf{R}^1 q_* \mathrm{DOL}^{\mathrm{par}}_{L^2}(H/S, \mathscr{E}, \varphi),$$

and the global Higgs field upstairs leads to a Higgs field ϕ on this $\mathcal{F}$

For sufficiently general Higgs bundles, as will be the case in this instance, the Dolbeault higher direct image takes a particularly nice form, with the cohomology along each fiber being localized at a finite set of points corresponding to the zeros of the relative Higgs field. This family of finite sets gives the spectral variety for the higher direct image.

There is a natural subscheme Crit of the projectivization $\mathbb{P}(\mathscr{E}/H)$, the ***relative critical locus*** consisting of the zeros of the relative Higgs field.

A point of Crit corresponds to a point $z \in H$ and a vector $e \in \mathscr{E}$ such that $\varphi(e)$ projects to zero in the $\mathscr{E}$-valued relative differentials $\mathscr{E} \otimes \Omega^1_{H/S}$.

In the case where Crit/S is finite (i.e. the zeros of the relative Higgs field are isolated in each fiber), then $\mathrm{Crit} \to S$ will be the spectral cover of the Dolbeault higher direct image $R^1 q_*(\mathscr{E}, \varphi)$. Over each point in the base, this statement says that the cohomology of the fiber localizes at the zeros of the Higgs field and decomposes naturally as a direct sum indexed by these zeros. It is a form of Witten's Morse theory [Wit82]. The proof is that the relative Dolbeault complex becomes quasiisomorphic to the cokernel sheaf that is supported on Crit.

The critical locus description matches up with the ***abelianized Hecke variety*** defined in general by the first two authors:

$$\widehat{\mathcal{H}}^{\mathrm{ab}} := \{(L, L', \alpha, A), \;\; L \xrightarrow{\alpha} L'\}$$

where L and L' are line bundles over the spectral curve $\widetilde{C}$ such that $U = \pi_*(L)$ has determinant $\mathcal{O}_C$ and $U' = \pi_*(L')$ has determinant $\mathcal{O}_C(t)$, and A is a square-root of $\mathcal{O}_C(t - \mathbf{p})$.

Thus $U' \otimes A^{-1}$ has determinant $\mathcal{O}_C(\mathbf{p})$. An alternate formulation is

$$\widehat{\mathcal{H}}^{\mathrm{ab}} := \{(L, \tilde{t}, A), \ \ \tilde{t} \in \widetilde{C}\}$$

where we put $L' := L(\tilde{t})$ and require $A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$ for $t := \pi(\tilde{t})$. The abelianized Hecke variety played a main role in the paper [DP12] on the classical limit.

The abelianized Hecke correspondence maps to the usual Hecke correspondence: $\widehat{\mathcal{H}}^{\mathrm{ab}} \to \overline{\mathcal{H}}$. Indeed, an abelianized Hecke transform between line bundles induces a usual Hecke transform between the rank 2 bundles they induce by pushforward from $\widetilde{C} \to C$.

In the situation of our application, $H \hookrightarrow \overline{\mathcal{H}}$. The Higgs bundle $\mathscr{E}$ on H is the pullback of the constructed Higgs bundle $\mathscr{F}_i$ that has spectral variety Y_i/X_i.

The relative critical locus then identifies with the abelianized Hecke: $\mathrm{Crit} = \widehat{\mathcal{H}}^{\mathrm{ab}}|_H$. The basic idea is to see a point of the full Hecke correspondence is a pair consisting of a rank two bundle and a rank 1 subspace over a point of C. When the bundle comes from a point of $\mathcal{P}^{\mathrm{vs}}$, that point is a line bundle L on $\widetilde{C}$ and the bundle over C is the pushforward. At a general point this pushforward has two distinguished directions coming from the two sheets of $\widetilde{C}$. One shows that our point in $\overline{\mathcal{H}}$ is in the zero set of the Higgs field exactly when the rank 1 quotient of the bundle goes in one of the distinguished directions. Hence, the Hecke point comes from $(L, \tilde{t}, A)$, that is to say a point of $\widehat{\mathcal{H}}^{\mathrm{ab}}$.

The pushforward computations applied in our case show that the higher direct image from H to $S = \ell$ is a Higgs bundle having $\widehat{\mathcal{H}}^{\mathrm{ab}}|_H$ as its spectral variety.

If we move back from the restriction to a line in X to looking at the direct image from $\overline{\mathcal{H}}$ to $X_j \times \overline{C}$, we see that the higher direct image Higgs bundle has spectral variety $\widehat{\mathcal{H}}^{\mathrm{ab}}$. Birationally,

$$\widehat{\mathcal{H}}^{\mathrm{ab}} \cong Y_j \times \widetilde{C} \longrightarrow X_j \times \overline{C}.$$

This is the statement needed on the level of spectral varieties to get the Hecke eigensheaf property saying that the higher direct image has the form $\mathscr{F} \boxtimes \Lambda$. To prove the full eigensheaf property, one needs to identify the spectral line bundle, and to deal with parabolic structures and various singularities.

This gives the basic idea of the proof of the Hecke property, although more technical discussion is needed in order to get the statement precisely.

1.3 Main theorems

This section contains consolidated statements of the main theorems. Recall that C is a smooth projective curve of genus 2. Suppose given a rank 2 flat bundle Λ_{dR} over C, corresponding to a Higgs bundle $\Lambda_{Dol} = (E, \theta)$ with spectral curve $\widetilde{C} \hookrightarrow T^{\vee}C$. Suppose that the spectral curve is corresponds to a general point in the Hitchin base.More precisely we will assume that $\widetilde{C}$ is smooth and unramified over any of the Weierstrass points of C

Use indices $i = 0, 1$ to indicate the moduli spaces X_0, X_1. Let $X_i^{\circ} := X_i - \mathrm{Wob}_i^{\mathrm{sing}}$ and $\mathrm{Wob}_i^{\circ} := \mathrm{Wob}_i - \mathrm{Wob}_i^{\mathrm{sing}}$. Let Y_i° be the preimage of X_i° in Y_i.

Theorem 1.3. *There is a tame purely imaginary harmonic bundle over $X_i - \mathrm{Wob}_i$, corresponding to a pure twistor $\mathscr{D}$-module whose Dolbeault fiber is a parabolic logarithmic Higgs bundle $\mathscr{F}_{i,Dol,\bullet}$ on $(X_i^{\circ}, \mathrm{Wob}_i^{\circ})$, such that the parabolic weights along Wob_i° are trivial for $i = 0$ and $0, 1/2$ for $i = 1$. The spectral data for $\mathscr{F}_{i,Dol,\bullet}$ consist of the spectral covering Y_i° together with a spectral line bundle $\mathscr{L}_i$ defined on Y_i as follows:*

- $\mathscr{L}_0 = \boldsymbol{\varepsilon}_0^*(\mathfrak{L}_0) \otimes f_0^* \mathcal{O}_{X_0}(2) \otimes \mathcal{O}_{Y_0}(\boldsymbol{E}_0)$
- $\mathscr{L}_1 = \boldsymbol{\varepsilon}_1^*(\mathfrak{L}_1) \otimes f_1^* \mathcal{O}_{X_1}(1),$

where $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are line bundles on the Prym variety associated to the spectral line bundle of $\Lambda_{Dol} = (E, \theta)$, and the spectral 1-form is given by the tautological form from the inclusion

$$\mathrm{Y}_i^{\mathrm{vs}} \cong \mathcal{P}_i^{\mathrm{vs}} \hookrightarrow \mathsf{Higgs}_i^{\mathrm{vs}} \cong T^*(X^{\mathrm{vs}}).$$

The parabolic structure is given from the spectral data by setting $\mathscr{F}_{i,Dol,0} = f_{i*}\mathscr{L}_i$ for $i = 0, 1$ and, in addition, in case $i = 1$, setting $\mathscr{F}_{1,Dol,1/2} = f_{1*}\mathscr{L}_i(\boldsymbol{E}_1)$.

In the twistor $\mathbb{P}^1$ we can consider the de Rham point $\lambda = 1$; let $\mathscr{F}_{i,dR}$ be the $\mathscr{D}$-module associated to the fiber over $\lambda = 1$. Let $\mathscr{F}_{i,B}$ be the perverse sheaf corresponding to this $\mathscr{D}$-module. Similarly, we will write Λ_{dR} and Λ_B for the flat bundle and local system on C corresponding to Λ_{Dol}.

We recall that the Hecke operations are defined for points in the curve $\overline{C}$ that maps to C by a $16:1$ etale covering $\mathsf{sq} : \overline{C} \to C$, so the big Hecke operation goes from sheaves on X_0 to sheaves on $X_1 \times \overline{C}$ and vice-versa.

Theorem 1.4. *The pair of perverse sheaves over $X_0 \sqcup X_1$ is a Hecke eigensheaf with Hecke eigenvalue Λ in the sense that the big Hecke operation applied to $\mathscr{F}_{0,B}$ is $\mathscr{F}_{1,B} \boxtimes \mathsf{sq}^*\Lambda_B$ and the big Hecke operation applied to $\mathscr{F}_{1,B}$ is $\mathscr{F}_{0,B} \boxtimes \mathsf{sq}^*\Lambda_B$.*

Drinfeld [Dri83] used a Radon transform to construct Hecke eigensheaves for rank 2 local systems Λ_B on smooth compact curves of any genus. His construction was put into a more geometric form by Laumon [Lau95].

Theorem 1.5. *The purely imaginary tame harmonic bundles associated to the Hecke eigensheaves constructed by Drinfeld in rank 2 have Dolbeault fiber, i.e. parabolic logarithmic Higgs bundles, that satisfies Conjecture 1.2 for compact curves of any genus.*

Theorem 1.6. *For a curve C of genus 2, Drinfeld's Hecke eigensheaves on X_0 and X_1 coincide with the $\mathscr{F}_{0,B}$ and $\mathscr{F}_{1,B}$ that come from the Higgs bundles constructed in Theorem 1.3.*

1.4 Structure of the paper

The case we consider in this paper, moduli of rank 2 bundles on a curve of genus 2, has very classical roots. The basic geometry involved may be viewed as coming from the expression of X_1 as the intersection of a pencil of quadrics in $\mathbb{P}^5$. This viewpoint is recalled and developed in Chapter 2, and we prove some of the properties needed, notably concerning the lines in X_1. Hecke correspondences make their appearance here from a synthetic point of view.

In Chapter 3 we introduce the basic notations of the modular approach in a more complete way than was done in the introduction, and discuss several different types of general considerations that will be used later in the discussion. These range from Chern class calculations (3.12) to the geometry of the $\mathbb{C}^\times$ flow on the Hitchin moduli space (3.6, 3.7), and include a discussion of the nonabelian Hodge correspondence (3.9) and parabolic structures (3.10) to be used in the basic construction. We state in (3.11) the results on Dolbeault higher direct images in a form that will be most useful for calculating the Hecke correspondences; their proofs are deferred to Chapter 12.

In Chapter 4 we construct the parabolic logarithmic Higgs bundle for our candidate Hecke eigensheaf over the moduli space X_1 of bundles of degree 1. This involves a precise description of the wobbly locus Wob_1 paired with a description of the blown-up Prym that forms the spectral covering of X_1, with exceptional divisor $\boldsymbol{E}_1$ above Wob_1. The main technical work is to arrive at a calculation of the parabolic Chern class, given that Wob_1 has cuspidal (as well as normal-crossings) singularities in codimension 2 in X_1. The technique used here is to pass to a finite covering of Kawamata type [Kaw88], with smooth total space and having ramification of order two along Wob_1. This works because the parabolic structure to be used here has weights $0, 1/2$ so it goes away upon pullback to the finite cover. It therefore does not matter that the inverse image of Wob_1 has a triple point where the cusp used to be: the Higgs bundle extends smoothly across the divisor and we can just compute its Chern class.

In Chapter 5 we construct the parabolic logarithmic Higgs bundle for our candidate Hecke eigensheaf over the moduli space X_0 of bundles of degree 0. This again involves a precise description of the wobbly locus Wob_0, which turns out to be Kummer's 16_6 configuration combining the Kummer surface with 16 nodes in $X_0 \cong \mathbb{P}^3$, with the 16 trope planes meeting the surface along trope conics that transversally make tacnodes. Once again, the main problem is how to compute the parabolic Chern class contributions from the singularities. In this case, at smooth points of Wob_0 there is no parabolic structure, but rather the Higgs field has nonzero nilpotent residues. The corresponding monodromies of the local system are unipotent. However, a naive extension of the bundle across the tacnodes has Higgs field that isn't logarithmic on a resolution. Our technique in this case, different from the case of degree 1, is to resolve the singularities of the tacnodes by two blow-ups, and then put an appropriate parabolic structure over the exceptional divisors. In other words, even if the Higgs bundle does not have parabolic structure along the smooth points of Wob_0, it does have a 'hidden' parabolic structure inside the tacnodes, and indeed the resulting local system will have nontrivial monodromy eigenvalues around components of the exceptional divisors. The good parabolic structure was found by trial and error using some computer calculations. Those were really bad so we don't reproduce them here, rather we just state what is the good parabolic structure, verify that it makes the Higgs field logarithmic, and verify that it yields vanishing of the parabolic Chern classes.

From Chapters 4 and 5 we thus obtain the constructions of local systems, and their associated purely imaginary tame harmonic bundles, on X_1^{vs} and X_0^{vs}, completing the proof of Theorem 1.3. The remainder of the paper is devoted to verifying the Hecke eigensheaf property plus a few other things.

In Chapter 6 we introduce the general setup of the Hecke correspondence in the modular viewpoint.

In Chapter 7 we introduce the ***abelianized Hecke*** variety, which is the main player in the proof of the Hecke property. In this chapter, we give a first approach by showing that the Hecke property holds at the level of spectral data using the abelianized Hecke. This part involves consideration of the "big Hecke correspondence" which is the total space of the family parametrized by points of the curve $\overline{C}$. Consideration of the Hecke correspondences at a single point is done in the next two chapters.

In Chapter 8 we fix a point $a \in \overline{C}$ and show that the Hecke correspondence at the point a takes the constructed Higgs bundle on X_0 to the constructed Higgs bundle on X_1. This is done by restricting to a general line, and applying the pushforward statements given in Subsection 3.11 and to be proven in Chapter 12. This section contains a subtle point about apparent singularities: the Hecke pushforward morphism seems to be singular along an additional subvariety of X_1, namely a Kummer K3 surface, known classically, and that depends on the point a. We need to show that the higher direct image harmonic bundle does not really have singularities there. This requires the discussion of Subsection 12.7.

In Chapter 9 for the fixed point $a \in \overline{C}$ we show that the Hecke correspondence at a takes the constructed Higgs bundle on X_1 to the constructed Higgs bundle on X_0. Again the pushforward statements are applied after restricting to a general line. In this direction, the difficulty with apparent singularities does not occur.

In Chapter 10 we go back to the big Hecke correspondence, and use the results of the previous chapters to prove the Hecke eigensheaf property. This completes the proof of Theorem 1.4.

In Chapter 11 we propose a third construction of a parabolic logarithmic Higgs bundle on X_0, having trivial parabolic structure and nilpotent residues over the trope planes, but with parabolic structure of weights $0, 1/2$ over the Kummer subvariety. We posit that this should be associated to a PGL_2 local system of odd degree, but that is not treated here.

In Chapter 12 we proceed in several stages to apply the L^2 Dolbeault direct image formulas of [DPS16] to obtain the pushforward statements required for computation of the Hecke transforms in our cases. The first step is to extend the general theory to the case when the parabolic divisors can have multiplicity. Then we consider the case where the horizontal divisor has a simple ramification point, by blowing up. Then we treat the case of "points of type 3.11.1(e)" that is needed to show that the higher direct image does not have singularities at the apparent singularities that show up in the $(X_0 \to X_1)$ direction of the Hecke operation. This chapter completes the proof of the pushforward statements made in Subsection 3.11.

In Chapter 13 we change gears and consider Drinfeld's Radon transform construction. We show how the Dolbeault direct image technology can apply to gain information about the Higgs bundles associated to the perverse sheaves constructed by Drinfeld. In particular, we obtain the spectral coverings of these HIggs bundles, and prove Conjecture 1.2 for them, i.e. Theorem 1.5. We also show that the Hecke eigensheaves that we construct coincide with those constructed by Drinfeld, which is Theorem 1.6. At the end of this chapter, we show that one can get an explicit description of the Hecke eigensheaf over X_1.

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2 Synthetic approach

A special advantage in studying the Geometric Langlands Conjecture for curves of genus 2 is that we can utilize two independent geometric approaches to the problem. We can study our various objects modularly, in term of their interpretation as moduli spaces of bundles with various decorations, or synthetically, in terms of the geometry of the intersection of two quadrics. In this section we follow the synthetic approach. In the rest of this work we follow the modular approach. The modular techniques we develop are considerably more difficult, but they are less dependent on specifics of our particular situation, so it is more likely that they can be extended to more general cases of GLC.

In this section we will describe synthetically the geometric correspondences between most fundamental objects of interest in this paper - the base locus X_1 of a general pencil of quadrics in $\mathbb{P}^5$ and a ruling X_0 of a general quadric in the pencil. We will also see synthetically how this geometry ties up with the genus two curve C parametrizing the rulings of the quadrics in the pencil and with the geometry of the Jacobian of C.

The significance of the synthetic considerations stems from the fact that, through the works of Narasimhan-Ramanan [NR69] and Newstead [New68], the objects X_1 and X_0 are identified with the moduli of semistable rank 2 bundles of fixed determinant of degree 1 and 0 on the curve C. Later on we will refine these identifications and will in particular show that synthetic constructions of this section reproduce the Hecke correspondences and their discriminant (wobbly) loci in moduli. This will give us easy geometric proofs of most of the important geometric properties needed for the analysis of the Hecke eigensheaf condition.

2.1 Pencils of quadrics in $\mathbb{P}^5$

We will study the geometry of a general pencil of quadrics in $\mathbb{P}^5$ by analyzing the families of linear subspaces contained in these quadrics.

2.1.1. Notation Our notation in this section is:

$X = X_1 = \cap_{x\in\mathbb{P}^1} Q_x \subset \mathbb{P}^5$ is the smooth intersection of a generic pencil of quadrics in $\mathbb{P}^5$.

$\ell \subset X$ is a line in X.

$v \in X$ is a point in X .

$\mathsf{Grass}(3,6)$ is the Grassmannian of projective planes $\Pi \subset \mathbb{P}^5$.

The universal ruling is $\mathscr{R} := \{(x, \Pi) \in \mathbb{P}^1 \times \mathsf{Grass}(3,6) \mid \Pi \subset Q_x\}$. The fiber of $\mathrm{pr}_1 : \mathscr{R} \to \mathbb{P}^1$ over a general $x \in \mathbb{P}^1$ has 2 components, the rulings of the quadric Q_x. We can thus consider the Stein factorization of $\mathrm{pr}_1 : \mathscr{R} \to \mathbb{P}^1$:

$$\begin{tikzcd} \mathscr{R} \arrow[r, "\mathsf{rul}"] \arrow[rr, bend right, "\mathrm{pr}_1"'] & C \arrow[r, "\mathsf{h}_C"] & \mathbb{P}^1. \end{tikzcd}$$

Then C is a hyperelliptic curve of genus 2, with 6 Weierstrass points $\mathbf{p}_i$, which are the ramification points of the hyperelliptic map h_C. They correspond to the rulings on the 6 singular quadrics in the pencil. We fix one of them, labelled $\mathbf{p} \in C$.

The fibers of rul, parametrizing planes in a given ruling $t \in C$, are spinor varieties, isomorphic to $\mathbb{P}^3$. The subvariety parametrizing planes in ruling t that pass through a specified point $v \in X$ is a line in this $\mathbb{P}^3$. Through a specified line $\ell \subset X$ there is a unique plane in each ruling.

We choose coordinates on $\mathbb{P}^5$ that are adapted to our pencil, meaning that the 6 coordinate points are the vertices of the 6 quadric cones in our pencil. All the quadrics Q_x become simultaneously diagonalized. If we take the matrix of one of the quadrics to be the identity and that of another quadric in the pencil to be $diag(\mathbf{x}_1, \dots, \mathbf{x}_6)$, the equation of C becomes $y^2 = \Pi_{i=1}^6 (x - \mathbf{x}_i)$, and $\mathsf{h}_C(\mathbf{p}_i) = \mathbf{x}_i$.

The modular approach studies bundles on this genus 2 hyperelliptic curve C. When we wish to compare the synthetic and modular approaches, the line ℓ will correspond to a line bundle $L \in \mathrm{Pic}^0(C)$, and the point v will correspond to a rank 2 vector bundle V on C with determinant $\det(V) \cong \mathcal{O}_C(\mathbf{p})$.

2.2 Basic geometry of the lines

Let $\mathbf{A}$ denote the variety of lines $\ell \subset X$ in X.

Lemma 2.1. *Through any point $v \in X$ there are four lines $\ell \in \mathbf{A}$ (counting multiplicities).*

Proof. For a point $v \in X$ let $\mathbb{T}_v X \cong \mathbb{P}^3 \subset \mathbb{P}^5$ denote the projective tangent space to X at v. A line $\ell \in \mathbb{P}^5$ passing through v is in X iff it is in $X \cap \mathbb{T}_v X$. But $X \cap \mathbb{T}_v X$ is a curve in $\mathbb{T}_v X \cong \mathbb{P}^3$ that has degree 4 and is a cone with vertex v, so it consists of 4 lines. $\square$

In any ruling of a quadric Q_x there is a unique plane Π containing any $\ell \subset Q_x$, and in particular any $\ell \subset X$. In the latter case, the intersection $\Pi \cap X$ consists of two lines, the given ℓ and another line $\ell' \subset X$. We get a natural map

$$\mathbf{i} : C \times \mathbf{A} \to \mathbf{A}$$

sending $(t, \ell) \mapsto \ell'$, where $t \in C$ labels a ruling $\mathscr{R}_t$ of the quadric $Q_{\mathsf{h}_C(t)}$, $\ell \subset X$ is a line, Π is the plane in $\mathscr{R}_t$ containing ℓ, and ℓ' is the other line in $\Pi \cap X$. Fixing $\ell \in \mathbf{A}$ we get a map $\mathbf{i}_\ell : C \to \mathbf{A}$. Fixing $t \in C$ we get an involution $\mathbf{i}_t : \mathbf{A} \to \mathbf{A}$. We will be particularly interested in $\mathbf{i}_\mathsf{p}$, where $\mathsf{p} \in C$ is our chosen Weierstrass point. The action of $C \times C$ on $\mathbf{A}$ sending $\ell \to \mathbf{i}_{t_1} \circ \mathbf{i}_{t_2}(\ell)$ is easily seen [Don80] to descend to $\mathrm{Sym}^2(C)$ and further to the Jacobian $\mathsf{J} = \mathrm{Jac}(C) = \mathrm{Pic}^0(C) \cong \mathrm{Pic}^2(C)$, where the last identification sends 0 to the canonical bundle $\omega_C = \mathcal{O}_C(2\mathsf{p})$. The resulting map

$$\mathsf{J} \times \mathbf{A} \to \mathbf{A}$$

turns $\mathbf{A}$ into a J-torsor. The choice of $\ell \in \mathbf{A}$ thus gives an isomorphism

$$\mathbf{j}_\ell : \mathsf{J} \xrightarrow{\cong} \mathbf{A}$$

sending the origin to ℓ and restricting to $\mathbf{i}_\ell$ on C, embedded in J via Abel-Jacobi with p mapped to the origin. Note that if $\ell, \mathsf{m} \in \mathbf{A}$ are two lines in X, then $\mathsf{m} = \mathbf{j}_\ell(M)$ for a unique $M \in \mathsf{J}$, and

$$\mathbf{j}_\mathsf{m} = \mathbf{j}_\ell \circ \boldsymbol{t}_M,$$

where $\boldsymbol{t}_M : \mathsf{J} \to \mathsf{J}$ is translation by M: $\boldsymbol{t}_M(N) := M \otimes N$.

As above, given a smooth point z of a subvariety $Z \subset \mathbb{P}^N$ in a projective space, we let $\mathbb{T}_z Z \cong \mathbb{P}^{\dim Z} \subset \mathbb{P}^N$ denote the projective tangent space at z, i.e. the linear subspace of $\mathbb{P}^N$ containing z and pointing in the direction of the usual tangent space $T_z Z$.

Lemma 2.2. *There are 16 lines $\mathsf{O} \in \mathbf{A}$ that are fixed by the involution $\mathbf{i}_\mathsf{p}$:*

$$\mathsf{O} = \mathbf{i}_\mathsf{p} \mathsf{O}.$$

Proof. We start with an initial $\ell \in \mathbf{A}$ and will see how to modify it to be a fixed point. Conjugating $\mathbf{i}_\mathsf{p}$ by $\mathbf{j}_\ell$ we get an involution

$$\mathbf{j}_\ell^{-1} \circ \mathbf{i}_\mathsf{p} \circ \mathbf{j}_\ell : \mathsf{J} \to \mathsf{J}.$$

This sends $L \in \mathsf{J}$ to $\mathfrak{U}_\ell \otimes L^{-1}$, where $\mathfrak{U}_\ell := \mathbf{j}_\ell^{-1}(\mathbf{i}_{\mathsf{p}}\ell) \in \mathsf{J}$. This involution of J has 16 fixed points, namely the square roots M of $\mathfrak{U}_\ell$. Replacing our initial ℓ by $\mathbf{O} := \mathbf{j}_\ell(M)$ gives a new line O such that $\mathbf{j}_{\mathbf{O}} = \mathbf{j}_\ell \circ \boldsymbol{t}_M$, so

$$\mathbf{j}_{\mathbf{O}}^{-1} \circ \mathbf{i}_{\mathsf{p}} \circ \mathbf{j}_{\mathbf{O}} = \boldsymbol{t}_M^{-1} \circ \mathbf{j}_\ell^{-1} \circ \mathbf{i}_{\mathsf{p}} \circ \mathbf{j}_\ell \circ \boldsymbol{t}_M$$

is just inversion, so $\mathbf{O} = \mathbf{i}_{\mathsf{p}}\mathbf{O}$ as desired. (The existence of 16 $\mathbf{O}$'s with this property was shown in [Don80] via a Schubert cycle calculation.) $\square$

If we choose one of these fixed points $\mathbf{O} \in \mathbf{A}$ as origin, then the composition

$$(\mathbf{j}_{\mathbf{O}})^{-1} \circ \mathbf{i} \circ (\mathrm{id} \times \mathbf{j}_{\mathbf{O}}) : C \times \mathsf{J} \to \mathsf{J}$$

becomes the Abel-Jacobi map

$$(t, L) \mapsto L(t - \mathsf{p}),$$

so its restriction:

$$\mathbf{j}_{\mathbf{O}}^{-1} \circ \mathbf{i}_{\mathbf{O}} : C \to \mathbf{A} \to \mathsf{J}$$

becomes the Abel-Jacobi map

$$t \mapsto \mathcal{O}_C(t - \mathsf{p}),$$

and the involutions

$$\mathbf{j}_{\mathbf{O}}^{-1} \circ \mathbf{i}_r \circ \mathbf{j}_{\mathbf{O}} : \mathsf{J} \to \mathsf{J}$$

$$L \mapsto L^{-1}(t - \mathsf{p}).$$

so as we have seen, $\mathbf{j}_{\mathbf{O}}^{-1} \circ \mathbf{i}_{\mathsf{p}} \circ become : \mathbf{j}_{\mathbf{O}} : \mathsf{J} \to \mathsf{J}$ is just inversion.

For the modular/synthetic dictionary, we set

$$\ell = \mathbf{j}_{\mathbf{O}}(L), \qquad L = \mathbf{j}_{\mathbf{O}}^{-1}(\ell).$$

Note that while $\mathbf{A}$ depends only on X, the isomorphism $\mathbf{j}_{\mathbf{O}}$ depends on the auxiliary choices of the Weierstrass point p and the origin $\mathbf{O}$.

2.2.1. Line incidence and special lines Given $\ell \in \mathbf{A}$, consider the family

$$I_\ell \subset \mathbf{A}$$

of lines $\mathsf{m} \in \mathbf{A}$ that intersect ℓ. By definition, this is the closure (in $\mathbf{A}$) of:

$$I_\ell^\circ := \{\mathsf{m} \in \mathbf{A} \mid \mathsf{m} \neq \ell, \mathsf{m} \cap \ell \neq \varnothing\}.$$

For general ℓ, $I_\ell^\circ = I_\ell$ is closed, so we do not need the closure. We say that the line ℓ is ***special*** if it "intersects itself", in the sense that $\ell \in I_\ell$, so $I_\ell^\circ \neq I_\ell$ (Compare with [GH94, page 792]).

Our isomorphism $\mathbf{j_O} : J \xrightarrow{\cong} \mathbf{A}$ identifies I_ℓ with the theta divisor

$$\Theta_\ell = \Theta_L = \left\{M \in \mathsf{J} \;\middle|\; h^0(L \otimes M(\mathbf{p}) > 0\right\} = \left\{\mathcal{O}_C(t - \mathbf{p}) \otimes L^{-1}) \;|t \in C\right\} \subset \mathsf{J},$$

where $\ell = \mathbf{j_O}(L)$.

I_ℓ is the image of our map $\mathbf{i}_\ell : C \to \mathbf{A}$, and in fact $\mathbf{i}_\ell$ induces an isomorphism $\mathbf{i}_\ell : C \xrightarrow{\cong} I_\ell$. The composed map $(\mathbf{j_O})^{-1} \circ \mathbf{i}_\ell : C \to \Theta_L$ is just the Abel-Jacobi map:

$$t \mapsto (L(\mathbf{p}))^{-1}(t).$$

Consider the curve

$$\overline{C} := \left\{ (L, t) \;\middle|\; L^2 \cong \mathcal{O}(t - \mathbf{p}) \right\} \subset \mathsf{J} \times C.$$

The second projection realizes $\overline{C}$ as the 16-sheeted cover $\mathsf{sq} : \overline{C} \to C$ induced from the doubling map $\mathsf{J} \to \mathsf{J}$, $\;L \mapsto L^2$ and the Abel-Jacobi map $C \to \mathsf{J}$, $\;t \mapsto \mathcal{O}_C(t - \mathbf{p})$. The first projection identifies $\overline{C}$ with its image in J. Composing with the isomorphism $\mathbf{j_O}$, we identify $\overline{C}$ as a subvariety of $\mathbf{A}$. In summary, this gives the following

Corollary 2.3. *The line ℓ is special iff $L = \mathbf{j_O}^{-1}(\ell) \in \overline{C} \subset \mathsf{J}$, iff $L^2(\mathbf{p})$ is effective.*

From now on we will think of $\overline{C}$ as the subvariety of $\mathbf{A}$ parametrizing special lines. It is intrinsic to X, independent of choices of $\mathbf{p}$ and $\mathbf{O}$.

A central object in our study is the Wobbly locus in X. We define it to be:

$$\mathrm{Wob} := \bigcup_{\ell \in \overline{C}} \ell \;\subset\; X,$$

i.e. the union of all the special lines.

2.2.2. Trigonal bundles Via the map $\mathbf{i}_\ell : C \to I_\ell$, we have identified C with the family I_ℓ of lines m meeting a given line ℓ, while $\mathbf{j_O}$ identifies I_ℓ with Θ_ℓ. The composition $\mathbf{j_O} \circ \mathbf{i}_\ell : C \to \Theta_\ell$ sends $t \mapsto \mathcal{O}_C(t - \mathbf{p}) \otimes L^{-1}$, where again $L = \mathbf{j_O}^{-1}(\ell)$.

Lemma 2.4. *Let* $L = \mathbf{j_O}^{-1}(\ell)$*. The map* $\mathsf{a}^\circ = \mathsf{a}^\circ_\ell : I^\circ_\ell \to \ell$ *sending* $\mathsf{m} \to \mathsf{m} \cap \ell$ *extends to* $\mathsf{a} = \mathsf{a}_\ell : C \cong I_\ell \to \ell$*, where it is given by the linear system of sections of the degree* 3 *line bundle* $L^2(3\mathbf{p})$*.*

Proof. We have a morphism $X \to \mathrm{Pic}^0(C)$ sending v to $\otimes_{i=1}^4 L_i$, where the L_i are the line bundles corresponding to the 4 lines ℓ_i through v, i.e. $L_i = \mathbf{j_O}^{-1}(\ell_i)$. Since X is unirational (in fact, rational), this map must be constant. Our choice of origin assures us that this constant value of this map is the origin in $\mathrm{Pic}^0(C)$. So for each point v of ℓ, the sum of the three lines other than ℓ through v (i.e. the product of the corresponding line bundles) must be $L^{-1} \in \mathsf{J}$. The identification of Θ_ℓ with C involves a translation by $L(\mathbf{p})$. Suppose that the four lines $\{\ell_i\}_{i=1}^4$ through $v \in \ell$ are labeled so that $\ell_4 = \ell$. So when we convert the three L_i to points $t_i \in C$, we see that their sum, in $\mathrm{Pic}^3(C)$, is:

$$\mathcal{O}_C(\Sigma_{i=1}^3 t_i) = L^{-1} \otimes (L(\mathbf{p})^{\otimes 3}) = L^2(3\mathbf{p})$$

as claimed. □

There are two possibilities: $L(3\mathbf{p})$ could be base-point free and therefore give a genuine trigonal map, or it could have a base point. This happens iff $L(3\mathbf{p}) = \omega_C(t)$ for some $t \in C$, or equivalently iff $L^2 \cong \mathcal{O}_C(t - \mathbf{p})$. We immediately get

Corollary 2.5. *The trigonal bundle* $L^2(3\mathbf{p})$ *has a base point iff* ℓ *is special, or equivalently* (L, t) *belongs to* $\overline{C}$*.*

In the case of a special ℓ, the map $\mathsf{a}_\ell : C \dashrightarrow \ell$ sending $\mathsf{m} \to \mathsf{m} \cap \ell$ is still given by the linear system of sections of $L^2(3\mathbf{p})$, but now this linear system has the base point t, so it is only a rational map. That means that the trigonal curve is now reducible, $C \cup \mathbb{P}^1$ with $t \in C$ glued to $\mathsf{h}_C(t) \in \mathbb{P}^1$, where $\mathsf{h}_C : C \to \mathbb{P}^1$ is the hyperelliptic double cover. The rational map a_ℓ lifts to a morphism $\widetilde{\mathsf{a}}_\ell : C \cup \mathbb{P}^1 \to \ell$. When restricted to $\mathbb{P}^1$ this gives a natural isomorphism $\mathsf{a}^\circ_\ell : \mathbb{P}^1 \to \ell$, and its restriction to C agrees with $\mathsf{h}_C : C \to \mathbb{P}^1$ followed by this isomorphism $\mathsf{a}^\circ_\ell : \mathbb{P}^1 \to \ell$. In this case the 3 lines meeting ℓ at any $v \in \ell$ consist of the fixed line ℓ, plus the moving hyperelliptic pair $\mathsf{h}_C^{-1}(v)$, or more precisely, their images under $\mathbf{i}_\ell$. So the 4 lines through any such v consist of twice ℓ, plus the pair $\mathsf{h}_C^{-1}(v)$.

Going back to a general ℓ, the trigonal map $\mathsf{a} : C \to \ell$ has 8 branch points $\mathbf{b}_i \in \ell$, with $\mathsf{a}^{-1}(\mathbf{b}_i) = 2r_i + s_i$ consisting of the ramification point r_i and one other point s_i. The corresponding 4 lines through such a $\mathbf{b}_i$ are then ℓ, m_i, and ℓ_i occurring with multiplicity 2. Here ℓ_i corresponds to the degree 0 line bundle $L^{-1}(r_i - \mathbf{p})$, and similarly m_i corresponds to the degree 0 line bundle $L^{-1}(s_i - \mathbf{p})$. Here ℓ is arbitrary, but the ℓ_i are in $\overline{C}$. In fact, the 8 ramification points are the intersection of $\overline{C}$ with an appropriate translate of the Θ divisor.

2.2.3. Patterns of lines We have now proved everything we need in order to describe all possible patterns of the 4 lines through some $v \in X$. These are summarized in the follwoing:

Theorem 2.6. *For a point v in a line $\ell \in \mathbf{A}$, the possible patterns of lines through v are:*

- $2\ell + \ell_1 + \ell_2$: *ℓ counts twice among the 4 lines through v iff $\ell \in I_\ell$, iff ℓ is special, iff the trigonal line bundle $L^2(3\mathbf{p})$ has a base point.*
- $\ell + 2\ell_1 + \ell_2$: *One of the lines in $(\mathsf{a}_\ell)^{-1}(v)$ counts twice iff v is a branch point of $\mathsf{a}_\ell : C \to \ell$.*
- $2\ell + 2\ell_1$: *The fiber $(\mathsf{a}_\ell)^{-1}(v)$ consists of two lines ℓ, ℓ_1, each counted twice, iff the lines $\ell, \ell_1 \subset X$ intersect at the point v, which is then a singular point of* Wob. *This occurs iff $\ell_1 = \mathbf{j}_\ell(\mathbf{p}_i) = \mathbf{i}(\mathbf{p}_i, \ell)$ where $\mathbf{p}_i := (\mathbf{j}_\ell)^{-1}(\ell_1)$ is one of the 6 Weierstrass points of the hyperelliptic C, and $v = \mathsf{h}_C(\mathbf{p}_i)$ is its image.*
- $3\ell + \ell_1$: *The line ℓ counts 3 times among the 4 lines through v iff ℓ is special, corresponding to $(L, t) \in \overline{C}$, and $v = \mathsf{h}_C(t)$.*
- $\ell + 3\ell_1$: *$(\mathsf{a}_\ell)^{-1}(v)$ consists of a single line ℓ_1 counted 3 times iff $(\mathbf{j}_\ell)^{-1}(\ell_1)$ is a total ramification point of $\mathsf{a}_\ell : C \to \ell$, and $v = \mathsf{h}_C((\mathbf{j}_\ell)^{-1}(\ell_1))$ is the corresponding total branch point.*
- 4ℓ: *The lines that count 4 times in a fiber are the 16 origin-candidates $\mathbf{O}$ from Lemma 2.2.*

Note in particular that pattern $2\ell + \ell_1 + \ell_2$ is independent of $v \in \ell$. We can prove this directly:

Lemma 2.7. *If ℓ counts twice through some $v \in \ell$, it does so for every $v \in \ell$.*

Proof. Given $\ell \subset X = \cap_{x \in \mathbb{P}^1} Q_x$, let $\mathbb{P} = \mathbb{P}^3_\ell$ be the quotient projective space $\mathbb{P}^5/\ell$, parametrizing planes in $\mathbb{P}^5$ through ℓ, and let $\mathbb{P}^\vee$ denote its dual, parametrizing hyperplanes in $\mathbb{P}^5$ through ℓ. Let the line $\mathfrak{l}_v$ denote the projection of $\mathbb{T}_v X$ to $\mathbb{P}$, and $\mathfrak{l}_v^\vee$ the dual line in $\mathbb{P}^\vee$.

Consider the morphism $\ell \times \mathbb{P}^1 \to \mathbb{P}^\vee$ sending $(v, x) \mapsto \mathbb{T}_v Q_x$. It is well-defined (each v is a non-singular point of each Q_x) and linear in each of its arguments v, x, i.e. it is given by a base-point free sublinear system of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^\vee}(1,1)$. There are only two possibilities: the map can be an embedding, identifying $\ell \times \mathbb{P}^1$ with a smooth quadric $\mathit{quad}^\vee \subset \mathbb{P}^\vee$, or it can be 2-to-1 onto a plane $\Pi \subset \mathbb{P}^\vee$, with branch locus a conic $\mathit{Conic} \subset \mathbb{P}^\vee$. In the first case, the lines $\mathfrak{l}_v^\vee$ form a ruling of $\mathit{quad}^\vee$, and dually the lines $\mathfrak{l}_v$ form a ruling of the dual quadric $\mathit{quad} \subset \mathbb{P}$. In the second case, the lines $\mathfrak{l}_v^\vee$ are the tangents in Π to Conic, and dually the lines $\mathfrak{l}_v$ form the ruling of a quadratic cone $\mathit{quad} \subset \mathbb{P}$.

Now the line ℓ counts more than once among the 4 lines in X through $v \in \ell$ if and only if the intersection $\mathbb{T}_X \cap X$ is singular at every $v' \in \ell$. This happens if and only if $\mathbb{T}_v X$ is not transversal to $\mathbb{T}_{v'} X$, if and only if the lines $\mathfrak{l}_v, \mathfrak{l}_{v'}$ intersect in $\mathbb{P}$. This happens if and only if our quadric $\mathit{quad} \subset \mathbb{P}$ is singular, i.e. the second case above. But that means that all the lines $\mathfrak{l}_{v'}$ intersect each other (at the vertex of quad). $\square$

2.3 The wobbly divisor

We defined Wob as the union in X of the special lines. For a general line $\ell \subset X$, the family I_ℓ of lines intersecting ℓ is given by $I_\ell \cong C$, which comes with a trigonal map $\mathsf{a}_\ell : I_\ell \to \ell$. According to the first pattern in Theorem 2.6, the 8 branch points of a_ℓ are the intersection $\ell \cap \mathrm{Wob}$. In particular, the class of Wob is $8H$, where H is the positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Consider the surface

$$D := \overline{C} \times \mathbb{P}^1 = \left\{ (\ell, x) \;\middle|\; \ell \text{ special}, \, x \in \mathbb{P}^1 \right\} \cong \left\{ (\ell, v) \;\middle|\; \ell \text{ special}, \, v \in \ell \right\},$$

where $v = \mathsf{a}^\circ_\ell(x)$, using the canonical identification $\mathsf{a}^\circ_\ell : \mathbb{P}^1 \to \ell$ of each special ℓ with the hyperelliptic $\mathbb{P}^1$.

Let $\Gamma \subset D$ be the graph of the map $\overline{C} \to \mathbb{P}^1$ sending $(A, t) \to \mathsf{h}_C(t)$. For each of the 6 Weierstrass points $\mathbf{p}_i$, let $\Gamma_i := \overline{C} \times \{\mathbf{x}_i\} \subset D = \overline{C} \times \mathbb{P}^1$, where $\mathbf{x}_i := \mathsf{h}_C(p_i)$. The involution $\mathbf{i}_{\mathbf{p}_i} : \mathbf{A} \to \mathbf{A}$ acts on Γ_i. In terms of the $\mathbf{p}$-dependent identification of $\overline{C}$ with its image in J, this involution sends $A \to A^{-1}(\mathbf{p}_i - \mathbf{p})$. So it has 16 fixed points, and the quotient $\Gamma_i/\mathbf{i}_{\mathbf{p}_i}$ has genus 5.

Theorem 2.8. *The map $\boldsymbol{\nu} : D \to X$ sending $(\ell, v) \to v$ is a finite morphism and maps D birationally onto* Wob*, so it gives its normalization. It is an isomorphism away from Γ and the Γ_i. The map $\boldsymbol{\nu}$ is 2-to-1 on Γ_i, with 16 fixed points. D has normal crossings along the genus 5 curve $\Gamma_i/\mathbf{i}_{\mathbf{p}_i}$, except at the 16 fixed points. The fixed points correspond to pattern 4ℓ, and the other points of $\Gamma_i/\mathbf{i}_{\mathbf{p}_i}$ correspond to pattern $2\ell + 2\ell_1$. The restriction of $\boldsymbol{\nu}$ to Γ is an embedding, and the image parametrizes pattern $3\ell + \ell_1$. But $\boldsymbol{\nu}$ is not immersive along Γ - the image* Wob $= \boldsymbol{\nu}(D)$ *has a curve of cusps along it.*

Proof. The two distinct points $(\ell, v), (m, v) \in D$ map to the same $v \in X$ iff the special lines ℓ, m intersect at $v \in X$. This means that $m = \mathbf{i}_r(\ell)$ for some $r \in C$, According to pattern $2\ell + 2m$ in Theorem 2.6, this occurs iff $m = \mathbf{i}_{\mathbf{p}_i}(\ell)$ for one of the Weierstrass points $\mathbf{p}_i \in C$. This means that we are on one of the Γ_i, and ℓ, m are related by the involution $\mathbf{i}_{\mathbf{p}_i}$.

Pattern $3\ell + m$ in Theorem 2.6 shows that Γ is the inverse image under $\boldsymbol{\nu}$ of the locus of (ℓ, v) such that (ℓ, v) counts 3 times among the lines through v. Finally we claim that special line $\ell \subset X$ is tangent to $\boldsymbol{\nu}(\Gamma) \subset X$ at the point $v \in \ell$ at which the patern of lines in X is $3\ell + m$. We will leave it to the reader to verify this tangency in the synthetic language but we will give a modular proof of this statement in section 4.4. □

2.4 Synthetic correspondences and rigidification

2.4.1. The connection The universal ruling $\mathscr{R}$ is a $\mathbb{P}^3$ bundle **rul** $: \mathscr{R} \to C$ over our hyperelliptic curve C.

Lemma 2.9. *The pullback $\overline{\mathscr{R}} := \mathscr{R} \times_C \overline{C}$ is a product, $\overline{\mathscr{R}} \cong \mathbb{P}^3 \times \overline{C}$.*

Proof. We work with a coordinate system adapted to our pencil, so the Q_x become diagonal matrices. For $x, x_0 \in \mathbb{P}^1 \setminus \{\mathsf{x}_1, \dots, \mathsf{x}_6\}$, let $S := S_{x_0,x}$ be a diagonal matrix such that $S^2 = Q_{x_0}^{-1} Q_x$. Then S acts on $\mathbb{P}^5$, taking Q_{x_0} to Q_x. The set of these square roots $S_{x_0,x}$ has cardinality 2^6, and forms a torsor under the action of the group $G := (\mathbb{Z}/2)^6$. If $t, t_0 \in C$ lie above x, x_0, i.e. $\mathsf{h}_C(t) = x$, $\mathsf{h}_C(t_0) = x_0$, then half of these S's take the ruling $\mathscr{R}_{t_0}$ of Q_{x_0} to the ruling $\mathscr{R}_t$ of Q_x, while the other half take $\mathscr{R}_{t_0}$ to the other ruling of Q_x. The set of square roots $S_{x_0,x}$ taking t_0 to t is a torsor under the subgroup $G_0 \cong (\mathbb{Z}/2)^5 \subset G$ which is the kernel of the sum map $(\mathbb{Z}/2)^6 \to \mathbb{Z}/2$. Further, we have the diagonal embedding $\mathbb{Z}/2 \to G_0$. Its non-zero element exchanges S with $-S$, so it does not affect the transformation of $\mathbb{P}^5$. The quotient $G_0/(\mathbb{Z}/2)$ is canonically identified with $\mathsf{J}[2]$, the 2-torsion subgroup of the Jacobian of C. We see that the family of quadrics $\{Q_{h_C(t)} \mid t \in C \setminus \{\mathsf{p}_1, \dots, \mathsf{p}_6\}\}$ has a flat connection with monodromy $J[2]$, and it becomes a product when pulled back to $\overline{C}$ minus the inverse image of the Weierstrass points. The same holds therefore for $\mathscr{R}$. But since $\rho : \mathscr{R} \to C$ is a $\mathbb{P}^3$ bundle (with no degenerations) over C, including the Weierstrass points, the flat connection extends to all of $\mathscr{R}$,with the same monodromy $\mathsf{J}[2]$, and therefore the pullback to $\overline{C}$ is a product as claimed. □

A fancier way to understand this is by identifying the $\mathbb{P}^3$ fiber of $\mathsf{rul} : \mathscr{R} \to C$ over $t \in C$ with a translated "2Θ" linear system on J. The right translation is seen to be by a square root of $\mathcal{O}(t - \mathsf{p})$, which gives rise to the $\mathsf{J}[2]$-monodromy and to the cover $\overline{C} \to C$. The projective connection we described here explicitly becomes a special case of the general theory of theta groups and their actions, due to Kummer, Heisenberg, Mumford etc. This approach is discussed in section 6.5.

2.4.2. The Hecke correspondence We will encounter several versions of the Hecke correspondence. The basic one is the incidence correspondence:

$$\mathcal{H} := \{(v, \Pi) \mid v \in \Pi\} \subset X \times \mathscr{R}$$

between points of X and planes contained in some quadric through X. Fixing $t \in C$, i.e. fixing the quadric and one of its rulings, we have

$$\mathcal{H}(t) := \mathcal{H} \cap (X \times \mathsf{rul}^{-1}(t)) = \{(v, \Pi) \mid v \in \Pi, \mathsf{rul}(\Pi) = t\} \ \subset \ X \times \mathsf{rul}^{-1}(t) \cong X \times \mathbb{P}^3.$$

The fiber of $\mathcal{H}$ over $\Pi \in \mathscr{R}$ (or the fiber of $\mathcal{H}(t)$ over $\Pi \in \mathsf{rul}^{-1}(t)$) is the conic $X \cap \Pi$, which could be smooth, a pair of intersecting lines, or a double line. The fiber of $\mathcal{H}(t)$ over $v \in X$

is a line in $\mathsf{rul}^{-1}(t) \cong \mathbb{P}^3$. The fiber of $\mathcal{H}$ over $v \in X$ is therefore a $\mathbb{P}^1$-bundle $\mathbb{P}E_v$ over C. (In the modular approach, points $v \in X$ will correspond to rank-2 vector bundles E_v on C whose determinant is $\mathcal{O}_C(\mathbf{p})$, and our $\mathbb{P}E_v$ will be the projectivization of E_v.)

Recall from Lemma 2.9 that the cover $\overline{\mathscr{R}} = \mathscr{R} \times_C \overline{C}$ is a product, $\overline{\mathscr{R}} \cong \mathbb{P}^3 \times \overline{C}$. A rigidified Hecke correspondence is obtained by pulling back to $\overline{\mathscr{R}}$ via $\overline{\mathscr{R}} \to \mathscr{R}$:

$$\overline{\mathcal{H}} := (X \times \overline{\mathscr{R}}) \times_{X \times \mathscr{R}} \mathcal{H} \ \subset \ X \times \overline{\mathscr{R}} \cong X \times \mathbb{P}^3 \times \overline{C}.$$

From the modular point of view, we will be interested in two moduli spaces of bundles: $X_1 = X$ parametrizes rank 2 bundles on C with determinant $\mathcal{O}_C(\mathbf{p})$, while $X_0 \cong \mathbb{P}^3$ parametrizes rank 2 bundles on C with determinant $\mathcal{O}_C$. The big Hecke correspondence can then be understood as a subvariety:

$$\overline{\mathcal{H}} \subset X_1 \times X_0 \times \overline{C}.$$

The group $\mathsf{J}[2]$ acts on all three factors. The action of $\mathsf{J}[2] = G_0/(\mathbb{Z}/2)$ on $X_1 = X$ was described in the proof of lemma 2.9. Let $\mathsf{M}_1 := [X_1/\mathsf{J}[2]]$ denote the quotient. Since $X_0 \cong \mathbb{P}^3$ was defined as the family of flat sections of $\overline{\mathscr{R}} \to \overline{C}$, we get an action of $\mathsf{J}[2]$ on X_0 induced from its action on $\overline{\mathscr{R}}$. Let M_0 denote the quotient $[X_0/\mathsf{J}[2]]$. The action of $\mathsf{J}[2]$ on $\overline{C}$ is the familiar one, with quotient C. The Hecke correspondence is compatible with these actions, so we get another version of the Hecke correspondence

$$\mathbf{Hecke} \subset \mathsf{M}_1 \times \mathsf{M}_0 \times C.$$

This is the Hecke correspondence for the group $\mathbb{P}GL(2)$.

2.5 The quadric line complex

Starting with the intersection of quadrics X we retrieve the curve C, its cover $\overline{C}$, the universal ruling $\mathscr{R}$, and its cover $\overline{\mathscr{R}} \to \overline{C}$. We can therefore recover 'the' $\mathbb{P}^3$ as the family of flat sections of $\overline{\mathscr{R}} \to \overline{C}$. We can also go in the opposite direction.

The Grassmannian $\mathsf{Grass}(2,4)$ of lines in $\mathbb{P}^3 = \mathbb{P}(V)$ can be identified, via the Plücker embedding, as a smooth quadric in $\mathbb{P}^5 = \mathbb{P}(\wedge^2 V)$. All smooth quadrics in $\mathbb{P}^5$ are isomorphic by a projective transformation, and the automorphisms of $\mathbb{P}^3$ correspond to automorphisms of $\mathbb{P}^5$ preserving the quadric, via the exceptional group isomorphism $PGL(4) \cong PSO(6)$. Thus, any smooth quadric may be viewed as being the Grassmanian $\mathsf{Grass}(2,4)$ up to adjustment of the identification $\mathbb{P}^5 = \mathbb{P}(\wedge^2 V)$, in a way that is unique up to projective transformations of $\mathbb{P}^3$.

The Grassmanian has two rulings: a point $p \in \mathbb{P}^3$ determines the plane $\Pi = \Pi_p \subset \mathsf{Grass}(2,4)$ of all lines in $\mathbb{P}^3$ through p, while a plane $P \subset \mathbb{P}^3$ determines the plane $\Pi = \Pi_P \subset \mathsf{Grass}(2,4)$ of all lines in P. Each of these rulings may be viewed as being specified, once chosen the identification of a certain quadric as the Grassmanian.

If we choose another quadric meeting $\mathsf{Grass}(2,4)$ transversally, then these two quadrics span a pencil giving the situation at the start of this chapter. The intersection of $\mathsf{Grass}(2,4)$ with the other quadric is known as the ***quadric line complex***. In the classical terminology cf [GH94], the "line complex" means the family of lines in $\mathbb{P}^3$ parametrized by this quadric section of $\mathsf{Grass}(2,4)$, the ***"lines of the quadric line complex"***.

Thus the quadric line complex is the intersection X of a general pencil of quadrics in $\mathbb{P}^5$. It carries a bit more information though: specifying our X is equivalent to specifying the hyperelliptic curve C, while specifying a quadric line complex is equivalent to specifying the hyperelliptic curve C together with one non-Weierstrass point $t \in C$: the point $\mathsf{h}_C(t) \in \mathbb{P}^1$ corresponds to the quadric we identify as the Grassmannian, the point $t \in C$ corresponds to its ruling by planes Π_p for points $p \in \mathbb{P}^3$.

In the other direction, saying that X is the quadric line complex depends on the choice of $t \in C$ or indeed on its lifting to $(A,t) \in \overline{C}$, because the correspondence between points of X and lines in $\mathbb{P}^3$ is exactly the Hecke correspondence depending on t, with furthermore the identification between the ruling and a fixed $\mathbb{P}^3$ being dependent upon the lifting to $\overline{C}$.

2.6 Hecke curves and Kummer surfaces

Consider the incidence:

$$\boldsymbol{\mathcal{J}} = \{\, (\ell, \Pi) \mid \ell \subset \Pi \,\} \;\subset\; \mathbf{A} \times \mathscr{R}.$$

We have an isomorphism

$$C \times \mathbf{A} \xrightarrow{\cong} \boldsymbol{\mathcal{J}}$$

sending $(t,\ell) \mapsto (\ell, \Pi)$, where $\Pi \subset Q_{\mathsf{h}_C(t)}$ is the plane spanned by ℓ and $\mathbf{i}(t,\ell)$. The involution $\ell \mapsto \mathbf{i}(t,\ell)$ acts on $\boldsymbol{\mathcal{J}}$, with quotient:

$$\boldsymbol{\mathcal{K}um} := \{\, \Pi \in \mathscr{R} \mid \Pi \cap X \text{ contains a line} \,\}\,.$$

We have maps

$$C \times \mathbf{A} \;\cong\; \boldsymbol{\mathcal{J}} \xrightarrow{2:1} \boldsymbol{\mathcal{K}um} \hookrightarrow \mathscr{R},$$

or, if we fix $t \in C$:

$$\mathbf{A} \overset{2:1}{\to} \mathsf{Kum} \hookrightarrow \mathbb{P}^3.$$

Here Kum is the Kummer surface, image of $\mathbf{A} \cong \mathsf{J}$ by the 2Θ linear system, which embeds $\mathsf{Kum} = \mathbf{A}/\mathbf{i}_t \cong \mathsf{J}/\pm 1$ into $\mathbb{P}^3$ as a singular hypersurface of degree 4. Note that the Kummer surface $\mathsf{Kum} \subset \mathbb{P}^3$ has 16 nodes, images of the 16 fixed points of the involution $\mathbf{i}_t : \mathbf{A} \to \mathbf{A}$. These are the points where the fiber $\ell \cup \mathbf{i}_t(\ell)$ becomes a double line.

A basic fact is that Kum is isomorphic to its dual [GH94, Keu97].

Given a point $p \in \mathbb{P}^3 = \mathsf{rul}^{-1}(t)$, the fiber of the Hecke correspondence over p is a subset of X that was denoted by X_p in [GH94]. In the terminology of the quadric line complex it is the set of lines in the line complex that pass through p.

Corollary 2.10. *Fix $t \in C$ and denote $\mathbb{P}^3 := \mathsf{rul}^{-1}(t)$.*

- *If p is a point of $\mathbb{P}^3 \setminus \mathsf{Kum}$, then X_p is a smooth conic $\Pi \cap X$.*
- *If p is a smooth point of Kum, then X_p is a union of two lines $\ell \cup \mathbf{i}(t, \ell)$ that touch.*
- *If p is one of the 16 nodes of Kum, then X_p is a double line 2ℓ.*

This was stated in [GH94, pp 762-763].

The locus of points in X that are intersection points of the two lines, for the second case of X_p, has closure that is a surface denoted $\Sigma \subset X$ in [GH94]. This is seen to be the ***Kummer K3 surface*** obtained by resolving the 16 nodes of $\mathsf{Kum} \subset \mathbb{P}^3$ (indeed, over the nodes we get not a single point but a line of points in Σ corresponding to the full double line).

The K3 surface will show up in our situation as a locus of apparent singularities for the Hecke transform from $X_0 = \mathbb{P}^3$ to $X_1 = X$. For the main local systems constructed here, it turns out that the singularities along Σ are removable. On the other hand for the third construction in Chapter 11, the Hecke transform will have singularities along Σ. We refer the reader to that chapter for more discussion.

In the synthetic picture, the trope planes may be characterized as planes in $\mathbb{P}^3$ that contain 6 of the 16 nodes. There are 16 of these. They are also the planes that correspond to the 16 nodes of the dual Kummer surface. The trope planes meet the Kummer surface in plane conics counted with multiplicity two; these conics are characterized also by passing through the 6 nodes that define the plane.

The 16_6 property, saying that each plane contains 6 nodes (by definition) and each node has 6 planes, was originally proven in the synthetic situation [GH94, Bea96, Keu97, Dol20].

3 General considerations

Throughout this paper, we consider a smooth projective curve C of genus 2. It is therefore hyperelliptic, with hyperelliptic involution denoted $\boldsymbol{\iota}_C : C \to C$, and the quotient projection denoted by $\mathsf{h}_C : C \to \mathbb{P}^1$. Let $\mathbf{p}_1, \ldots, \mathbf{p}_6 \in C$ denote the Weierstrass points, i.e. the ramification points for the map h_C, and let $\mathbf{x}_1, \ldots, \mathbf{x}_6 \in \mathbb{P}^1$ denote the corresponding branch points. For brevity we will choose one of the Weierstrass points and drop the index, writing $\mathbf{p} := \mathbf{p}_1$. Thus the canonical line bundle of C is given by $\omega_C = \mathcal{O}_C(2\mathbf{p})$.

If $x \in C$ we will usually denote by $x' := \boldsymbol{\iota}_C(x)$ its image by the hyperelliptic involution; then $\mathcal{O}_C(x + x') \cong \mathcal{O}_C(2\mathbf{p}) = \omega_C$.

Fix a line bundle $\mathbf{d}$ to be either $\mathcal{O}_C$ or $\mathcal{O}_C(\mathbf{p})$. We will look at the moduli space X of polystable rank 2 vector bundles E provided with an isomorphism $\det(E) \cong \mathbf{d}$. The choice of $\mathbf{d}$, that is to say degree 0 or 1, will be made according to section in the paper—we prefer not to overload the notations by indexing on this choice. When it becomes necessary to distinguish them, we will denote the two moduli spaces by X_0 and X_1 and similarly for their associated constructions (e.g. Y_0, Y_1).

By Narasimhan-Ramanan [NR69], for degree 0 we have $X = \mathbb{P}^3$ and for degree 1 we have $X \subset \mathbb{P}^5$ is a smooth complete intersection of two quadrics. These descriptions will be developed in more detail below.

Let Higgs denote the coarse moduli space of semistable Higgs bundles (E, θ) of rank 2 with isomorphism $\det(E) \cong \mathbf{d}$ and satisfying $\operatorname{tr}\theta = 0$. In particular, $X \subset \mathsf{Higgs}$ is the subset of Higgs bundles with Higgs field equal to zero. The ***Hitchin map*** on rank 2 Higgs bundles was defined by Hitchin [Hit87a, Hit87b] and is hiven by

$$\boldsymbol{h} : \mathsf{Higgs} \to \mathcal{B} := H^0(\omega_C^{\otimes 2}), \quad (E, \theta) \mapsto \det(\theta)$$

The space of quadratic differentials $\mathcal{B} := H^0(\omega_C^{\otimes 2})$ is called the ***Hitchin base*** and in our case is a complex vector space of dimension 3.

We will write $T^\vee C$ for the total space of ω_C. $\boldsymbol{\pi} : T^\vee C \to C$ for the natural projection, and $\lambda \in H^0(T^\vee C, \boldsymbol{\pi}^*\omega_C)$ for the tautological section. For any quadratic differential $\mathsf{b} \in \mathcal{B}$, the associated ***spectral curve*** is defined as the curve $\widetilde{C}_{\mathsf{b}} \subset T^\vee C$ given by the equation

$$\widetilde{C}_{\mathsf{b}} \,:\, \lambda^2 - \boldsymbol{\pi}^*\mathsf{b} = 0.$$

In other words, the spectal curve correcsponding to b is the zero divisor of the holomorphic section $\lambda^2 - \boldsymbol{\pi}^*\mathsf{b} \in H^0(T^\vee C, \boldsymbol{\pi}^*\omega_C)$. By definition, the spectral curve associated to a Higgs bundle (E, θ) is the spectral curve corresponding to $\boldsymbol{h}(E, \theta)$. It is given explicitly the equation

$$\det(\lambda \cdot \mathrm{id} - \boldsymbol{\pi}^*\theta) = \lambda^2 - \det(\theta) = 0,$$

where $\lambda \cdot \mathrm{id} - \boldsymbol{\pi}^*\theta$ is viewed as a map

$$\lambda \cdot \mathrm{id} - \boldsymbol{\pi}^*\theta \,:\, \boldsymbol{\pi}^*E \longrightarrow \boldsymbol{\pi}^*E \otimes \boldsymbol{\pi}^*\omega_C.$$

For future reference we record the following

Proposition 3.1. *Let* Higgs *be the moduli space of rank* 2 *Higgs bundles with fixed determinant* $\mathbf{d}$*. Then*

(a) The Hitchin map $\boldsymbol{h} : \mathsf{Higgs} \to \mathcal{B}$ *is proper and surjective.*

(b) For any $(E, \theta) \in \mathsf{Higgs}$ *the spectral curve* $\widetilde{C} \subset T^\vee C$ *of* θ *is a curve of arithmetic genus* 5 *and the projection*

$$\pi = \boldsymbol{\pi}_{|\widetilde{C}} : \widetilde{C} \to C$$

is the degree 2 *covering branched over the bicanonical divisor* $\det(\theta) = 0$*. For a general* (E, θ) *the spectral curve* $\widetilde{C}$ *is smooth and connected.*

(c) For any $\mathsf{b} \in \mathcal{B}$*, the branch divisor of the associated spectral curve* $\pi : \widetilde{C}_{\mathsf{b}} \to C$ *is the* h_C*- pullback of a degree* 2 *effective divisor* $\mathsf{y} + \mathsf{z}$ *on* $\mathbb{P}^1$*. For a general* b *the branch divisor* $\mathsf{zero}(\mathsf{b}) = \mathsf{h}_C^{-1}(\mathsf{y} + \mathsf{z}) = \tilde{y} + \tilde{y}' + \tilde{z} + \tilde{z}'$ *consists of two* $\boldsymbol{\iota}_C$*-conjugate pairs of points in* C*.*

(d) For general $\mathsf{b} \in \mathcal{B}$ *the Hitchin fiber* $\boldsymbol{h}^{-1}(\mathsf{b})$ *is identified with the* 3*-dimensional abelian variety* $\mathcal{P}$ *of line bundles* L *on* $\widetilde{C}$ *such that* $\det(\pi_*(L)) \cong \mathbf{d}$*, or equivalently* $\mathsf{Nm}_\pi(L) = \omega_C \otimes \mathbf{d}$*. The Higgs bundle corresponding to* L *is* $(E, \theta) = (\pi_*L, \pi_*(\lambda \otimes (-)))$*.*

Proof. Parts (a), (b), and (d) are standard and are proven in many classical sources, e.g. [Hit87a, Hit87b, BNR89]. For part (c) we only need to note that $\omega_C = \mathsf{h}_C^*\mathcal{O}(1)$, and that $\mathsf{h}_C^* : H^0(\mathbb{P}^1, \mathcal{O}(2)) \to H^0(C, \omega_C^{\otimes 2})$ is injective, Since both spaces are 3-dimensional this shows that the pullback map is an isomorphism, and so any quadratic differential on C is a pullback from an effective degree 2 divisor on $\mathbb{P}^1$. This proves (c) and completes the proof of the proposition. $\square$

Throughout the paper, we will fix a point b and the corresponding spectral curve $\widetilde{C}_{\mathsf{b}}$. Since b will be fixed we will drop the subscript from the notation and will simply write $\widetilde{C}$ finstead of $\widetilde{C}_{\mathsf{b}}$. We will also fix Λ the rank 2 local system corresponding to a rank 2 Higgs bundle with trivial determinant (E, θ).

3.1 Curves

Let us make the following notations and definitions.

C is a curve of genus $g(C) = 2$ with hyperelliptic map $\mathsf{h}_C : C \to \mathbb{P}^1$.

$\mathbf{p} \in C$ is a fixed Weierstrass point.

$\overline{C} := \left\{ (A, t) \;\middle|\; A \in \mathrm{Jac}^0(C),\ \ t \in C,\ \ A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p}) \right\}$.

Note that the map

$$\mathsf{sq} : \overline{C} \to C, \quad \mathsf{sq}(A, t) = t$$

is a 16-sheeted étale cover of C, while the map

$$\boldsymbol{\imath}_{\overline{C}} : \overline{C} \hookrightarrow \mathrm{Jac}^0(C), \quad \boldsymbol{\imath}_{\overline{C}}(A, t) = A$$

is a closed embedding. In particular $\overline{C}$ is a smooth connected curve of genus $g(\overline{C}) = 17$.

$\pi : \widetilde{C} \to C$ is a fixed spectral curve, assumed to be smooth. We also assume that the spectral curve has two branches over $\mathbf{p}$.

$\widehat{C} := \widetilde{C} \times_C \overline{C}$ is a smooth connected curve of genus $g(\widehat{C}) = 65$ which fits in the fiber square

$$\begin{tikzcd} \widehat{C} \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{\mathsf{sq}}"'] & \overline{C} \arrow[d, "\mathsf{sq}"] \\ \widetilde{C} \arrow[r, "\pi"'] & C \end{tikzcd}$$

Suppose (E, θ) is a Higgs bundle with trivial determinant on C, i.e. $\det(E) \cong \mathcal{O}_C$ and $tr(\theta) = 0$. Our main condition is that $\widetilde{C}$ is the spectral cover of (E, θ), this means that there is a line bundle L on $\widetilde{C}$ such that $E = \pi_* L$ and $\theta = \pi_*(\lambda \otimes (-))$.

The choice of (E, θ) will determine (via the construction we are going to do, using its spectral data $(\widetilde{C}, L)$) two tame parabolic Higgs bundles of rank 8,

$$(\mathscr{F}_{0,\bullet}, \Phi_0)/X_0, \quad (\mathscr{F}_{1,\bullet}, \Phi_1)/X_1. \tag{1}$$

Our goal is to give a detailed construction of these Higgs bundles and to show that they satisfy the Hecke eigensheaf property on $X_0 \sqcup X_1$.

3.2 Moduli spaces

The coarse moduli spaces of semistable bundles will be denoted by X with, if necessary, a subscript depending on the degree. These have coverings destined to become the modular spectral covers of the parabolic Higgs bundles we are going to construct. The notations are as follows.

X_0 is the moduli space of rank 2 bundles F with $\det(F) = \mathcal{O}_C$. We have $X_0 \cong \mathbb{P}^3$ [NR69].

X_1 is the moduli space of rank 2 bundles E with $\det(E) = \mathcal{O}_C(p)$. We have [NR69, New68]

$$X_1 \subset \mathbb{P}^5, \quad X_1 = \bigcap_{x \in \mathbb{P}^1} Q_x,$$

where $\{Q_x\}_{x \in \mathbb{P}^!}$ is a pencil of quadrics in $\mathbb{P}^5$ parametrized by the hyperelliptic $\mathbb{P}^1$ of C, with a discriminant divisor being exactly the ramification divisor of the hyperelliptic map $\mathsf{h}_C : C \to \mathbb{P}^1$.

$\mathcal{P}_2$ is the degree two Prym for $\widetilde{C}$, that is to say

$$\mathcal{P}_2 := \left\{ L \in \mathrm{Jac}^2(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(L) \cong \omega_C \right\}.$$

This is identified with the Hitchin fiber over $[\lambda : \widetilde{C} \hookrightarrow T^\vee C] \in \mathcal{B}$ (corresponding point in the Hitchin base) for the $SL(2)$ Hitchin fibration $\boldsymbol{h} : \mathsf{Higgs}_0 \to \mathcal{B}$.

$\mathcal{P}_3$ is the degree three Prym:

$$\mathcal{P}_3 := \left\{ L \in \mathrm{Jac}^3(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(L) \cong \omega_C(\mathbf{p}) \right\}.$$

It is the Hitchin fiber over $[\lambda : \widetilde{C} \hookrightarrow T^\vee C] \in \mathcal{B}$ for the odd degree Hitchin fibration $\boldsymbol{h} : \mathsf{Higgs}_1 \to \mathcal{B}$ (Higgs bundles whose determinant Higgs bundle is $(\mathcal{O}_C(\mathbf{p}), 0)$).

Y_0 is the blow-up of $\mathcal{P}_2$ at the pullbacks of the theta characteristics on C. More precisely, consider the un(semi)stable locus in $\mathcal{P}_2$,

$$\begin{aligned}\mathcal{P}_2^{\mathrm{unss}} &= \{L \in \mathcal{P}_2 \mid \pi_* L \text{ is un(semi)stable } \} \\ &= \left\{\pi^*\kappa \in \mathcal{P}_2 \;\middle|\; \kappa \in \mathrm{Jac}^1(C), \text{ s.t. } \kappa^{\otimes 2} = \omega_C \right\} \;\; = \pi^*\mathsf{Spin}(C)\end{aligned}$$

which has 16 points, and $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ is the blow-up of $\mathcal{P}_2^{\mathrm{unss}} \cong \mathsf{Spin}(C)$.

$\boldsymbol{E}_0 \subset Y_0$ denotes the exceptional divisor, it has 16 connected components

$$\boldsymbol{E}_0 = \sqcup_\kappa \boldsymbol{E}_{0,\kappa}.$$

f_0 is a morphism $f_0 : Y_0 \to X_0$ such that if $L \in \mathcal{P}_2$ is a point not on $\mathcal{P}_2^{\mathrm{unss}}$ we have $f_0(L) = \pi_*(L)$. In Theoroem 3.6 the morphism f_0 is constructed as the minimal resolution of the rational map $\pi_*(-) : \mathcal{P}_2 \dashrightarrow X_0$. From the construction it follows that for a general spectral curve $f_0 : Y_0 \to X_0$ is finite and by Lemma 4.9 and Remark 4.10 has degree 8.

Y_1 is the blow-up of $\mathcal{P}_3$ in the un(semi)stable (=unstable) locus

$$\mathcal{P}_3^{\mathrm{unss}} = \{L \in \mathcal{P}_3 \mid \pi_* L \text{ is un(semi)stable } (= \text{is unstable})\} \cong \widehat{C}.$$

To see that the unstable locus is related to $\widehat{C}$, suppose we have a line bundle $L \in \mathcal{P}_3$ such that $\pi_* L$ is unstable. Then $\pi_* L$ will have a destabilizing line subbundle $M \subset \pi_* L$ of degree one. By adjunction we get an injective map of locally free rank one sheaves $\pi^* M \hookrightarrow L$ and for degree reasons we must have a short exact sequence

$$0 \to \pi^* M \to L \to \mathcal{O}_{\tilde{t}} \to 0$$

for some point $\tilde{t} \in \widetilde{C}$. Writing $M = A^{-1}(\mathbf{p})$ for some line bundle of degree zero, we see that $\mathsf{Nm}_\pi(L) \otimes \mathcal{O}_C(-\pi(\tilde{t})) = \mathsf{Nm}_\pi(\pi^* A^{-1}(\mathbf{p})) = A^{-2}(2\mathbf{p})$. Since $\mathsf{Nm}_\pi(L) = \mathcal{O}_C(3\mathbf{p})$ we have that $A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(\pi(\tilde{t}))$. Thus $(A, \tilde{t}) \in \widehat{C}$ and $L = \pi^*(A^{-1}(\mathbf{p}))(\tilde{t})$. In Lemma 3.2 below we show that the map $\widehat{C} \to \mathcal{P}_3$, $(A, \tilde{t}) \mapsto \pi^*(A^{-1}(\mathbf{p}))(\tilde{t})$ is a closed embedding which gives the identification $\mathcal{P}_3^{\mathrm{unss}} = \widehat{C}$.

Again we will write $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$ for the blow-up map.

f_1 is a morphism $f_1 : Y_1 \to X_1$ such that if $L \in \mathcal{P}_3$ is a point not on $\mathcal{P}_3^{\mathrm{unss}}$ we have $f_1(L) = \pi_* L$. In Theoroem 3.6 the morphism f_1 is constructed as the minimal resolution of the rational map $\pi_*(-) : \mathcal{P}_3 \dashrightarrow X_1$ and by construction is finite for a general spectral cover. By Lemma 4.9 f_1 has degree 8.

$\boldsymbol{E}_1 \subset Y_1$ denotes the exceptional divisor, it is a $\mathbb{P}^1$-bundle

$$\boldsymbol{E}_1 = \mathbb{P}(N_{\widehat{C}/\mathcal{P}_3}) \to \widehat{C}.$$

In Theoroem 3.6 we will see that in fact

$$\boldsymbol{E}_1 \cong \widehat{C} \times \mathbb{P}^1,$$

where the second factor is naturally idenitified with the hyperelliptic $\mathbb{P}^1$ of C, that is it is identified with the projective line $\mathbb{P}(H^0(C, \omega_C)^\vee)$. We have a diagram

$$\begin{tikzcd} \boldsymbol{E}_1 \arrow[r, hook] \arrow[d] & Y_1 \arrow[d] \\ \overline{C} \times \mathbb{P}^1 \arrow[r] & X_1 \end{tikzcd}$$

and in section 4 we will see that the map on the bottom factors as

$$\overline{C} \times \mathbb{P}^1 \to \mathrm{Wob}_1 \hookrightarrow X_1$$

where the surface Wob_1 is the wobbly divisor discussed in the next section, and the map $\overline{C} \times \mathbb{P}^1 \to \mathrm{Wob}_1$ is the normalization.

To understand the map

$$\overline{C} \times \mathbb{P}^1 = \overline{C} \times \mathbb{P}(H^0(C, \omega_C)^\vee) \to X_1. \tag{2}$$

in more concrete terms fix points $(A, t) \in \overline{C}$ and $x \in \mathbb{P}(H^0(C, \omega_C)^\vee)$. Any non-split extension of $A^{-1}(\mathbf{p})$ by A will be a stable rank two bundle with determinant $\mathcal{O}_C(\mathbf{p})$, i.e. will give us a point in X_1. However the space of extensions of $A^{-1}(\mathbf{p})$ by A is canonically identified with the space $H^0(C, \omega_C)^\vee$. Indeed we have

$$\begin{aligned} \mathrm{Ext}^1(A^{-1}(\mathbf{p}), A) &= H^1(C, A^{\otimes 2}(-\mathbf{p})) = H^1(C, \mathcal{O}_C(t - 2\mathbf{p})) \\ &= H^1(C, \mathcal{O}_C(-t')) = H^1(C, \mathcal{O}_C) \\ &= H^0(C, \omega_C)^\vee, \end{aligned} \tag{3}$$

where t' is the hyperelliptic conjugate of t, and the identification $H^1(C, \mathcal{O}_C(-t')) = H^1(C, \mathcal{O}_C)$ is induced by the natural inclusion of sheaves $\mathcal{O}_C(-t') \subset \mathcal{O}_C$.

With this picture in mind we can now describe the map (2). It sends a point $((A, t), x) \in \overline{C} \times \mathbb{P}(H^0(C, \omega_C)^\vee)$ to the vector bundle $E \in X_1$, where E is defined as the unique up to isomorphism extension

$$0 \to A \to E \to A^{-1}(\mathbf{p}) \to 0$$

which corresponds to the extension class $x \in \mathbb{P}(H^0(C,\omega_C)^\vee) \cong \mathbb{P}(\mathrm{Ext}^1(A^{-1}(\mathbf{p}), A))$ under the identification (3).

Finally, note that the curve $\overline{C}$ embeds in $\overline{C} \times \mathbb{P}^1$ as the graph of the composition

$$\overline{C} \xrightarrow{\mathsf{sq}} C \xrightarrow{\mathsf{h}_C} \mathbb{P}^1,$$

with h_C denoting the hyperelliptic map. It is easy to check that restricting the map (2) to this embedded copy of $\overline{C}$ yields a closed embedding

$$\mathsf{ql}: \quad \overline{C} \hookrightarrow X_1 \subset \mathbb{P}^5$$

of $\overline{C}$ in the intersection of two quadrics X_1.

We conclude this subsection with the promised check that the curve $\widehat{C} = \widetilde{C} \times_C \overline{C}$ embeds in $\mathcal{P}_3$.

Lemma 3.2. *The map*

$$\boldsymbol{\imath}_{\widehat{C}} : \widehat{C} \to \mathcal{P}_3, \quad (A, \tilde{t}) \mapsto \pi^*(A^{-1}(\mathbf{p})) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t}). \tag{4}$$

embeds the curve $\widehat{C}$ inside $\mathcal{P}_3$. In particular $\boldsymbol{E}_1 = \mathbb{P}(N_{\widehat{C}/\mathcal{P}_3})$.

Proof. Suppose $(A_1, \tilde{t}_1), (A_2, \tilde{t}_2) \in \widehat{C}$. If these two points map to the same point in $\mathcal{P}_3$, then we will have

$$(\pi^* A_1^{-1})(\tilde{t}_1) = (\pi^* A_2^{-1})(\tilde{t}_2). \tag{5}$$

Squaring this identity and using the fact that $A_i^{-2}(\pi(\tilde{t}_i)) = \mathcal{O}_C(\mathbf{p})$ for $i = 1, 2$ we get

$$\begin{aligned}
\mathcal{O}_{\widetilde{C}} &= (\pi^* A_2^2)(-2\tilde{t}_2) \otimes (\pi^* A_1^{-2})(2\tilde{t}_1) \\
&= \left(\pi^*(A_2^2(-\pi(\tilde{t}_2))\right) \left(\pi^*(\pi(\tilde{t}_2) - 2\tilde{t}_2\right) \otimes \left(\pi^* A_1^{-2}(\pi(\tilde{t}_1)\right) \left(-\pi^*(\pi(\tilde{t}_1) + 2\tilde{t}_1\right) \\
&= \mathcal{O}_{\widetilde{C}}(\tilde{t}_1 + \tau(\tilde{t}_2) - \tau(\tilde{t}_1) - \tilde{t}_2),
\end{aligned}$$

where, as usual, $\tau : \widetilde{C} \to \widetilde{C}$ denotes the covering involution for the map $\pi : \widetilde{C} \to C$.

Thus either $\tilde{t}_1 + \tau(\tilde{t}_2)$ and $\tau(\tilde{t}_1) + \tilde{t}_2$ are equal as divisors or $\tilde{t}_1 + \tau(\tilde{t}_2)$ and $\tau(\tilde{t}_1) + \tilde{t}_2$ span a g^1_2 linear system on $\widetilde{C}$, that is $\tilde{t}_1 + \tau(\tilde{t}_2)$ and $\tau(\tilde{t}_1) + \tilde{t}_2$ are disjoint divisors in the hyperelliptic linear system on $\widetilde{C}$. This gives the following possibilities

Case 1. We have $\tilde{t}_1 = \tilde{t}_2$. In this case the equality (5) implies $\pi^* A_1 = \pi^* A_2$. Since π is ramified, the pullback $\pi^* : \mathrm{Jac}^0(C) \to \mathrm{Jac}^0(\widetilde{C})$ is injective, and so $A_1 = A_2$.

Case 2. We have $\tilde{t}_1 = \tau(\tilde{t}_1)$ and $\tilde{t}_2 = \tau(\tilde{t}_2)$, and $\tilde{t}_1 \neq \tilde{t}_2$, i.e. $\tilde{t}_1$ and $\tilde{t}_2$ are two distinct ramification points of $\pi : \widetilde{C} \to C$. This violates the identity (5) which implies that

$$\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \tilde{t}_2) = \pi^* \left(A_1 \otimes A_2^{-1} \right).$$

However the line bundle $\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \tilde{t}_2)$ can not be a pullback of a line bundle from C. Indeed, every line bundle $\mathscr{L}$ on $\widetilde{C}$ which is a pullback from a line bundle on C admits a τ-equivariant structure which acts trivially on the fibers of $\mathscr{L}$ at all ramification points of $\pi : \widetilde{C} \to C$. On the other hand, if $\tilde{r}$ is a fixed point of τ, then τ preserves the ideal subsheaf $\mathcal{O}_{\widetilde{C}}(-\tilde{r})$ and so the locally free sheaf $\mathcal{O}_{\widetilde{C}}(-\tilde{r})$ is equipped with a canonical τ-equivariant structure in which τ acts as multiplication by (-1) on the fiber of $\mathcal{O}_{\widetilde{C}}(-\tilde{r})$ at $\tilde{r}$ and acts as multiplication by 1 on the of $\mathcal{O}_{\widetilde{C}}(-\tilde{r})$ at any fixed point different from $\tilde{r}$. By duality $\mathcal{O}_{\widetilde{C}}(\tilde{r})$ has an equivariant structure with the same exact property and so by tensoring we see that for two distinct ramification points $\tilde{t}_1$ and $\tilde{t}_2$ we see that $\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \tilde{t}_2)$ has a τ-equivariant structure in which τ acts by multiplication by (-1) on the fibers at $\tilde{t}_1$ and $\tilde{t}_2$ and by multiplication by 1 on the fibers at the other two ramification points. But the only other τ-equivariant structure on this line bundle will be obtained by multiplying the given equivariant structure by the sign charater $\langle \tau \rangle \to \mathbb{C}^\times$, $\tau \mapsto -1$. In either of these structures τ acts non-trivially on the fibers of $\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \tilde{t}_2)$ at two of the ramification points, and so this line bundle is not a pullback of a line bundle on C. This shows that in this case equation (5) does not have a solution.

Case 3. We have that $\tilde{t}_1 + \tau(\tilde{t}_2)$ and $\tilde{t}_2 + \tau(\tilde{t}_1)$ are disjoint divisors in the hyperelliptic linear system on $\widetilde{C}$. To analyze this case better we first recall the basic diagram governing the geometry of the curve $\widetilde{C}$. To construct the spectral curve $\widetilde{C}$ we start with a quadratic differential $\beta \in H^0(C, \omega_C^{\otimes 2})$ having simple zeroes and we take $\widetilde{C} \subset \mathrm{tot}(\omega_C)$ to be the unique double cover of C branched at the zeroes of β. Let $\mathsf{h}_C : C \to \mathbb{P}^1$ be the hyperelliptic map. Since $\omega_C = \mathsf{h}_C^* \mathcal{O}_{\mathbb{P}^1}(1)$ and it is easy to see that every section of $\omega_C^{\otimes 2}$ is a pullback of a unique section in $\mathcal{O}_{\mathbb{P}^1}(2)$. Thus the divisor of β is the pullback of a degree two divisor $\mathbf{y} + \mathbf{z}$ in

$\mathbb{P}^1$, where $\mathbf{y}, \mathbf{z} \in \mathbb{P}^1$ are two distinct points, neither of which is a branch point of h_C. In particular, the degree 4 cover $\mathsf{h}_C \circ \pi : \widetilde{C} \to \mathbb{P}^1$ factors as $\widetilde{C} \to \mathbb{P}^1 \to \mathbb{P}^1$, where $\mathbb{P}^1 \to \mathbb{P}^1$ is the double cover branched at $\mathbf{y} + \mathbf{z}$, and $\widetilde{C} \to \mathbb{P}^1$ is the hyperelliptic map on $\widetilde{C}$ which we will denote by $\mathsf{h}_{\widetilde{C}}$. This also implies that the covering involution $\tau : \widetilde{C} \to \widetilde{C}$ for $\pi : \widetilde{C} \to C$ commutes with the hyperelliptic involution $\sigma : \widetilde{C} \to \widetilde{C}$ and that the composition $\rho = \sigma \circ \tau$ is a fixed point free involution with quotient $D = \widetilde{C}/\langle \rho \rangle$ which is a smooth hyperelliptic curve of genus 3. All these data can be organized in the commutative diagram

$$\begin{tikzcd}
 & \widetilde{C} \arrow[dl, "\mathsf{h}_{\widetilde{C}}"'] \arrow[d, "\mathsf{a}"] \arrow[dr, "\pi"] & \\
\mathbb{P}^1 \arrow[dr, "\mathsf{h}_{\mathbb{P}^1}"'] & D \arrow[d, "\mathsf{h}_D"] & C \arrow[dl, "\mathsf{h}_C"] \\
 & \mathbb{P}^1 &
\end{tikzcd} \tag{6}$$

where the maps $\mathsf{h}_{\widetilde{C}}$, a, and π have covering involutions σ, ρ, and τ respectively.

Therefore the divisor $\tilde{t}_1 + \tau(\tilde{t}_2)$ is a fiber of the map $\mathsf{h}_{\widetilde{C}}$ if and only if $\tilde{t}_1 = \sigma \circ \tau(\tilde{t}_2) = \rho(\tilde{t}_2)$. So in this case the identity (5) becomes

$$\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \rho(\tilde{t}_1)) = \pi^*(A_1 \otimes A_2^{-1}). \tag{7}$$

Note that the line bundle $\mathcal{O}_{\widetilde{C}}(\tilde{t}_1 - \rho(\tilde{t}_1))$ is a ρ-antiinvariant line bundle on $\widetilde{C}$, i.e. as a point in $\mathrm{Jac}^0(\widetilde{C})$ it belongs to the subgroup

$$\ker \left[\mathrm{Jac}^0(\widetilde{C}) \xrightarrow{1+\sigma} \mathrm{Jac}^0(\widetilde{C}) \right] = \ker \left[\mathrm{Jac}^0(\widetilde{C}) \xrightarrow{\mathsf{Nm}_\mathsf{a}} \mathrm{Jac}^0(C) \right] \subset \mathrm{Jac}^0(\widetilde{C}).$$

From the classical theory of Prym varieties [Mum74] of étale double covers, it is known that $\ker(1+\rho) = \ker \mathsf{Nm}_\mathsf{a}$ is a disconnected abelian subgroup with two connected components, and a connected component of the identity equal to the (degree zero) Prym variety for the pair $(\widetilde{C}, D)$:

$$\mathsf{Prym}(\widetilde{C}, D) = \mathrm{im} \left[\mathrm{Jac}^0(\widetilde{C}) \xrightarrow{1-\sigma} \mathrm{Jac}^0(\widetilde{C}) \right].$$

Next consider the Abel-Prym map

$$\mathfrak{ap}^k_{(\widetilde{C},D)} : \mathrm{Sym}^k \widetilde{C} \longrightarrow \ker \mathsf{Nm}_\mathsf{a} \subset \mathrm{Jac}^0(\widetilde{C}), \ \mathfrak{d} \mapsto \mathcal{O}_{\widetilde{C}}(\mathfrak{d} - \rho(\mathfrak{d})).$$

As explained in [Mum74] the image of this map lands in the identity component

$$\mathsf{Prym}(\widetilde{C}, D) = (\ker \mathsf{Nm}_{\mathsf{a}})_o \subset \ker \mathsf{Nm}_{\mathsf{a}}$$

if and only if k is even. Thus the image of Abel-Prym map

$$\mathfrak{ap}^1_{(\widetilde{C},D)} : \widetilde{C} \longrightarrow \ker \mathsf{Nm}_{\mathsf{a}} \subset \mathrm{Jac}^0(\widetilde{C}),\ \tilde{t} \mapsto \mathcal{O}_{\widetilde{C}}(\tilde{t} - \rho(\tilde{t})),$$

is contained in the non-neutral component of $\ker \mathsf{Nm}_{\mathsf{a}}$ and so the curve $\mathfrak{ap}^1_{(\widetilde{C},D)}(\widetilde{C}) \subset \mathrm{Jac}^0(\widetilde{C})$ is disjoint[4] from the two dimensional abelian subvariety $\mathsf{Prym}(\widetilde{C}, D) \subset \mathrm{Jac}^0(\widetilde{C})$ inside $\mathrm{Jac}^0(\widetilde{C})$. But the configuration (6) of double covers implies that

$$\mathsf{Prym}(\widetilde{C}, D) = \pi^* \mathrm{Jac}^0(C) \ \subset \ \mathrm{Jac}^0(\widetilde{C}),$$

and hence the equation (7) has no solution.

This completes the analysis of the third case and shows that the map (4) is injective. To check that the map (4) also separates tangent directions, consider the degree six version of the Prym for $(\widetilde{C}, C)$:

$$\mathcal{P}_6 = \left\{ M \in \mathrm{Jac}^6(\widetilde{C}) \,\middle|\, \mathsf{Nm}_\pi(M) = \omega_C^{\otimes 3} \right\}.$$

We have a natural multiplication-by-2 map

$$\mathsf{mult}_2 : \mathcal{P}_3 \to \mathcal{P}_6, \quad M \mapsto M^{\otimes 2},$$

which fits in a commutative diagram

$$\begin{tikzcd} \widehat{C} \arrow[r] \arrow[d, "\widehat{\mathsf{sq}}"'] & \mathcal{P}_3 \arrow[d, "\mathsf{mult}_2"] \\ \widetilde{C} \arrow[r] & \mathcal{P}_6 \end{tikzcd} \tag{8}$$

[4] In our setting we can see this fact directly without appealing to Mumford's parity analysis [Mum74]. Indeed, by definition the curve D is the double cover of $\mathbb{P}^1$ branched at the 8 points which are the union of the 6 branch points of $\mathsf{h}_C : C \to \mathbb{P}^1$ and the two points $\mathbf{y}, \mathbf{z} \in \mathbb{P}^1$. Let Y denote the ramification point of $\mathsf{h}_D : D \to \mathbb{P}^1$, sitting over $\mathbf{y} \in \mathbb{P}^1$. Then $\mathsf{a}^{-1}(\mathsf{Y})$ consists of two distinct points $\tilde{y}, \rho(\tilde{y})$, which are both ramification points of $\pi : \widetilde{C} \to C$. But in **Case 2.** above we argued that the line bundle

$$\mathcal{O}_{\widetilde{C}}(\tilde{y} - \rho(\tilde{y}))$$

cannot be a pullback from $\mathrm{Jac}^0(C)$. Thus the point $\mathfrak{ap}^1_{(\widetilde{C},D)}(\tilde{t})$ must belong to the non-neutral component of $\ker \mathsf{Nm}_{\mathsf{a}}$. Since $\mathfrak{ap}^1_{(\widetilde{C},D)}(\widetilde{C})$ is connected, this implies that $\mathfrak{ap}^1_{(\widetilde{C},D)}(\widetilde{C})$ is entirely contained in the non-neutral component.

where the top horizontal map is the map (4), while the bottim horizontal map $\widetilde{C} \to \mathcal{P}_6$ is given by $\tilde{t} \mapsto \pi^*(\omega_C(\mathbf{p})) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t} - \tau(\tilde{t}))$.

Now observe that for any $\tilde{t} \in \widetilde{C}$ the divisor $\tilde{t} + \sigma(\tilde{t})$ is in the hyperelliptic linear system on $\widetilde{C}$. By the same token $\rho(\tilde{t}) + \sigma \circ \rho(\tilde{t}) = \rho(\tilde{t}) + \tau(\tilde{t})$ is in the hyperelliptic linear system on $\widetilde{C}$. Thus we have a linear equivalence

$$\tilde{t} + \sigma(\tilde{t}) \sim \rho(\tilde{t}) + \tau(\tilde{t})$$

and so

$$\mathcal{O}_{\widetilde{C}}(\tilde{t} - \tau(\tilde{t})) \cong \mathcal{O}_{\widetilde{C}}(\rho(\tilde{t}) - \sigma(\tilde{t})) = \mathcal{O}_{\widetilde{C}}(\rho(\tilde{t}) - \tau(\rho(\tilde{t}))).$$

Hence the map $\widetilde{C} \to \mathcal{P}_6$ factors through D. The induced map $D \to \mathrm{Jac}^0(\widetilde{C})$ is given by $s \mapsto a^*\mathcal{O}_D(s) \otimes \mathsf{h}_{\widetilde{C}}\mathcal{O}_{\mathbb{P}^1}(-1)$ for all $s \in D$. This shows that the map $\widetilde{C} \to \mathcal{P}_6$ factors as

$$\widetilde{C} \xrightarrow{\mathsf{a}} D \xrightarrow{\mathfrak{aj}} \mathrm{Jac}^1(D) \xrightarrow{a^*(-)\otimes \mathsf{h}_{\widetilde{C}}\mathcal{O}_{\mathbb{P}^1}(-1)\otimes \pi^*(\omega_C(\mathbf{p}))} \mathcal{P}_6.$$

The map $\mathrm{Jac}^1(D) \to \mathcal{P}_6$ is an étale double cover which is isomorphic to the quotient of $\mathrm{Jac}^1(D)$ by the the translation action of the 2-torsion line bundle defining the cover $\widetilde{C} \to D$ [Mum74]. From the diagram (6) we see that this 2-torsion line bundle is given explicitly as $\mathcal{O}_D(\mathsf{Y} - \mathsf{Z})$, where Y and Z are the two Weierstrass points in D that sit over $\mathbf{y}, \mathbf{z} \in \mathbb{P}^1$. Since the Abel-Jacobi map $\mathfrak{aj} : D \to \mathrm{Jac}^1(D)$ is a closed embedding, this shows that the map $D \to \mathcal{P}_6$ embeds $D - \{\mathsf{Y}, \mathsf{Z}\}$ and glues Y and Z into a node. If we write $\underline{D}$ for the iimage of this nodal curve in $\mathcal{P}_6$ we get that the map $\widetilde{C} \to \underline{D} \subset \mathcal{P}_6$ is injective on tangent spaces. But the map $\widehat{\mathsf{sq}} : \widehat{C} \to \widetilde{C}$ is étale, and so the composition $\widehat{C} \to \widetilde{C} \to \underline{D} \subset \mathcal{P}_6$ is injective on tangent spaces. From the diagram (8) we see that this map also factors as $\widehat{C} \to \mathcal{P}_3 \to \mathcal{P}_6$ and so the $\widehat{C} \to \mathcal{P}_3$ is injective on tangent spaces. This completes the proof of the lemma. □

3.3 The wobbly locus

We recall that Laumon defines the notion of ***very stable*** vector bundle as one that does not admit a non-zero nilpotent Higgs field. Such bundles are automatically stable [Lau88]. The first two authors therefore introduced the complementary notion of ***wobbly*** as a semistable vector bundle that isn't very stable. The ***wobbly locus*** $\mathrm{Wob} \subset X$ thus consists of those polystable vector bundles that admit a nonzero nilpotent Higgs field.

With indexation on the degree our notations for the wobbly loci are:

$$\mathrm{Wob}_0 \subset X_0, \quad \mathrm{Wob}_1 \subset X_1.$$

These will be the supports for the parabolic structures and logarithmic poles of the Higgs fields for the Higgs bundles (1).

In section 5 we show that the wobbly locus Wob_0 is a divisor in the 2-theta space of C comprising several familiar players in the classical geometry of the quadric line complex. Specifically the divisor Wob_0 in $X_0 \cong \mathbb{P}^3$ has 17 irreducible components: the quartic Kummer surface $\mathsf{Kum} \subset \mathbb{P}^3$ and its 16 trope planes Trope_κ, labeled by the theta characteristics of C. Thus

$$\mathrm{Wob}_0 = \mathsf{Kum} \cup \left[\cup_{\kappa \in \mathsf{Spin}(C)} \mathsf{Trope}_\kappa\right].$$

From the classical 16_6 configuration in the quadratic line complex [GH94] we know that each trope plane Trope_κ is tangent to the Kummer surface Kum along a trope conic $\mathfrak{C}_\kappa$. Hence Wob_0 fails to be normal crossing along the trope conics $\{\mathfrak{C}_\kappa\}_{\kappa \in \mathsf{Spin}(C)}$.

It is also easy to characterize the components of the wobbly divisor from the moduli point of view. The Kummer surface Kum parametrizes S-equivalence classes of semistable bundles with polystable representatives of the form $\mathfrak{a} \oplus \mathfrak{a}^{-1}$ for $\mathfrak{a} \in \mathrm{Jac}^0(C)$. The trope plane Trope_κ can be canonically identified with $\mathbb{P}(H^1(C, \kappa^{-\otimes 2})) = \mathbb{P}(H^1(C, \omega_C^{-1}))$ and parametrizes all bundles that can be realized as non-split extensions $0 \to \kappa^{-1} \to F \to \kappa \to 0$.

Similarly, in section 4 we show that $\mathrm{Wob}_1 \subset X_1$ is an irreducible divisor. Specifically, as mentioned in the previous section, Wob_1 parametrizes all vector bundles $E \in X_1$ which arise as non-split extensions $0 \to A \to E \to A^{-1}(\mathbf{p}) \to 0$ for some point $(A, t) \in \overline{C}$. In fact we will see that Wob_1 is the tangent developable of the map $\mathsf{ql} : \overline{C} \to X_1 \subset \mathbb{P}^5$, i.e. the union of all projective tangent lines to points in the curve $\mathsf{ql}(\overline{C})$. This shows that Wob_1 is also a non normal crossings divisor in X_1. It has the curve $\mathsf{ql}(\overline{C})$ as its curve of cusps.

3.4 Hecke correspondences

The ***big Hecke correspondence*** $\overline{\mathcal{H}} \to X_1 \times X_0 \times \overline{C}$ is the moduli of quadruples $(E, F, (A, t), \beta)$ such that $(E, F, (A, t)) \in X_1 \times X_0 \times \overline{C}$ and $\beta : F \otimes A^{-1} \to E$ is a map that fits in a short exact sequence

$$0 \longrightarrow F \otimes A^{-1} \xrightarrow{\beta} E \longrightarrow \mathbb{C}_t \longrightarrow 0.$$

We will view $\overline{\mathcal{H}}$ either as a correspondence from X_1 to $X_0 \times \overline{C}$ or as a correspondence from X_0 to $X_1 \times \overline{C}$. We will label the respective projections as

$$
\begin{tikzcd}
 & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_1 & & X_0 \times \overline{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\
X_0 & & X_1 \times \overline{C}
\end{tikzcd}
\tag{9}
$$

where $p = \mathrm{pr}_{X_1}$, $d = \mathrm{pr}_{X_0}$, $q = \left(\mathrm{pr}_{X_0}, \mathrm{pr}_{\overline{C}}\right)$, and $b = \left(\mathrm{pr}_{X_0}, \mathrm{pr}_{\overline{C}}\right)$. In fact the moduli $\overline{\mathcal{H}}$ is a subvariety in the triple product $X_0 \times X_1 \times \overline{C}$. To see this, note that when β exists for a given point $(E, F, (A, t)) \in X_0 \times X_1 \times \overline{C}$, then it is unique up to scale: $\mathcal{O}_C(t)$, and hence t, is determined as the ratio of determinants, and then for this given t, the Hecke fiber is a line in one direction, a conic in the other which excludes any self-intersections.

In some parts of the discussion we will look at Hecke transformations supported at a single point. In those situations we will write $\overline{\mathcal{H}}(a) = \mathrm{pr}_{\overline{C}}^{-1}(\{a\})$ for the preimage of $a = (A, t) \in \overline{C}$ in $\overline{\mathcal{H}}$ and will view $\overline{\mathcal{H}}(a)$ as a correspondence between X_0 and X_1.

The (big) Hecke correspondences appeared in Chapter 2, Sections 2.4 and 2.6, from the synthetic viewpoint. We will review the comparison between the synthetic and modular viewpoints in Section 6.3.

The ***big abelianized Hecke correspondence*** $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is the blow-up of $Y_0 \times \widehat{C}$ along a copy of $\widehat{C} \times \widehat{C}$. As we will see in section 7.2 the map

$$
\begin{aligned}
\boldsymbol{\imath}_{\widehat{C}\times\widehat{C}} : \qquad \widehat{C} \times \widehat{C} & \hookrightarrow \mathcal{P}_2 \times \widehat{C} \\
((A_1, \tilde{t}_1), (A_2, \tilde{t}_2)) & \longrightarrow \left(\pi^* \left(A_2 \otimes A_1^{\vee}(\mathbf{p})\right)(\tilde{t}_1 - \tilde{t}_2), (A_2, \tilde{t}_2)\right),
\end{aligned}
$$

is a closed embedding. This embedding clearly preserves the second projections to $\widehat{C}$ and the strict transform of $\widehat{C} \times \widehat{C}$ in $Y_0 \times \widehat{C}$ is isomorphic copy of $\widehat{C} \times \widehat{C}$ embedded as a subvariety of $Y_0 \times \widehat{C}$. The blow-up of $Y_0 \times \widehat{C}$ along this copy of $\widehat{C} \times \widehat{C}$ is the big abelianized Hecke correspondence $\widehat{\mathcal{H}}^{\mathrm{ab}}$.

The variety $\widehat{\mathcal{H}}^{\mathrm{ab}}$ maps to $Y_0 \times Y_1 \times \widehat{C}$ and thus gives correspondences

$$\begin{tikzcd} & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "p^{\mathrm{ab}}"'] \arrow[dr, "q^{\mathrm{ab}}"] & \\ Y_1 & & Y_0 \times \widehat{C} \end{tikzcd} \qquad \begin{tikzcd} & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "d^{\mathrm{ab}}"'] \arrow[dr, "b^{\mathrm{ab}}"] & \\ Y_0 & & Y_1 \times \widehat{C} \end{tikzcd}$$

There is also map

$$g : \widehat{\mathcal{H}}^{\mathrm{ab}} \to \overline{\mathcal{H}},$$

such that altogether we get a map of abelianized and usual Hecke correspondences. The formulas for the maps p^{ab}, q^{ab}, d^{ab}, b^{ab}, and g are given in section 7.2.

Again, most of the time, we will work with a single fiber of the abelianized Hecke correspondence. If $\tilde{a} = (A, \tilde{t}) \in \widehat{C}$ is a fixed point we will focus on the fiber $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tilde{a}) = \mathrm{pr}_{\widehat{C}}^{-1}(\tilde{a})$.

3.5 Spectral line bundles

In the abelianization strategy to the Hecke eigensheaf problem we use spectral data on C to describe the eigenvalue Higgs bundle (E, θ) and spectral data on X_0 and X_1 to describe the eigensheaf parabolic Higgs bundles $(\mathscr{F}_{0,\bullet}, \Phi_0)$ and $(\mathscr{F}_{1,\bullet}, \Phi_1)$. The spectral covers for these spectral data are $\pi : \widetilde{C} \to C$ and $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ respectively.

Suppose $(\widetilde{C} \subset T^\vee C, \boldsymbol{N})$ is the spectral data for (E, θ), i.e. $(E, \theta) = (\pi_* \boldsymbol{N}, \pi_* (\lambda \otimes (-)))$, where $\lambda : \widetilde{C} \to T^\vee C$ is the embedding, and $\boldsymbol{N} \in \mathcal{P}_2$ is the spectral line bundle. We will use $\boldsymbol{N}$ to construct the ***spectral line bundles*** $\mathscr{L}_0$ on Y_0, and $\mathscr{L}_1$ on Y_1 which will define our parabolic eigensheaf Higgs bundles.

An appropriately normalized Fourier-Mukai transform on the Jacobian of $\widetilde{C}$ (see section 7.2 for the precise details of the normalization) sends the skyscraper sheaf $\mathcal{O}_{\boldsymbol{N}} \in D^b_{\mathrm{coh}}(\mathrm{Jac}^2(\widetilde{C}))$ to a line bundle on $\mathrm{Jac}^2(\widetilde{C})$ with a vanishing first Chern class. We denote the restriction of this line bundle to $\mathcal{P}_2 \subset \mathrm{Jac}^2(\widetilde{C})$ by $\mathfrak{L}_0$ and define a spectral line bundle $\mathscr{L}_0$ on the modular spectral cover $f_0 : Y_0 \to X_0$ by setting

$$\mathscr{L}_0 = (\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0) (\boldsymbol{E}_0) \otimes f_0^* \mathcal{O}_{X_0}(2) \text{ in } \mathrm{Pic}(Y_0).$$

Using the modular spectral data $(Y_0, \mathscr{L}_0)$ we can now define a meromorphic Higgs bundle on X_0 by

$$(\mathscr{F}_{0,0}, \Phi_0) := (f_{0*} \mathscr{L}_0, f_{0*}(\boldsymbol{\alpha}_0 \otimes (-))),$$

where $\boldsymbol{\alpha}_0 : Y_0 \to T^\vee_{X_0}(\log \mathrm{Wob}_0)$ is the tautological map, defined away from the tacnodes of Wob_0. We will later provide this with a parabolic structure to build a parabolic rank 8 bundle $\mathscr{F}_{0,\bullet}$. Away from the non normal crossing codimension two strata of Wob_0, the parabolic structures are trivial; the bundle needs to have a parabolic structure on a blown-up version of X_0 at the tacnodes. The geometry of the blow-up and the construction of the parabolic structure will be explained in section 5 and section 9.

In this analysis we will also see that the Higgs field of $\Phi_0 : \mathscr{F}_{0,\bullet} \to \mathscr{F}_{0,\bullet} \otimes \Omega^1_{X_0}(\log \mathrm{Wob}_0)$ has logarithmic poles on both the trope planes and the Kummer surface. The residues are nilpotent and have Jordan blocks as follows: two Jordan blocks of size 2 and four of size 1 over the tropes, and four Jordan blocks of size 2 over the Kummer.

To describe the spectral line bundle $\mathscr{L}_1$ on Y_1 it is convenient to fix a point $\widetilde{\mathsf{p}} \in \widetilde{C}$ lying over p. Recall the assumption that the spectral curve has two branches over p.

The choice of $\widetilde{\mathsf{p}}$ gives an isomorphism $\boldsymbol{t}_{-\widetilde{\mathsf{p}}} : \mathcal{P}_3 \xrightarrow{\cong} \mathcal{P}_2$ sending L to $L(-\widetilde{\mathsf{p}})$ and we can pullback $\mathfrak{L}_0$ by this isomorphism to define a line bundle $\mathfrak{L}_1 = \boldsymbol{t}^*_{-\widetilde{\mathsf{p}}}\mathfrak{L}_0$ on $\mathcal{P}_3$ with vanishing first Chern class. With this notation we now set

$$\mathscr{L}_1 := \boldsymbol{\varepsilon}_1^*\mathfrak{L}_1 \otimes \mathcal{O}_{X_1}(1),$$

and

$$(\mathscr{F}_{1,0}, \Phi_1) := \left(f_{1*}\mathscr{L}_1, f_{1*}\left(\boldsymbol{\alpha}_1 \otimes (-)\right)\right).$$

In section 4 this is going to be given a parabolic structure over Wob_1, with parabolic weights 0 and $1/2$, such that the associated graded bundle of grading $1/2$ has rank 2. This means that

$$\mathscr{F}_{1,s} = \mathscr{F}_1 \quad \text{for} \quad s \in [0, 1/2).$$

3.6 Orbits of the $\mathbb{C}^\times$-action

An important aspect of the geometry of the Hitchin moduli space is the $\mathbb{C}^\times$-action. In this section, Higgs will denote a moduli space of Higgs bundles of some degree and fixed determinant. In our situation, it means $\mathsf{Higgs} = \mathsf{Higgs}_0$ or Higgs_1. Some of this discussion takes place in more general settings so the notation is left non-specific when possible.

An element $z \in \mathbb{C}^\times$ sends $(E, \theta) \in \mathsf{Higgs}$ to $(E, z\theta)$. The fixed-point locus $\mathsf{Higgs}^{\mathbb{C}^\times}$ decomposes

$$\mathsf{Higgs}^{\mathbb{C}^\times} = \mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{u}} \sqcup \mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$$

into the "unitary piece" that is just the moduli of bundles (= moduli of bundles equipped with zero Higgs fields), i.e. $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{u}} \cong X$, and the disjoint union of remaining pieces that are moduli spaces of polystable Hodge bundles with nonzero Higgs field.

We recall (see [Sim92]) that a Higgs bundle is called a ***Hodge bundle*** if $E = \bigoplus E^p$ and $\theta : E^p \to E^{p-1} \otimes \omega_C$. These correspond under the nonabelian Hodge correspondence to complex variations of Hodge structure. In our case, we consider bundles of rank 2, so a non-unitary Hodge bundle is a direct sum of two line bundles

$$E \cong L^1 \oplus L^0, \quad \theta : L^1 \to L^0 \otimes \omega_C.$$

We remark that a Hodge bundle with a nonzero Higgs field that is semistable but not stable could be S-equivalent to a polystable unitary Hodge bundle, that is a to a Hodge bundle with a zero Higgs field. This will happen in our case if L^p are both line bundles of degree 0. Those don't count as points in $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$ since the moduli space parametrizes S-equivalence classes and the polystable representative would be the same bundle with trivial Higgs field.

Proposition 3.3. *For the case of rank* 2 *Higgs bundles on a curve of genus* 2 *considered in this paper, the non-unitary fixed point loci in* Higgs_0 *and* Higgs_1 *are described as follows:*

(a) $\mathsf{Higgs}_0^{\mathbb{C}^\times,\mathrm{nu}}$ *is a disjoint union of* 16 *points parametrizing the uniformizing Higgs bundles* $\{(E_\kappa, \theta_\kappa)\}_{\kappa \in \mathsf{Spin}(C)}$*, where* $E_\kappa = \kappa \oplus \kappa^{-1}$*, and* $\theta_\kappa : \kappa \to \kappa^{-1} \otimes \omega_C = \kappa$ *is an isomorphism;*

(b) $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}}$ *is connected, isomorphic to the curve* $\overline{C}$ *that is a* 16*-sheeted étale covering of* C*.*

Proof. Suppose $E \cong L^1 \oplus L^0$ is a Hodge bundle. Polystability with nontrivial Higgs field implies (since we are in the rank 2 case) that the Higgs bundle is in fact stable, and $(L^0, 0)$ is a sub-Higgs bundle, so $\deg(L^0) < (\deg E)/2$. If E has degree 0 with determinant $\mathcal{O}_C$ then $L^0 = (L^1)^\vee$, $\deg L^0 < 0$, and the only possibility is that $\deg(L^0) = -1$, the Higgs field $\theta : L^1 \to L^0 \otimes \omega_C$ is an isomorphism, and hence $L^1 = \kappa$ is one of the 16 square-roots of the canonical bundle. If E has degree 1 with determinant $\mathcal{O}_C(\mathbf{p})$ then $E \cong L \oplus L^{-1}(\mathbf{p})$, $\deg L \leq 0$, and $\theta : L^{-1}(\mathbf{p}) \to L \otimes \omega_C$ is a non-zero map. Therefore we must have $\deg L = 0$ and so the Higgs field $\theta \in H^0(C, L^{\otimes 2}(\mathbf{p}))$ has a single zero at some point $t \in C$, and we get $L^{\otimes -2} \cong \mathcal{O}_C(\mathbf{p} - t)$. This corresponds to a point of the connected curve $\overline{C}$. $\square$

The Hitchin fibration $\boldsymbol{h} : \mathsf{Higgs} \to \mathcal{B} = \mathbb{A}^N$ is equivariant for the $\mathbb{C}^\times$-action, where the weights of the action on the base are strictly positive, determined by the degrees of the invariant polynomials on the Lie algebra or, equivalently, by the expression in terms of sections of powers of ω_C. In the present case, the Hitchin base is simply $H^0(C, \omega_C^{\otimes 2})$ so there is a single weight 2.

Properness of the Hitchin fibration implies that if $y \in \mathsf{Higgs}$ is any point, then the limit $\lim_{z\to 0}(zy)$ exists and is a $\mathbb{C}^\times$-fixed point. This fixed point is a unitary Higgs bundle, i.e. a semistable vector bundle with zero Higgs field, if and only if $y = (E, \theta)$ with semistable underlying vector bundle E, and in this case $\lim_{z\to 0}(zy) = (E, 0)$. Let $\mathsf{Higgs}^{\mathrm{su}} \subset \mathsf{Higgs}^{\mathrm{seu}} \subset \mathsf{Higgs}$ denote the open subsets of Higgs bundles whose underlying vector bundle is stable or semistable respectively. The limiting construction provides a regular map $\mathsf{Higgs}^{\mathrm{seu}} \to X$ to the moduli space of semistable bundles.

Over the open subset of stable bundles, the map that we will write as $\mathsf{Higgs}^{\mathrm{su}} \to X^{\mathrm{s}}$ may be identified with the cotangent bundle

$$\mathsf{Higgs}^{\mathrm{su}} \cong T^\vee(X^{\mathrm{s}}),$$

compatibly with $\mathbb{C}^\times$ actions, the one on the right being the scaling action on the total space of the cotangent bundle. This identification preserves the symplectic structure [Hit87b], in particular the fiber over any point of X^{s} is a Lagrangian subspace.

Let $X^{\mathrm{vs}} \subset X$ be the open subset of very stable bundles, and let $\mathsf{Higgs}^{\mathrm{vs}}$ be the open subset of Higgs bundles whose underlying bundle is very stable. We have $\mathsf{Higgs}^{\mathrm{vs}} \subset \mathsf{Higgs}^{\mathrm{su}}$ and also $\mathsf{Higgs}^{\mathrm{vs}} \cong T^\vee(X^{\mathrm{vs}})$.

The limiting map over the open subset $\mathsf{Higgs}^{\mathrm{seu}}$ provides a rational map

$$\mathsf{Higgs} \dashrightarrow X.$$

One may think of this in terms of ***broken orbits***. Those are defined as maximal $\mathbb{C}^\times$-invariant subsets of the form

$$Z_0 \cup Z_1 \cup \cdots \cup Z_k \subset \mathsf{Higgs}$$

such that $Z_0 \cong \mathbb{A}^1$ and $Z_i \cong \mathbb{P}^1$ for $i = 1, \ldots, k$, provided with points $I_i, O_i \in Z_i$ (except I_0) such that $O_i = I_{i+1}$, where O_i corresponds to the origin and I_i to ∞ in $Z_i \cong \mathbb{P}^1$ (or just the origin for $Z_0 \cong \mathbb{A}^1$), such that everything is compatible with the $\mathbb{C}^\times$ action. The input point y corresponding to $1 \in \mathbb{A}^1 = Z_0$ is mapped by the correspondence of the rational map, to the output point $O_k \in \mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{u}}$. The intermediate points $O_i = I_{i+1}$ are non-unitary fixed points. The data needed to determine such a broken orbit consists of fixing the downward or outgoing direction (see below) of the next orbit Z_{i+1} whenever the limiting point O_i is a non-unitary fixed point. The process stops when we get to a unitary fixed point.

If we start at a general point y_ε nearby to y, the orbit of y_ε limiting to a unitary fixed point, will be near to a broken orbit starting at y.

Let $\mathfrak{Q} \subset \mathsf{Higgs}$ be the complement of $\mathsf{Higgs}^{\mathrm{seu}}$. It is the set of points y such that $\lim_{z\to 0}(zy) \in \mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$. The limiting map provides a constructible map from $\mathfrak{Q}$ to the non-unitary fixed point set. In general, this will not be a regular map. However, in our case there are not very many non-unitary fixed points so broken orbits have at most one break. Points of $\mathfrak{Q}$ are those having one break which is a single well-defined point of $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$.

Lemma 3.4. *In our case, the map $y \mapsto \lim_{z\to 0}(zy)$ is a regular map from $\mathfrak{Q}$ to the fixed-point set $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$.*

Proof. In the general situation the map can be non-regular if there is a broken orbit with two non-unitary fixed points joined by an orbit. However, in view of the description of Proposition 3.3, this does not happen in our case. □

In order to understand how broken orbits can work, or equivalently what their nearby orbits could look like, we need to consider the local picture of Higgs at a fixed point. Assume we are at a fixed point y that is a stable Higgs bundle, so it is a smooth point of Higgs. The general theory of Bialynicki-Birula [BB73] tells us that the fixed point set is smooth at y. The $\mathbb{C}^\times$-action on Higgs determines an action of $\mathbb{C}^\times$ on its tangent space at the fixed point y.

Lemma 3.5. *The tangent space decomposes into pieces according to weights of the $\mathbb{C}^\times$-action:*

$$T_y(\mathsf{Higgs}) = \bigoplus T_y(\mathsf{Higgs})^p$$

such that $T_y(\mathsf{Higgs})^p$ is Serre-dual to $T_y(\mathsf{Higgs})^{1-p}$ and in particular they have the same dimension. The tangent space of the fixed point set is $T_y(\mathsf{Higgs})^0$. In our case, at a non-unitary fixed point, the weights that occur are -1 and 2 in the case of Higgs_0, and $-1, 0, 1, 2$ in the case of Higgs_1.

Proof. The weights can be understood by thinking of the Hodge bundle associated to y as being associated to a complex variation of Hodge structure V. The hermitian structure of V (given by a flat but indefinite hermitian form) leads to a real structure on the VHS $\mathrm{End}^0(V)$

of trace-free endomorphisms. Thus [Zuc79] the first cohomology group has a real Hodge structure of weight 1:

$$T_y(\mathsf{Higgs}) \cong H^1(C, \mathrm{End}^0(V)) = \bigoplus_{p+q=1} T^{p,q}$$

with $T^{p,q} = \overline{T^{q,p}}$). This complex conjugation may be identified, using the real structure, with Serre duality at the level of Dolbeault cohomology. The $\mathbb{C}^\times$-action acts by weight p on the piece $T^{p,1-p}$.

More concretely if we write $E = \bigoplus E^p$ for the general case, then

$$T_y(\mathsf{Higgs})^p = T^{p,1-p} = \mathbb{H}^1 \left[\bigoplus_q \mathrm{Hom}(E^q, E^{p+q}) \xrightarrow{[-,\theta]} \bigoplus_q Hom(E^q, E^{p+q-1}) \otimes \omega_C \right]$$

(where one should also include the trace-free condition but we didn't put that in so as not to complicate the notation). The complex is Serre-dual to itself with a shift by 1. The tangent space of the fixed point set is the set of fixed points in the tangent space, as may be seen by the linearization result to be mentioned shortly below. In our particular case, the VHS at a fixed point has only two adjacent Hodge weights, so the VHS of trace-free endomorphisms has Hodge weights $(-1, 1)$, $(0, 0)$ and $(1, -1)$. Thus, the Hodge structure on $\mathbb{H}^1$ has weights obtained by adding $(1, 0)$ and $(0, 1)$, namely $(-1, 2)$, $(0, 1)$, $(1, 0)$ and $(2, -1)$. The $\mathbb{C}^\times$-weights are therefore $-1, 0, 1, 2$. The piece of weight 1 is complex conjugate or Serre-dual to the piece of weight 0, and this is the tangent space of the fixed point set. Thus, in our case of Higgs_0 this vanishes whereas for Higgs_1 it has dimension 1. We see that the piece of weight -1 has dimension 3 for Higgs_0 and dimension 2 for Higgs_1. □

Define the ***incoming*** or ***upward*** directions to be the directions on which the weight is > 0, and the ***outgoing*** or ***downward*** directions to be those on which the weight is < 0. The general theory [BB73] of $\mathbb{C}^\times$ actions on complex manifolds tells us that there is an analytic open neighborhood around a fixed point z that is isomorphic, as a complex manifold with $\mathbb{C}^\times$-action, to a neighborhood of the origin in the tangent space $T_y(\mathsf{Higgs})$. In particular, the fixed points, and incoming and outgoing manifolds are smooth and identified with the corresponding subspaces of the tangent space. The incoming manifold is the local subset of points whose limit as $z \to 0$ is equal to the fixed point, whereas the outgoing manifold is the local subset of points whose $z \to \infty$ is equal to the fixed point.

The terminologies "upward" and "downward" come from Hitchin's picture [Hit87a] of the energy, or L^2-norm of the Higgs field, as a Morse function on Higgs. We note that the orbit structure of the $\mathbb{C}^\times$-action has been identified, by work of Collier, Wentworth, Wilkin and others, as being the same as the structure of flow lines for the gradient of the Morse function [CW19, Wil20]. It is because of this picture that we call the incoming directions "upward" (they are the directions where one goes upwards by following the gradient of the energy function) and the outgoing directions "downward" (they go downwards following the gradient of the energy function). The flow stops at the "bottom" which is the unitary fixed point set, the moduli space of bundles with zero Higgs field which is obviously the minimum for the L^2 norm.

From Lemma 3.5 we see that the space of incoming directions (weights $p > 0$) is Serre dual to the tangent of the fixed points plus the space of outgoing directions (weights $p \leq 0$). Thus, the space of incoming directions has $1/2$ the dimension. In fact, each of the fibers of the projection over a fixed point, is a Lagrangian subspace of the symplectic moduli space. The space of outgoing directions, plus the fixed point directions, together form the tangent space of the manifold of points that flow out of somewhere in that fixed point set. This is locally (at a general point of the fixed point set) one of the components of the nilpotent cone in Higgs, also a Lagrangian subspace.

In general, the space of directions that are outgoing from a given fixed point will have dimension smaller than half, although it is also half the dimension if the fixed point set is 0-dimensional as is our case in Higgs_0. Similarly, the total space $\mathfrak{Q}$ of directions incoming to the local piece of the fixed point set, will generally have dimension greater than half, but again it is half if the fixed point set is 0-dimensional.

In general one will be considering a Prym variety $\mathcal{P} \subset \mathsf{Higgs}$ that is a general fiber of the Hitchin map. The rational map $\mathsf{Higgs} \dashrightarrow X$ therefore provides a rational map $\mathcal{P} \dashrightarrow X$, and to resolve it one should blow up $\mathfrak{Q} \cap \mathcal{P}$ to get $Y \to \mathcal{P}$. The dimension of $\mathfrak{Q} \cap \mathcal{P}$ is equal to the dimension of the fixed point locus.

In our cases, for Higgs_0 the dimension of $\mathfrak{Q}$ is 3 and it is Lagrangian, while for Higgs_1 the dimension of $\mathfrak{Q}$ is 4 and it fibers over $\overline{C}$ with fibers that are Lagrangian. The locus $\mathfrak{Q} \cap \mathcal{P}$ to be blown up has dimension 0 for $\mathcal{P} \subset \mathsf{Higgs}_0$ and dimension 1 for $\mathcal{P} \subset \mathsf{Higgs}_1$.

One can understand from the local description at the fixed point that a single blowing-up will be sufficient in our case, yielding a morphism $Y \to X$. We will denote the exceptional divisor by $\boldsymbol{E} \subset Y$. In principle, one can use the description of the $\mathbb{C}^\times$ action to understand the resulting ramification of the map $Y \to X$ along $\boldsymbol{E}$. We have chosen instead to give more direct proofs (Lemma 5.6, Proposition 5.12) that the map is simply ramified along the exceptional locus.

In the general case when the $\mathbb{C}^\times$-action can have a wider range of weights, and when there can be multiply broken flow lines, understanding the birational transformation needed to resolve the rational map, and understanding the resulting ramification, seem to be difficult questions.

3.7 The logarithmic property

As before we will write X for the coarse moduli space of semistable rank two bundles on C with fixed determinant $\mathbf{d}$ and Higgs for the moduli space of semistable Higgs bundles with fixed determinant $\mathbf{d}$. Recall that Higgs is equipped with an algebraic $\mathbb{C}^\times$-action, scaling the Higgs fields, i.e. $z \in \mathbb{C}^\times$ acts by $(E, \theta) \mapsto (E, z\theta)$. Note that with this definition the Hitchin map $\boldsymbol{h} : \mathsf{Higgs} \to \mathcal{B}$ becomes $\mathbb{C}^\times$-equivariant once we equip $\mathcal{B} = H^0(\omega_C^{\otimes 2})$ with a scaling action of degree 2.

We will also keep using the notation $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}} \subset \mathsf{Higgs}$ for the union of components of the fixed point locus that are disjoint from X, and let $\mathfrak{Q} \subset \mathsf{Higgs}$ denote the incoming variety to $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$, that is

$$\mathfrak{Q} = \left\{ (E, \theta) \in \mathsf{Higgs} \;\middle|\; \lim_{z \to 0} (E, z\theta) \in \mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}} \right\}.$$

In the degree 1 case, $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}} = \overline{C}$ is an irreducible curve that is a 16-sheeted etale covering of C, whereas in the degree 0 case $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$ consists of 16 distinct points.

In our situation, the fixed point locus has only one higher level, that is to say that the limit for $z \to 0$ along any outgoing direction at $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$ lies in X. This will no longer be the case for higher genus or higher rank. It allows for some simplification.

Theorem 3.6. *The subvariety $\mathfrak{Q}$ is smooth and contained in the smooth locus of* Higgs. *It coincides with the locus of polystable Higgs bundles whose underlying vector bundle is un(semi)stable.*

(a) Let $\widetilde{\mathsf{Higgs}}$ denote the blow-up of Higgs *along $\mathfrak{Q}$. Then this resolves the projection map to X, in other words the rationally defined map* $\mathsf{Higgs} \dashrightarrow X$ *extends to a morphism $\mathfrak{f} : \widetilde{\mathsf{Higgs}} \to X$.*

(b) $\widetilde{\boldsymbol{h}} : \widetilde{\mathsf{Higgs}} \to \mathcal{B}$ *denote the composition of the blow-up morphism* $\widetilde{\mathsf{Higgs}} \to \mathsf{Higgs}$ *with the Hitchin map. Let* $\mathsf{b} \in \mathcal{B}$ *be a point corresponding to a smooth spectral curve* $\widetilde{C}$ *and let* $Y = \widetilde{\boldsymbol{h}}^{-1}(\mathsf{b}) \subset \widetilde{\mathsf{Higgs}}$ *be the fiber of* $\widetilde{\boldsymbol{h}}$*. Then* Y *is a smooth compact threefold which is a blow-up of of the Hitchin fiber* $\mathcal{P} = \boldsymbol{h}^{-1}(\mathsf{b})$ *in a smooth center. The induced map* $f := \mathfrak{f}_{|Y} : Y \to X$ *is finite and maps each connected component of the exceptional divisor of the blow-up* $Y \to \mathcal{P}$ *onto some irreducible component of the wobly divisor* $\mathrm{Wob} \subset X$*.*

Proof. Note that Higgs is smooth in the degree 1 case. In the degree 0 case, the singular locus consists of the reducible Higgs bundles. A point of $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$ is a Hodge bundle, for which the underlying bundle is of the form $\kappa \oplus \kappa^{-1}$ with $\kappa^{\otimes 2} = \omega_C$ and for which the Higgs field sends κ^{-1} to zero, and maps κ isomorphically to $\kappa^{-1} \otimes \omega_C$. Such Higgs bundles are stable. The stable locus in Higgs is $\mathbb{C}^\times$-invariant and contains a neighborhood of $\mathsf{Higgs}^{\mathbb{C}^\times,\mathrm{nu}}$ so it contains $\mathfrak{Q}$. In the previous section we checked that $\mathfrak{Q}$ itself is smooth as it is a rank three affine bundle over $\overline{C}$ in the degree one case and is the disjoint union of 16 copies of a three dimensional affine space in the case of degree zero. This proves the first statement.

For the second statement we will first deal with the degree zero case. For this discussion we will write Higgs_0, $\mathfrak{Q}_0$, X_0, etc. By definition and by the stability comment in the previous paragraph we have that a Higgs bundle (E,θ) belongs to $\mathfrak{Q}_o$ if and only if $\lim_{z\to 0}(E, z\theta)$ exists as a stable Higgs bundle and is isomoprphic to

$$\left(\kappa \oplus \kappa^{-1}, \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\right), \text{ with } \gamma : \kappa \xrightarrow{\cong} \kappa^{-1} \otimes \omega_C \ .$$

But if (E,θ) is a semistable Higgs bundle for which E is not semistable as bundle, then we can find a saturated line sub bundle $\kappa \subset E$, such that $\deg \kappa > \deg E/2$, and $\theta(\kappa) \not\subset \kappa \otimes \omega_C$. Since E has trivial determinant, this means that E fits in a short exact sequence

$$0 \to \kappa \to E \to \kappa^{-1} \to 0, \tag{10}$$

$\deg \kappa > 0$, and the composite map

$$\kappa \hookrightarrow E \xrightarrow{\theta} E \otimes \omega_C \longrightarrow \kappa^{-1} \otimes \omega_C \tag{11}$$

is non-zero. Thus we must have $0 < \deg \kappa \leq g(C) - 1 = 1$, Since (11) is a non-zero map between two line bundles of degree 1, it must be an isomorphism and κ must be a theta characteristic on C. In particular this implies that $\theta(\kappa) \otimes \omega_C^{-1} \subset E$ is a line sub bundle which projects isomorphically onto κ^{-1}. Thus the short exact sequence (10) is split and we have an isomorphism

$$(E,\theta) \;\cong\; \left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right), \tag{12}$$

where $\alpha \in H^0(C,\omega_C)$, $\beta \in H^0(C,\omega_C^{\otimes 2})$, and $\gamma : \kappa \xrightarrow{\cong} \kappa^{-1}\otimes\omega_C$. The shape of the Higgs field in (12) can be rigidified somewhat by choosing the identification $E \cong \kappa\oplus\kappa^{-1}$ more carefully. Indeed, the automorphisms of the bundle $\kappa\oplus\kappa^{-1}$ that act trivially on the determinant line bundle are given by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad a \in \mathbb{C}^\times,\; b \in H^0(C,\omega_C).$$

Conjugating by the authomorphism $\begin{pmatrix} u & -u^{-1}\alpha \\ 0 & u^{-1} \end{pmatrix}$ with $u^2 = \gamma$, we get an isomorphism

$$(E,\theta) \;\cong\; \left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} 0 & -\mathsf{b} \\ 1 & 0 \end{pmatrix}\right)$$

where $\mathsf{b} = -\alpha^2 - \gamma\beta = \boldsymbol{h}(E,\theta) \in \mathcal{B}$ is the quadratic differential defining the spectral cover of (E,θ). This shows that the locus $\mathsf{Higgs}_0^{\mathrm{unss}} \subset \mathsf{Higgs}_0$ of all trivial determinant Higgs bundles whose underlying vector bundle is not semi-stable is the disjoint union $\mathsf{Higgs}_0^{\mathrm{unss}} = \sqcup_\kappa \mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}$ of the 16 Hitchin sections

$$\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}} = \left\{ \left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} 0 & -\mathsf{b} \\ 1 & 0 \end{pmatrix}\right) \;\middle|\; \mathsf{b} \in H^0(C,\omega_C^{\otimes 2}\right\}$$

of $\boldsymbol{h} : \mathsf{Higgs}_0 \to \mathcal{B} = \mathbb{A}^3$ labeled by the theta characteristics $\kappa \in \mathsf{Spin}(C)$. Also, for any

$$(E,\theta) \;\cong\; \left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} 0 & -\mathsf{b} \\ 1 & 0 \end{pmatrix}\right) \in \mathsf{Higgs}_0^{\mathrm{unss}}$$

we have that whenever $z \neq 0$ the Higgs bundle $(E, z\theta)$ is isomorphic to

$$\left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} 0 & -z^2\mathsf{b} \\ 1 & 0 \end{pmatrix}\right)$$

where the isomorphism is given by conjugation by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, with $u^2 = z$. Thus

$$\lim_{z\to 0}(E, z\theta) = \left(\kappa\oplus\kappa^{-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

and so $\mathsf{Higgs}_0^{\mathrm{unss}} \subset \mathfrak{Q}_0$. Since both the un(semi)stable locus and the incoming variety are conical Lagrangian submanifolds which intersect at the 16 non-unitary $\mathbb{C}^\times$-fixed points, we get that $\mathsf{Higgs}_0^{\mathrm{unss}} = \mathfrak{Q}_0$. To shorten the notation we will write $\mathfrak{Q}_{0,\kappa} = \mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}$ for the connected components of this locus.

Note that each of the Hitchin sections $\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}$ has a trivial normal bundle inside Higgs_0. Indeed, the Hitchin sections intersect each Hitchin fiber at smooth points, and by the Lagrangian property we have that the normal bundle of $\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}$ inside Higgs_0 is isomorphic to the cotangent bundle of $\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}$. Since $\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}} \cong \mathcal{B} = H^0(C, \omega_C^{\otimes 2})$ we conclude that $N_{\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}/\mathsf{Higgs}_0} \cong \mathcal{O}_{\mathsf{Higgs}_{0,\kappa}^{\mathrm{unss}}} \otimes H^0(C, \omega_C^{\otimes 2})^\vee$. This implies that when we form the blow-up $\widetilde{\mathsf{Higgs}_0} = \mathsf{Bl}_{\mathfrak{Q}_0}\mathsf{Higgs}_0$ the exceptional divisor $\widetilde{\mathfrak{Q}}_0$ is a product

$$\widetilde{\mathfrak{Q}}_0 = \mathfrak{Q}_0 \times \mathbb{P}\left(H^0(C, \omega_C^{\otimes 2})^\vee\right) \cong \bigsqcup_{\kappa \in \mathsf{Spin}(C)} \mathfrak{Q}_{0,\kappa} \times \mathbb{P}^2.$$

The question of whether the rational map $\widetilde{\mathsf{Higgs}_0} \dashrightarrow X_0$ extends to a morphism $\widetilde{\mathsf{Higgs}_0} \to X_0$ is local near $\mathfrak{Q}_0 \subset \mathsf{Higgs}_0$. Therefore to check that this happens it suffices to check that if S is a scheme and $({}^S E, {}^S\theta)$ is a relative semistable Higgs bundle on $S \times C$ that gives an étale map $S \to \mathsf{Higgs}_0$, then the composite rational map $S \to \mathsf{Higgs}_0 \dashrightarrow X_0$ extends to a morphism $\widetilde{S} = S \times_{\mathsf{Higgs}_0} \widetilde{\mathsf{Higgs}_0} \to X_0$. This is equivalent to checking that the vector bundle ${}^S E_{|(S-\mathfrak{Q}_0)\times C}$ has a canonical (unique up to unique isomorphism) extension to a vector bundle ${}^{\widetilde{S}}E$ on $\widetilde{S} \times C$ which is semistable of trivial determinant on all geometric fibers over $\widetilde{S}$.

As a first approximmation to ${}^{\widetilde{S}}E$ one can take the pullback

$${}^{\widetilde{S}}E^{\mathrm{naive}} := (\widetilde{S} \times C \to S \times C)^* \left({}^S E\right).$$

This is a rank two vector bundle on $\widetilde{S} \times C$ which has trivial determinant on all geometric fibers over $\widetilde{S}$ and is semistable on all geometric fibers over points in $\widetilde{S} - \widetilde{\mathfrak{Q}}_0 = S - \mathfrak{Q}_0$. However by construction the restriction of ${}^{\widetilde{S}}E^{\mathrm{naive}}$ to $\{\tilde{y}\} \times C$ for any closed point $\tilde{y} \in \mathfrak{Q}_{0,\kappa} \times \mathbb{P}^2 \subset \widetilde{\mathfrak{Q}}$ is isomorphic to $\kappa \oplus \kappa^{-1}$ which is unstable.

To construct the actual bundle ${}^{\widetilde{S}}E$ let ${}^S\mathfrak{Q}_0 = S \times_{\mathsf{Higgs}_0} \mathfrak{Q}_0$ and let ${}^{\widetilde{S}}\mathfrak{Q}_0$ be the Cartier divisor $\widetilde{S} \times_S ({}^S\mathfrak{Q}_0)$, i.e. the exceptional divisor of the blow-up $\widetilde{S} \to S$. We will also write ${}^S\mathfrak{Q}_{0,\kappa} = \times_{\mathsf{Higgs}_0}\mathfrak{Q}_{0,\kappa}$ and ${}^{\widetilde{S}}\mathfrak{Q}_{0,\kappa} = \widetilde{S} \times_S ({}^S\mathfrak{Q}_{0,\kappa})$ for the connected components of these loci.

Shrinking S if necessary we get that the restriction of the bundle ${}^{\widetilde{S}}E^{\mathrm{naive}}$ to $({}^{\widetilde{S}}\mathfrak{Q}_{0,\kappa}) \times C$ has a natural surjective homomorphism to $\mathrm{pr}_C^* \kappa^{-1}$ which is unique up to multiplication by an invertible function on ${}^S\mathfrak{Q}_{0,\kappa}$. Define ${}^{\widetilde{S}}E$ to be the Hecke transform of ${}^{\widetilde{S}}E^{\mathrm{naive}}$ centered at this homomorphism, i.e.

$$0 \longrightarrow {}^{\widetilde{S}}E \longrightarrow {}^{\widetilde{S}}E^{\text{naive}} \longrightarrow \bigoplus_{\kappa \in \mathsf{Spin}(C)} \left({}^{\widetilde{S}}\mathfrak{Q}_{0,\kappa} \times C \to \widetilde{S} \times C \right)_* \operatorname{pr}_C^* \kappa^{-1} \longrightarrow 0.$$

Note that scaling the last map in the sequence by an invertible function on ${}^{S}\mathfrak{Q}_0$ does not change the kernel sheaf ${}^{\widetilde{S}}E$ and so it is canonically defined as a subsheaf in ${}^{\widetilde{S}}E^{\text{naive}}$.

The fact that ${}^{\widetilde{S}}E$ is a relative semistable bundle on $\widetilde{S} \times C$ now comes from the following

Lemma 3.7. *Suppose $\tilde{y} = (y, \mathfrak{e}) \in {}^{\widetilde{S}}\mathfrak{Q}_{0,\kappa} = ({}^{S}\mathfrak{Q}_{0,\kappa}) \times \mathbb{P}(H^0(\omega_C^{\otimes 2})^\vee)$. Then the vector bundle ${}^{\widetilde{S}}E_{|\{\tilde{y}\} \times C}$ is isomorphic to the semi-stable rank two bundle $E_\mathfrak{e}$ which is the extension*

$$0 \to \kappa^{-1} \to E_\mathfrak{e} \to \kappa \to 0$$

corresponding to the extension class $\mathfrak{e} \in \mathbb{P}(H^0(\omega_C^{\otimes 2})^\vee) = \mathbb{P}(H^1(\omega_C^{-1})) = \mathbb{P}(H^1(\kappa^{\otimes -2}))$.

Proof. Since $\mathfrak{e} \in \mathbb{P}(H^1(\kappa^{\otimes -2}))$ the extension $E_\mathfrak{e}$ is non-split. But if $L \subset E_\mathfrak{e}$ is a destibilazing line sub bundle, then $\deg L > 0$ and hence as subsheaves in $E_\mathfrak{e}$ we must have $L \cap \kappa^{-1} = 0$ this shows that L maps injectively into κ, and hence we will have $\deg L = 1$ and $L \cong \kappa$. This will split the extension which is contradiction. This proves that $E_\mathfrak{e}$ is semi-stable.

The identification of ${}^{\widetilde{S}}E_{|\{\tilde{y}\} \times C}$ and $E_\mathfrak{e}$ is a general fact in deformation theory which we explain next. Suppose S is the spectrum of a DVR with uniformizer x and closed point $o \in S$. Let $\mathcal{E} \to S \times C$ be an algebraic vector bundle and let $E = \mathcal{E}_{|\{o\} \times C}$. Assume that E fits in a short exact sequence

$$0 \longrightarrow \mathsf{A} \xrightarrow{\imath} E \xrightarrow{\jmath} \mathsf{B} \longrightarrow 0 \tag{13}$$

of vector bundles on C. Consider the Hecke transform

$$0 \to \mathcal{E}' \to \mathcal{E} \to (\{o\} \times C \to S \times C)_* \mathsf{B} \to 0.$$

Then the restriction $E' = \mathcal{E}'_{|\{o\} \times C}$ is a vector bundle that fits in a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{B} \otimes \mathcal{O}_{S \times C}(-\{o\} \times C)_{|\{o\} \times C} & \longrightarrow & E' & \longrightarrow & \mathsf{A} & \longrightarrow & 0 \\ & & \cong\big\downarrow & & & & & & \\ & & \mathsf{B} & & & & & & \end{array} \tag{14}$$

Write

$$\mathsf{ks}^{\mathcal{E}} : H^0(S, T_S) \to H^1(C, \operatorname{End}_0(E)) \subset \operatorname{Hom}(E, E[1])$$

for the Kodaira-Spencer map of $\mathcal{E}$. A direct calculation with either square zero extenions or Čech cocycles gives now that the extension class of the short exact sequence (14) is

$$\jmath \circ \mathsf{ks}^{\mathcal{E}}(\partial_{\mathsf{x}}) \circ \imath \in \mathrm{Hom}(\mathsf{A}, \mathsf{B}[1]) = \mathrm{Ext}^1(\mathsf{A}, \mathsf{B}).$$

If in addition the sequence (13) is split, i.e. $E = \mathsf{A} \oplus \mathsf{B}$, then we can write the Kodaira-Spencer class $\mathsf{ks}^{\mathcal{E}}(\partial_{\mathsf{x}})$ as a block matrix

$$\mathsf{ks}^{\mathcal{E}}(\partial_{\mathsf{x}}) = \begin{pmatrix} \eta_{\mathsf{AA}} & \eta_{\mathsf{AB}} \\ \eta_{\mathsf{BA}} & \eta_{\mathsf{BB}} \end{pmatrix}$$

where $\eta_{\mathsf{AA}} : \mathsf{A} \to \mathsf{A}[1]$, $\eta_{\mathsf{AB}} : \mathsf{B} \to \mathsf{A}[1]$, $\eta_{\mathsf{BA}} : \mathsf{A} \to \mathsf{B}[1]$, and $\eta_{\mathsf{BB}} : \mathsf{B} \to \mathsf{B}[1]$. In particular, the extension class of (14) will be $\eta_{\mathsf{BA}} : \mathsf{A} \to \mathsf{B}[1]$.

Suppose next that S is the spectrum of a DVR which maps to Higgs_0 so that the closed point $o \in S$ maps to a closed point $y \in \mathfrak{Q}_{0,\kappa}$ and the differential of the map $S \to \mathsf{Higgs}_0$ maps ∂_{x} to a tangent vector in $T_y\mathsf{Higgs}_0$ which projects to the normal line $\mathfrak{e} \subset H^1(\kappa^{\otimes -2})$. In particular the map $S \to \mathsf{Higgs}_0$ is given by a relative Higgs bundle $({}^SE, {}^S\theta)$ on $S \times C$ and the component $\eta_{\kappa,\kappa^{-1}}$ of $\mathsf{ks}^{({}^SE)}(\partial_{\mathsf{x}})$ is just equal to $\mathfrak{e}$. This concludes the proof of the lemma. $\square$

These considerations prove assertion (a) in the statement of Theorem 3.6 for moduli of Higgs bundles of determinant $\mathbf{d} = \mathcal{O}_C$. In fact, the above discussion also proves assertion (b) in the trivial determinant case. Indeed, when $\mathbf{d} = \mathcal{O}_C$ the discussion above shows that the connected component $\mathfrak{Q}_{0,\kappa}$ of the center of teh blow-up $\widetilde{\mathsf{Higgs}} \to \mathsf{Higgs}$ intesects the Hitchin fiber $\boldsymbol{h}^{-1}(\mathsf{b}) = \mathcal{P}_2 = \{L \in \mathrm{Jac}^2(\widetilde{C}) \mid \mathsf{Nm}_\pi(L) \cong \omega_C\}$ at a single point, namely the point $\pi^*\kappa \in \mathcal{P}_2$. By the universal property of the blow-up it follows that Y is the blow-up of $\mathcal{P}_2$ at the 16 distinct points $\pi^*\kappa$, $\kappa \in \mathsf{Spin}(C)$. But the morphism $\mathcal{P}_2 - \{\pi^*\kappa\}_{\kappa \in \mathsf{Spin}(C)} \to X$ is quasi-finite, and also by the construction in Lemma 3.7 the map $f = \mathfrak{f}_{|Y} : Y \to X$ maps the exceptional divisor $\boldsymbol{E}_{0,\kappa} \subset Y$ corresponding to the point $\pi^*\kappa \in \mathcal{P}_2$ isomorphically to the plane in $X = X_0$ parametrizing all semistable bundles which are non-trivial extensions of κ by κ^{-1}. Thus $f : Y \to X$ is everywhere quasi-finite and since it is proper, it is finite. Furthermore note that the plane parametrizing non-trivial extensions of κ by κ^{-1} is precisely the trope component Trope_κ of Wob_0 which completes the proof of assertion (b) in this case

Finally we analyze the degree one case. Again we will write Higgs_1, $\mathfrak{Q}_1$, X_1, etc.

Lemma 3.8. *Suppose $(E,\theta) \in \mathsf{Higgs}_1$ is a semistable Higgs bundle, such that E is not semistable as a bundle. Then*

$$(E,\theta) \;\cong\; \left(A^{-1}(\mathbf{p}) \oplus A, \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right)$$

where $A \in \mathrm{Jac}^0(C)$ satisfying $A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(t)$ for some $t \in C$, $\alpha \in H^0(C,\omega_C)$, $\beta \in H^0(C,\omega_C(t'))$, $\gamma \neq 0 \in H^0(C,\mathcal{O}_C(t))$, and t' is the image of t under the hyperelliptic involution on C.

Proof. Let $L \subset E$ be a saturated destabilizing line bundle, i.e. a saturated line subbundle s.t. $\deg L \geq \deg E/2 = 1/2$. Since (E,θ) is semistable as a Higgs bundle, L can not be θ-invariant. In other words, we must have $\theta(L) \neq 0 \subset E \otimes \omega_C$ which is equivalent to the composite map $\theta : L \to E \otimes \omega_C \to (E/L) \otimes \omega_C$ being non-zero. Thus we must have $1 \leq \deg L \leq 1 - \deg L + \deg \omega_C = 3 - \deg L$, i.e. $\deg L = 1$. Writing $L = A^{-1}(\mathbf{p})$ with $\deg A = 0$ we get a non-zero (hence injective) map of locally free sheaves $A^{-1}(\mathbf{p}) \to A(2\mathbf{p})$ and so we must have $A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(t)$ for some point $t \in C$, i.e. we must have that $(A,t) \in \overline{C}$.

Also E fits in a short exact sequence

$$0 \to A^{-1}(\mathbf{p}) \to E \to A \to 0, \tag{15}$$

and viewing θ as a map $E \otimes \omega_C^{-1} \to$ we get a subsheaf $\theta\left(A^{-1}(\mathbf{p}) \otimes \omega_C^{-1}\right) \subset E$ which maps injectively into A via the map $E \to A$. This implies that the extension (15) splits over the subsheaf $\theta : A^{-1}(\mathbf{p}) \otimes \omega_C^{-1} \hookrightarrow A$. Since $(A,t) \in \overline{C}$ we have that $A^{-1}(\mathbf{p}) \otimes \omega_C^{-1} \cong A(-t)$ and under this isomorphism the map θ gets identified with the natural inclusion $A(-t) \subset A$.

By the same token we have

$$\mathrm{Ext}^1(A, A^{-1}(\mathbf{p})) = H^1(C, A^{\otimes -2})(\mathbf{p})) = H^1(C, \mathcal{O}_C(2\mathbf{p} - t)) = H^1(C, \mathcal{O}_C(t')),$$

while

$$\mathrm{Ext}^1(A(-t), A^{-1}(\mathbf{p})) = H^1(C, A^{\otimes -2}(t + \mathbf{p})) = H^1(C, \mathcal{O}_C(2\mathbf{p})) = H^1(C, \mathcal{O}_C(t' + t)),$$

and under these identifications the map

$$\mathrm{Ext}^1(A, A^{-1}(\mathbf{p})) \to \mathrm{Ext}^1(A(-t), A^{-1}(\mathbf{p}))$$

gets identified with the map on cohomology

$$H^1(C, \mathcal{O}_C(t')) \to H^1(C, \mathcal{O}_C(t' + t)) \tag{16}$$

induced from the natural inclusion $\mathcal{O}_C(t') \subset \mathcal{O}_C(t'+t)$.

From the long exact sequence in cohomology associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_C(t') \to \mathcal{O}_C(t'+t) \to \mathcal{O}_t \to 0$$

we see that the map (16) is surjective. on the other hand both $H^1(C, \mathcal{O}_C(t'))$ and $H^1(C, \mathcal{O}_C(t'+t))$ are one dimensional and so the map (16) is an isomorphism. This implies that if an extension class in $\mathrm{Ext}^1(A, A^{-1}(\mathbf{p}))$ maps to zero in $\mathrm{Ext}^1(A(-t), A^{-1}(\mathbf{p}))$, then this class is zero to begin with. Hence the extension (15) is split as claimed. The statement about the matrix entries of θ under the decomposition $E \cong A^{-1}(\mathbf{p}) \oplus A$ now follows tautologically. Note that the entry γ is precisely the composite map

$$A^{-1}(\mathbf{p}) \hookrightarrow E \xrightarrow{\theta} E \otimes \omega_C \longrightarrow A \otimes \omega_C = A(2\mathbf{p})$$

and hence is non-zero. This finishes the proof of the lemmma. $\square$

Again, the shape of the Higgs field in the pair (E, θ) can be simplified by choosing the isomorphism $E \cong A^{-1}(\mathbf{p}) \oplus A$ more carefully. The global automorphisms of the bundle $A^{-1}(\mathbf{p}) \oplus A$ are all of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

where $a \in \mathbb{C}^\times$, and $b \in H^0(C, \mathcal{O}_C(t'))$. Conjugating $z \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ with $u^2 = z$ we get

$$\begin{pmatrix} z\alpha & z^2\beta \\ \gamma & -z\alpha \end{pmatrix}.$$

Hence

$$\lim_{z \to 0} (E, z\theta) = \left(A^{-1}(\mathbf{p}) \oplus A, \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \right),$$

and so the locus $\mathsf{Higgs}_1^{\mathrm{unss}}$ parametrizing stable Higgs bundles with unstable underlying vector bundles is contained in the incoming variety $\mathfrak{Q}_1$. By the explicit description both of these are four dimensional irreducible subvarieties in Higgs_1 and therefore $\mathsf{Higgs}_1^{\mathrm{unss}} = \mathfrak{Q}_1$.

Similarly to the degree zero case the identifictaion $\mathsf{Higgs}_1^{\mathrm{unss}} = \mathfrak{Q}_1$ is compatible both with the Hitchin map $\boldsymbol{h} : \mathsf{Higgs}_1 \to \mathcal{B}$ and with the flow limit map $\mathfrak{Q}_1 \to \mathsf{Higgs}_1^{\mathbb{C}^\times, \mathrm{nu}}$. Indeed,if

$$\left(A^{-1}(\mathbf{p}) \oplus A, \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right) \in \mathsf{Higgs}_1^{\mathrm{unss}},$$

then the determinant of the Higgs field is $\mathsf{b} = -\alpha^2 - \gamma\beta$. This shows that $\mathsf{Higgs}_1^{\mathrm{unss}} = \mathfrak{Q}_1$ surjects onto the Hitchin base $\mathcal{B}$. In Section 3.2 we also checked that the fiber of $\mathsf{Higgs}_1^{\mathrm{unss}}$ over a general point $\mathsf{b} \in \mathcal{B}$ of the Hitchin base is a copy of $\widehat{C}_{\mathsf{b}} = \overline{C} \times_C \widetilde{C}_{\mathsf{b}}$ embedded in the corresponding Hitchin fiber $\mathcal{P}_{3,\mathsf{b}} = \boldsymbol{h}^{-1}(\mathsf{b})$. Thus we can view the smooth variety $\mathfrak{Q}_1$ as a family of curves over $\mathcal{B}$.

At the same time, the above caluclation of $\lim_{z\to 0} zy$ for $y \in \mathfrak{Q}_1$ identifies the limiting flow map $\mathbf{fl}_1 : \mathfrak{Q}_1 \to \overline{C}$ with the natural projection of "forgetting the Higgs field", i.e. for every

$$y = \left(A^{-1}(\mathbf{p}) \oplus A, \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right) \in \mathfrak{Q}_1$$

we have $\mathbf{fl}_1(y) = A \in \overline{C}$.

With this observation at hand we can now compute the normal bundle of smooth subvariety $\mathfrak{Q}_1 \subset \mathsf{Higgs}_1$ and the exceptional divisor of the blow up $\widetilde{\mathsf{Higgs}}_1 = \mathrm{Bl}_{\mathfrak{Q}_1} \mathsf{Higgs}_1 \to \mathsf{Higgs}_1$. Specifically we have

Lemma 3.9. *(a) The normal bundle of* $\mathfrak{Q}_1 \subset \mathsf{Higgs}_1$ *is given by*

$$N_{\mathfrak{Q}_1/\mathsf{Higgs}_1} = H^0(C, \omega_C)^\vee \otimes \mathbf{fl}_1^* \mathsf{sq}^* \omega_C.$$

Thus the exceptional divisor $\widetilde{\mathfrak{Q}}_1$ *of the blow up* $\widetilde{\mathsf{Higgs}}_1 \to \mathsf{Higgs}_1$ *is*

$$\widetilde{\mathfrak{Q}}_1 = \mathbb{P}(N_{\mathfrak{Q}_1/\mathsf{Higgs}_1}) = \mathfrak{Q}_1 \times \mathbb{P}(H^0(C, \omega_C)^\vee) = \mathfrak{Q}_1 \times \mathbb{P}^1.$$

(b) Suppose $\pi : \widetilde{C} \to C$ *is a smooth spectral curve. Let* $\mathcal{P}_3$ *denote the corresponding Hitchin fiber, and let* $\widehat{C} \subset \mathcal{P}_3$ *be the correponding fiber of the map* $\boldsymbol{h}_{|\mathfrak{Q}_1} : \mathfrak{Q}_1 \to \mathcal{B}$. *Then*

$$N_{\widehat{C}/\mathcal{P}_3} = H^0(C, \omega_C)^\vee \otimes \widehat{\mathsf{sq}}^* \pi^* \omega_C,$$

and the exceptional divisor $\boldsymbol{E}_1 \subset Y_1$ *of the blow up* $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$ *is given by*

$$\boldsymbol{E}_1 = \widehat{C} \times \mathbb{P}(H^0(C, \omega_C)^\vee) = \widehat{C} \times \mathbb{P}^1.$$

Proof. To prove (a) note that since $\mathfrak{Q}_1$ is $\mathbb{C}^\times$-stable, we have that the normal bundle $N_{\mathfrak{Q}_1/\mathsf{Higgs}_1}$ is $\mathbb{C}^\times$-equivariant. Furthermore, as we explained in the previous section, in our situation there are no broken orbits in $\mathfrak{Q}_1$ and therefore the normal bundle will be the pullback of the bundle of outgoing directions to the non-unitary fix point set $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}} = \overline{C}$, i.e.

$$N_{\mathfrak{Q}_1/\mathsf{Higgs}_1} = \mathbf{fl}_1^* \left(T_{\mathsf{Higgs}_1}\right)^{(-1)}_{|\overline{C}}$$

is the piece in the tangent bundle to Higgs_1 of $\mathbb{C}^\times$-weight -1 along the fixed point curve $\overline{C}$. Note that here $\overline{C}$ is embedded in Higgs_1 by the map sending $A \in \overline{C}$ to the isomorphism class of the Higgs bundle

$$(E_A, \theta_A) := \left(A^{-1}(\mathbf{p}) \oplus A, \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \right)$$

with $\gamma : A^{-1}(\mathbf{p}) \to A \otimes \omega_C$ being the unique (up to scale) non-zero map.

From the cohomological description of the tangent space of Higgs_1 at a point we see that the piece of $\mathbb{C}^\times$-weight -1 of the tangent space at a point $A \in \overline{C} \subset \mathsf{Higgs}_1$ is precisely the image of the composition map

$$T_{\mathsf{Higgs}_1,(E_A,\theta_A)} \to \mathbb{H}^1\left(C, \mathrm{End}_0(E_A) \xrightarrow{[\theta_A,-]} \mathrm{End}_0(E_A) \otimes \omega_C \right) \to H^1(C, \mathrm{End}_0(E_A)).$$

This image is precisely the matrix coefficient piece $H^1(C, A^{\otimes 2}(-\mathbf{p})) \subset H^1(C, \mathrm{End}_0(E_A))$ which gives a canonical identification

$$T^{(-1)}_{\mathsf{Higgs}_1,(E_A,\theta_A)} = H^1(C, A^{\otimes 2}(-\mathbf{p})).$$

Next recall, that A is a point in $\overline{C}$ precisely when $A^{\otimes 2} \cong \mathcal{O}_C(t - \mathbf{p})$ for some point $t \in C$. Thus $A^{\otimes 2}(-\mathbf{p})$ is isomorphic to $\mathcal{O}_C(-t')$ where t' is the hyperelliptic conjugate of t. But the inclusion $\mathcal{O}_C(-t') \subset \mathcal{O}_C$ induces an isomorphism on H^1's and so we conclude that we get a natural, unique up to scale, isomorphism

$$T^{(-1)}_{\mathsf{Higgs}_1,(E_A,\theta_A)} \cong H^1(C, \mathcal{O}_C). \tag{17}$$

More precisely, the isomorphism (17) is the map induced on H^1's from the map of locally free rank one sheaves $A^{\otimes 2}(-\mathbf{p}) \cong \mathcal{O}_C(-t') \hookrightarrow \mathcal{O}_C$. Since both the isomorphism $A^{\otimes 2}(-\mathbf{p}) \cong \mathcal{O}_C(-t')$ and the inclusion $\mathcal{O}_C(-t') \hookrightarrow \mathcal{O}_C$ are uniquely defined up to scale it follows that isomorphism (17) is also unique up to scale.

This implies that we have a line bundle L on $\overline{C}$ so that

$$(T^{(-1)}_{\mathsf{Higgs}_1})_{|\overline{C}} \cong H^1(C, \mathcal{O}_C) \otimes L = H^0(C, \omega_C)^\vee \otimes L.$$

To finish the proof of (a) we need to compute the line bundle L. But note that the restriction of the normal bundle of $\mathfrak{Q}_1 \subset \mathsf{Higgs}_1$ to a general fiber $\widehat{C}$ of $\boldsymbol{h}_{|\mathfrak{Q}_1}$ is the normal bundle of $\widehat{C} \subset \mathcal{P}_3$. Since the map $\widehat{\pi} : \widehat{C} \to \overline{C}$ induces an injective map on Picard varieties, we only need to compute $\widehat{\pi}^* L$. Thus (a) will follow immediately once we prove (b).

To prove (b) fix a smooth spectral cover $\pi : \widetilde{C} \to C$ whose branch locus does not include any Weirstrass point on C. Let $\mathcal{P}_3 = \mathrm{Prym}^3(\widetilde{C}, C) \subset \mathsf{Higgs}_1$ denote the corresponding Hitchin fiber and let $\widehat{C} = \mathfrak{Q}_1 \cap \mathcal{P}_3$ be the unstable locus in $\mathcal{P}^3$. Recall that $\widehat{C} = \overline{C} \times_C \widetilde{C}$ and the embedding $\widehat{C} \hookrightarrow \mathcal{P}_3$ is given by $(A, \tilde{t}) \mapsto \pi^*(A^{-1}(\mathbf{p}))(\tilde{t})$. The tangent bundle to $\mathcal{P}_3$ is trivial with fiber identified with the anti-invariants $H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})^- \subset H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ for the covering involution of $\pi : \widetilde{C} \to C$. By Serre duality we can identify $H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})^-$ with the anti-invariants in $H^0(\widetilde{C}, \omega_{\widetilde{C}})^\vee$. As $\widetilde{C} \to C$ is a spectral cover we have

$$\begin{aligned} H^0(\widetilde{C}, \omega_{\widetilde{C}}) &= H^0(\widetilde{C}, \pi^*\omega_C^{\otimes 2}) \\ &= H^0(C, \pi_*\pi^*\omega_C^{\otimes 2}) \\ &= H^0(C, \omega_C^{\otimes 2} \oplus \omega_C) \\ &= H^0(C, \omega_C^{\otimes 2}) \oplus H^0(C, \omega_C), \end{aligned}$$

and so the anti-invariants in $H^0(\widetilde{C}, \omega_{\widetilde{C}})^\vee$ are identified intirinsically with

$$H^0(C, \omega_C^{\otimes 2}).$$

With this identification the normal sequence for the embedding $\widehat{C} \subset \mathcal{P}_3$ can be written as

$$0 \to T_{\widehat{C}} \to H^0(C, \omega_C^{\otimes 2})^\vee \otimes \mathcal{O}_{\widehat{C}} \to N_{\widehat{C}/\mathcal{P}_3} \to 0,$$

where the first map is the Kodaira-Spencer map for the varying family of line bundles $\pi^*(A^{-1}(\mathbf{p}))(\tilde{t})$ on $\widetilde{C}$. Dually, the conormal sequence reads

$$0 \to N^\vee_{\widehat{C}/\mathcal{P}_3} \to H^0(C, \omega_C^{\otimes 2}) \otimes \mathcal{O}_{\widehat{C}} \to \omega_{\widetilde{C}} \to 0, \tag{18}$$

where the last map is the composition

$$H^0(C, \omega_C^{\otimes 2}) \otimes \mathcal{O}_{\widehat{C}} \xhookrightarrow{\pi^*} H^0(\widetilde{C}, \omega_{\widetilde{C}})) \otimes \mathcal{O}_{\widetilde{C}} \xrightarrow{\mathrm{ev}} \omega_{\widetilde{C}}.$$

But $\omega_{\widetilde{C}} = \pi^*\omega_C^{\otimes 2}$ is a pull-back of $\mathcal{O}(2)$ from the hyperelliptic $\mathbb{P}^1$ and $H^0(C, \omega_C^{\otimes 2})$ is the pullback of $H^0(\mathbb{P}^1, \mathcal{O}(2))$. Thus the conormal sequence (18) is a pullback of the canonical evaluation sequence of vector bundles on the hyperelliptic $\mathbb{P}^1$

$$0 \longrightarrow \mathcal{K} \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(2)) \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

But this is a standard Koszul sequence and so we have a canonical identification

$$\mathcal{K} = H^0(\mathbb{P}^1, \mathcal{O}(1)) \otimes \mathcal{O}(-1).$$

Equivalently we can note that by definition the bundle $\mathcal{K}$ has no cohomology and so is isomorphic to a two dimensional vector space tensored with $\mathcal{O}(-1)$. Since the whole sequence is $SL(2,\mathbb{C})$ equivariant, this two dimensional vector space has to be the fundamental representation as claimed.

Altogether we get that

$$N^{\vee}_{\widehat{C}/\mathcal{P}_3} = H^0(\mathbb{P}^1, \mathcal{O}(1)) \otimes \widehat{\mathsf{sq}}^* \pi^* \mathsf{h}_C^* \mathcal{O}(-1) = H^0(C, \omega_C) \otimes \widehat{\mathsf{sq}}^* \pi^* \omega_C^{-1}$$

which finishes the proof of part (b) and the lemma. $\square$

Again, the question of whether the rational map $\mathsf{Higgs}_1 \dashrightarrow X_1$ is resolved by blowing up $\mathfrak{Q}_1 \subset \mathsf{Higgs}_1$ is local near $\mathfrak{Q}_1$. Similarly to the degree zero case it suffices to check that if S is a scheme and $({}^S E, {}^S\theta)$ is a relative semistable Higgs bundle on $S \times C$, which corresponds to an étale map $S \to \mathsf{Higgs}_1$, then the composite rational map $S \to \mathsf{Higgs}_1 \dashrightarrow X_1$ extends to a morphism $\widetilde{S} = S \times_{\mathsf{Higgs}_1} \widetilde{\mathsf{Higgs}}_1 \longrightarrow X_1$. Thus we must show that the vector bundle ${}^S E_{|(S-\mathfrak{Q}_1)\times C}$ has a canonical extension to a vector bundle ${}^{\widetilde{S}}E$ on $\widetilde{S} \times C$ which is semistable and of determinant $\mathcal{O}_C(\mathbf{p})$ on all geometric fibers over $\widetilde{S}$.

Let ${}^S\mathfrak{Q}_1 = S \times_{\mathsf{Higgs}_1} \mathfrak{Q}_1$ and let ${}^{\widetilde{S}}\mathfrak{Q}_1 = \widetilde{S} \times_{\widetilde{\mathsf{Higgs}}_1} \mathfrak{Q}_1$ denote the exceptional divisor for the blow-up $\widetilde{S} \to S$. Shrinking S if necessary we can assume without loss of generality that the restriction of ${}^S E$ to ${}^S\mathfrak{Q}_1 \times C$ is a direct sum

$${}^S E = \left({}^S\!\mathcal{A}\right)^{-1} (\mathrm{pr}_C^* \mathbf{p}) \oplus \left({}^S\!\mathcal{A}\right)$$

where ${}^S\!\mathcal{A}$ denotes the pullback

$${}^S\!\mathcal{A} = \left({}^S\mathfrak{Q}_1 \times C \longrightarrow \mathfrak{Q}_1 \times C \xrightarrow{\mathbf{fl}_1 \times \mathrm{id}} \overline{C} \times C \right)^* \mathcal{A}$$

of the normalized Poincaré line bundle $\mathcal{A} \to \overline{C} \times C$, which is uniquely characterized by the conditions $\mathcal{A}_{\{A\}\times C} \cong A$ for all $A \in \overline{C}$, and $\mathcal{A}_{\overline{C}\times\{\mathbf{p}\}} \cong \mathcal{O}_{\overline{C}}$.

Writing ${}^{\widetilde{S}}\!\mathcal{A}$ for the pullback line bundle

$${}^{\widetilde{S}}\!\mathcal{A} = \left({}^{\widetilde{S}}\mathfrak{Q}_1 \times C \to {}^S\mathfrak{Q}_1 \times C\right)^* {}^S\!\mathcal{A}$$

we can repeat the reasoning we used in the degree zero case we define ${}^{\widetilde{S}}E$ to be the Hecke transform

$$0 \longrightarrow {}^{\widetilde{S}}E \longrightarrow \left(\widetilde{S} \times C \to S \times C\right)^* \left({}^S E\right) \longrightarrow \left({}^{\widetilde{S}}\mathfrak{Q}_1 \times C \to \widetilde{S} \times C\right)_* \left({}^{\widetilde{S}}\!\mathcal{A}\right) \longrightarrow 0.$$

By the calculation in the previous lemma we see that the fiber of the normal bundle to $\mathfrak{Q}_1$ at a point (E, θ) with $E = A^{-1}(\mathsf{p}) \oplus A$ are precisely the parametrizing the extension space

$$\mathrm{Ext}^1(A^{-1}(\mathsf{p}), A) = H^1(C, A^{\otimes 2}(-\mathsf{p})) = H^1(C, \mathcal{O}_C(-t')) = H^1(C, \mathcal{O}_C).$$

Suppose now that S is the spectrum of a DVR which maps to Higgs_1 so that the closed point $o \in S$ maps to $y = (E, \theta) \in \mathfrak{Q}_1$ with $E = A^{-1}(\mathsf{p}) \oplus A$, and such that the differential of $S \to \mathsf{Higgs}_1$ maps the tangent vector ∂_{x} to a normal vector $\mathfrak{e} \neq 0 \in H^1(C, \mathcal{O}_C(-t')) = H^1(C, \mathcal{O}_C)$. Again the map $S \to \mathsf{Higgs}_1$ is given by a relative Higgs bundle $({}^S E, {}^S\theta)$ on $S \times S$ for which the lower left corner of the matrix representing the Kodairra-Spencer class $\mathsf{ks}^{({}^S E)}(\partial_{\mathsf{x}})$ is equal to $\mathfrak{e}$. The deformation theory calculation in Lemma 3.7 now implies that the bundle ${}^{\widetilde{S}}E_{|\{[\mathfrak{e}]\}\times C}$ is the extension of $A^{-1}(\mathsf{p})$ by A given by the class $\mathfrak{e}$. This showas that for every $\mathfrak{e}$ the bundle ${}^{\widetilde{S}}E_{|\{[\mathfrak{e}]\}\times C}$ is stable and completes the proof of the Theorem 3.6(a) in the case of Higgs bundles with determinant $\mathbf{d} = \mathcal{O}_C(\mathsf{p})$. We have also proven Theorem 3.6(b) in this case. Indeed, in Lemma 3.9(b) we saw that $Y = Y_1$ is the blow-up of $\mathcal{P}_3$ centered at a copy of $\widehat{C} \subset \mathcal{P}_3$. Thus Y is a smooth projective threefold. Furthermore, Lemma 3.9(b) and the discussion that follows immediately the proof of Lemma 3.9 we see that the map $f = \mathfrak{f}_{|X} : Y \to X$ sends the exceptional divisor $\boldsymbol{E}_1$ onto the wobbly divisor $\mathrm{Wob}_1 \subset X_1$ and the map $f_1 : \boldsymbol{E}_1 \to \mathrm{Wob}_1$ factors as a composition $\boldsymbol{E}_1 = \widehat{C} \times \mathbb{P}^1 \to \overline{C} \times \mathbb{P}^1 \to \mathrm{Wob}_1$, where $\widehat{C} \times \mathbb{P}^1 \to \overline{C} \times \mathbb{P}^1$ is the natural double cover, and $\overline{C} \times \mathbb{P}^1 \to \mathrm{Wob}_1$ is the normalization map. In other words, in this case $\boldsymbol{E}_1$ and Wob_1 are irreducible and $f_{|\boldsymbol{E}_1} : \boldsymbol{E}_1 \to \mathrm{Wob}_1$ is a double cover. Similarly to the degree zero case this implies that $f : Y \to X$ is quasi-finite everywhere and hence is finite since it is a proper map. The theorem is proven. $\square$

3.8 The main construction

The description of the resolution in Theorem 3.6 shows in particular that the exceptional divisor of the blowup maps onto the wobbly locus in the moduli of bundles. This allows us to understand the relationship between the moduli of Higgs bundles and the cotangent bundle of the moduli of vector bundles not only over the very stable locus but also in codimension one in the stable locus, i.e. over the generic point of the wobbly divisor.

Let $\widetilde{\mathfrak{Q}} \subset \widetilde{\mathsf{Higgs}}$ denote the exceptional divisor, that is - the inverse image of $\mathfrak{Q}$. Recall that we have a natural morphism

$$\psi : T^\vee X \to \mathsf{Higgs}$$

inducing an isomorphism on dense open subsets.

Let $\mathrm{Wob}^{\mathrm{sing}}$ denote the singular locus of the wobbly divisor Wob, and set

$$X^\circ := X - \mathrm{Wob}^{\mathrm{sing}}, \quad \mathrm{Wob}^\circ := \mathrm{Wob} \cap X^\circ = \mathrm{Wob} - \mathrm{Wob}^{\mathrm{sing}}.$$

We note that $\mathrm{Wob}^\circ \subset X^\circ$ is now a smooth divisor.

Let $\widetilde{\mathsf{Higgs}}^\circ := \mathfrak{f}^{-1}(X^\circ) \subset \widetilde{\mathsf{Higgs}}$ and let $\widetilde{\mathfrak{Q}}^\circ$ be the intersection of $\widetilde{\mathfrak{Q}}$ with $\widetilde{\mathsf{Higgs}}^\circ$. With this notation we now have

Proposition 3.10. *Let $T^\vee X^\circ(\log \mathrm{Wob}^\circ)$ denote the total space of the locally free sheaf $\Omega^1_{X^\circ}(\log \mathrm{Wob}^\circ)$. There is a commutative diagram*

$$\begin{tikzcd} T^\vee X^\circ(\log \mathrm{Wob}^\circ) \arrow[rr, "\psi^{\log}"] \arrow[dr] & & \widetilde{\mathsf{Higgs}}^\circ \arrow[dl, "\mathfrak{f}"] \\ & X^\circ & \end{tikzcd}$$

such that the map $\psi^{\log}$ coincides with ψ over a dense open subset, and such that $\psi^{\log}$ maps the zero-section of $T^\vee X^\circ(\log \mathrm{Wob}^\circ)$ isomorphically to $\widetilde{\mathfrak{Q}}^\circ$.

Proof. This follows tautologically from the Hecke description of the families of Higgs bundles describing the map $\mathfrak{f}$ over the exceptional $\widetilde{\mathfrak{Q}}$ that we gave in the proof of Theorem 3.6. $\square$

The construction: Again we write Higgs for the moduli space od Higgs bundles of determinant $\mathbf{d} = \mathcal{O}_C(k \cdot \mathbf{p})$ where k isfixed to be either 0 or 1. Let $\boldsymbol{h} : \mathsf{Higgs} \to \mathcal{B}$ denote the corresponding Hitchin map and let $\mathsf{b} \in \mathcal{B}$ be a general point in the Hitchin base. Concretely, choose b so that the correspondng spectral cover $\pi : \widetilde{C} = \widetilde{C}_{\mathsf{b}} \to C$ is smooth and unramified above the Weierstrass points of C. Write $\mathcal{P} = \boldsymbol{h}^{-1}(\mathsf{b})$ for the corresponding fiber of the Hitchin map. Then $\mathcal{P}$ is smooth, and isomorphic to the Prym variety of the cover $\widetilde{C} \to C$. Let Y denote the pullback of $\mathcal{P}$ in the blown up moduli space $\widetilde{\mathsf{Higgs}}$. Then Y is a smooth projective threefold, which by Theorem 3.6 is obtained by blowing up $\mathcal{P}$ in the smooth subvariety $\mathcal{P} \cap \mathfrak{Q}$ which in the case of degree zero consists of 16 points - the 16 line bundles $\{\pi^*\kappa\}_{\kappa \in \mathsf{Spin}(C)}$ and in the case of degree one is a copy of the curve $\widehat{C}$. Furthermore from the proof of Theorem 3.6(b) we see that the restriction $f := \mathfrak{f}_{|Y} : Y \to X$ is a finite morphism which maps the exceptional divisor in Y onto a union of components of the wobbly divisor in X. In the degree zero case the exceptional divisor obtained by blowing up the point $\pi^*\kappa \in \mathcal{P}$ maps birationally onto the trope plane $\mathsf{Trope}_\kappa \subset \mathrm{Wob}_0 \subset X_0$ labeled by the theta characteristic κ. In the degree one case the exceptional divisor obtained by blowing up $\widehat{C} \subset \mathcal{P}$ maps two-to-one onto the wobbly divisor $\mathrm{Wob}_1 \subset X_1$. Recall also that as explained in the introduction, the finite morphism $f : Y \to X$ has degree $2^{3g(C)-3} = 8$.

Definition 3.11. *Let $\Omega^1_X(\log \mathrm{Wob})^+$ denote the unique reflexive sheaf on X whose restriction to X° is the vector bundle $\Omega^1_{X^\circ}(\log \mathrm{Wob}^\circ)$.*

If we let $j^\circ : X^\circ \hookrightarrow X$, then we can set more precisely

$$\Omega^1_X(\log \mathrm{Wob})^+ := \left(j^\circ_* \Omega^1_{X^\circ}(\log \mathrm{Wob}^\circ)\right)^{\vee\vee}.$$

Note that $\Omega^1_X(\log \mathrm{Wob})^+$ is locally free over X° and there it is equal to $\Omega^1_{X^\circ}(\log \mathrm{Wob}^\circ)$.

Let $\mathcal{L}$ be a line bundle on Y. Then set $\mathcal{V} := f_*(\mathcal{L})$. It is a rank 8 vector bundle on X. It has a meromorphic Higgs field coming from the fact that a dense open subset of Y may be identified with a subvariety of $T^\vee X$ via the inverse to ψ.

Corollary 3.12. *Under the main construction $\mathcal{V} = f_*(\mathcal{L})$ comes equipped with a meromorphic Higgs field which comes from a morphism*

$$\Phi : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X(\log \mathrm{Wob})^+.$$

Viewed as a sheaf with operators, $\mathcal{V}$ has no nontrivial Φ-invariant subsheaves.

Proof. Since $\Omega^1_X(\log \mathrm{Wob})^+$ is defined as a reflexive hull, it suffices to see this over X°, and there it comes from the commutative diagram of Proposition 3.10.

The spectral variety of Φ is birational to the covering $f : Y \to X$, and this is irreducible, so there can be no Φ-invariant subsheaves of $\mathcal{V}$, indeed such a subsheaf would correspond to a nontrivial decomposition of Y as a union of closed subsets. $\square$

3.9 Nonabelian Hodge outside codimension 2

In Corollary 3.12, we are not able to say that we construct a "logarithmic Higgs field" because the divisor Wob does not have normal crossings. The objective of this section is to investigate how the data of a logarithmic Higgs bundle defined outside of codimension 2 determines a local system by the nonabelian Hodge correspondence. We will also include parabolic structures.

In this section we consider a general variety X, a divisor W, and an open subset $X^\circ \subset X$ complement of a closed subset of codimension ≥ 2, such that $W^\circ := W \cap X^\circ$ is smooth.

Suppose we are given a reflexive sheaf $\mathcal{V}$ over X that is locally free over X°, and a parabolic structure for $\mathcal{V}$ along W°, denoted $\mathcal{V}_\bullet$. Suppose $\Phi : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X(\log W)^+$ is an operator with values in the sheaf of differentials that is the reflexive extension of $\Omega^1_{X^\circ}(\log W^\circ)$ as defined previously. We suppose that the residues of Φ along components of W°, acting on the parabolic associated graded pieces, are nilpotent.

Consider a projective morphism $g : Z \to X$ and let $Z^\circ := g^{-1}(X^\circ)$. Suppose $D \subset Z$ is a simple normal crossings divisor such that $D^\circ := D \cap Z^\circ$ contains $g^{-1}(W^\circ)$. The following statement is a recasting of Mochizuki's extension theorem [Moc07b] to our setting.

Theorem 3.13. *Suppose that $g : (Z, D) \to (X, W)$ exists as above, and suppose that there exists a semistable logarithmic parabolic Higgs sheaf $\mathcal{F}$ on (Z, D) whose restriction to Z° is isomorphic to $g^*((\mathcal{V}_\bullet, \Phi)|_{X^\circ})$. Suppose furthermore that $c_1^{\mathrm{par}}(\mathcal{F}) = 0$ and $c_2^{\mathrm{par}}(\mathcal{F}) = 0$. Then there is a mixed twistor $\mathscr{D}$-module $\mathbf{E}$ over X, with singularities along W, such that the restriction of $\mathbf{E}$ to X° corresponds to $(\mathcal{V}_\bullet, \Phi)|_{X^\circ}$. Furthermore, $\mathbf{E}$ is unique up to isomorphism, and the pullback $g^*(\mathbf{E})$ corresponds to $\mathcal{F}$.*

If $S \subset X^\circ$ is any projective curve then the restriction of $\mathbf{E}$ to S is the mixed twistor $\mathscr{D}$-module that corresponds to the logarithmic parabolic Higgs bundle on a curve $(\mathcal{V}_\bullet, \Phi)|_S$.

The mixed twistor $\mathscr{D}$-module $\mathbf{E}$ has Betti realization that restricts to a local system on $X - W$, and this local system has quasi-unitary monodromy transformations around the components of W° such that the arguments of eigenvalues of the monodromy are the parabolic weights and the unipotent part of the monodromy for each eigenvalue has the same Jordan form as the residue of Φ on the corresponding graded piece. This comes from the restriction to curves $S \subset X^\circ$.

Stated more compactly, the conclusion of the above theorem is that in order to construct such a local system on $X - W$ it suffices to know the parabolic structure and logarithmic

Higgs field over an open subset $X^\circ \subset X$ whose complement has codimension two, provided we can find some covering Z/X where the divisor has normal crossings and where there exists an extension with vanishing parabolic Chern classes.

This is what we will be doing in our concrete situations.

3.10 Parabolic structures

In this subsection we discuss some aspects of parabolic structures. We refer to the numerous available references for the general theory of parabolic bundles and their Chern classes, including for example [Bis97b, Bis97a, Bod91, BH95, Bor07, BV12, Kon93, MY92, IS08, IS07, Tah10, Tah13]. The present discussion will be tailored to our specific needs. Suppose (X, D) is a pair consisting of a smooth variety and a reduced divisor. Let $X^\circ \subset X$ be the complement of a subset of codimension ≥ 2, and let $D^\circ := D \cap X^\circ$ (with this notation extended in a similar way to other objects). We assume that D° is smooth. That is notably attained by taking X° to be the complement of the non-smooth locus of D, but one might want to throw out other subsets of codimension ≥ 2 as well.

Define the notion of ***quasi-parabolic bundle*** on (X, D) to consist of a reflexive sheaf $\mathcal{V}$ on X together with a filtration by strict subsheaves $0 = F_0 \subset F_1 \subset \cdots \subset F_k = \mathcal{V}|_{D^\circ}$ of the restriction of $\mathcal{V}$ to D°. A ***parabolic bundle*** is a quasi-parabolic bundle together with an assignment of real parabolic weights to the subquotients of the filtration over connected components of D°.

If D has simple normal crossings, we obtain by extension filtrations over each smooth irreducible component of D. Assuming D has simple normal crossings, a ***locally abelian parabolic bundle*** on (X, D) is a parabolic bundle such that $\mathcal{V}$ is locally free, the filtrations on divisor components have locally free subquotients, and locally at any point there exists a frame adapted to the filtrations of all divisor components passing through that point. If, in addition, the parabolic weights are rational, then these objects correspond to vector bundles on a root stack of (X, D) [Bis97b, Bod91, Bor07, BV12].

For the purposes of Theorem 3.13, we are interested in parabolic Chern numbers of the form $c_1^{\mathrm{par}} \cdot [\omega]^{n-1}$ and $c_2^{\mathrm{par}} \cdot [\omega]^{n-2}$, and for those it suffices to know the parabolic structure outside of a subset of codimension 3 in X. If D has normal crossings outside of codimension 3, any parabolic structure satisfies the locally abelian condition outside of codimension 3.

Our particular application is to the case where $D = \mathrm{Wob}$ is the wobbly divisor in $X = X_1$, and furthermore the parabolic weights are $-1/2$ and 0. We will be using a technique of pullback to a ramified covering to compensate for the fact that Wob has non-normal crossings singularities, namely cuspidal singularities as well as nodes up to X-codimension 2. The fact that there are only two distinct parabolic weights means that the filtration involves a single subsheaf.

In this setting, the notion of parabolic structure can be simplified further. Given (X, D) a ***crude quasi-parabolic structure*** consists of a reflexive sheaf $\mathcal{V}$ and a subsheaf $\mathcal{V}' \subset \mathcal{V}$ such that $\mathcal{V}$ is locally free on X°, $\mathcal{V}'$ is also reflexive, and $\mathcal{U} := \mathcal{V}/\mathcal{V}'$ is a locally free sheaf over W°. We let $r_{\mathcal{U}} := \mathrm{rank}(\mathcal{U})$ be the rank of this quotient.

A ***crude parabolic structure*** consists of a crude quasi-parabolic structure plus a ***parabolic weight*** which is a single number $\alpha \in (0, 1]$. To these data we associate a parabolic vector bundle on (X°, W°) whose weights are α for the associated-graded piece $\mathcal{U}$, and 0 for other the associated-graded piece $\mathcal{U}' := \mathcal{V}'/\mathcal{V}(-W)$.

The filtered sheaf $\mathcal{E}^\circ_\cdot$ on X° is given by

$$\mathcal{E}^\circ_a = \mathcal{V}', \;\; 0 \le a < \alpha, \quad \mathcal{E}^\circ_a = \mathcal{V}, \;\; \alpha \le a < 1.$$

Let $\Omega^1_X(\log D)^+$ be the reflexive extension to X of $\Omega^1_{X^\circ}(\log D^\circ)$ which is well-defined since we are assuming that D° is smooth. This isn't really the correct "sheaf of logarithmic differentials" as we will confront when dealing with the tacnodes of the wobbly locus in the degree 0 moduli space, but let us leave that discussion for later.

Say that a map $\Phi : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X(\log D)^+$ is a ***pre-logarithmic Higgs field*** if it satisfies $\Phi \wedge \Phi = 0$ generically. It is called ***nil-compatible*** with the parabolic structure if $\Phi(\mathcal{V}') \subset \mathcal{V}' \otimes \Omega^1_X(\log W)^+$ and if the residue of Φ along D° is nilpotent. This nilpotence is equivalent to demanding that the two associated-graded pieces of the residue, operating on $\mathcal{U}$ and $\mathcal{U}'$, are nilpotent.

Given such a logarithmic Higgs field, then on the parabolic structure associated to $(\mathcal{V}, \mathcal{V}', \alpha)$ we obtain a parabolic logarithmic Higgs field with nilpotent residues.

The following construction shows how to get a crude logarithmic parabolic structure from a spectral covering.

Theorem 3.14. *Suppose $f : Y \to X$ is a finite morphism, and suppose we are given a factorization through an inclusion $\psi : Y^\circ \to T^\vee X^\circ(\log D^\circ)$ over an open subset X° complement of a subset of codimension ≥ 2. Set $Y^\circ := f^{-1}(X^\circ)$ and so forth. Let $B \subset Y$ be a divisor such that $B^\circ = B \cap Y^\circ$ is smooth and $f(B) \subset D$. Assume that ψ° maps the reduced inverse image $f^{-1}(D^\circ)^{\mathrm{red}}$ to the zero-section of $T^\vee X^\circ(\log D^\circ)$. Let $\mathcal{L}$ be a line bundle on Y, and let*

$\mathcal{L}' := \mathcal{L}(-B)$. Then $\mathcal{V} := f_*(\mathcal{L})$ has a subsheaf $\mathcal{V}' := f_*(\mathcal{L})$. For any choice of parabolic weight α, and letting Φ be the pre-logarithmic Higgs field determined by ψ, we obtain a crude parabolic pre-logarithmic Higgs bundle $(\mathcal{V}, \mathcal{V}', \alpha, \Phi)$ on (X, D) with nilpotent residues.

Proof. Indeed, the quotient $\mathcal{U} := \mathcal{V}/\mathcal{V}'$ is isomorphic to $f_*(\mathcal{L}|_B)$ and over D° this is a locally free quotient of $\mathcal{V}|_D$. The fact that ψ maps Y° to $T^\vee X^\circ(\log D^\circ)$ implies that Φ is logarithmic over X°, for both $\mathcal{V}$ and its subsheaf $\mathcal{V}'$. On the reflexive extensions this yields a crude parabolic pre-logarithmic Higgs bundle. The condition on the zero-section insures that the residues of Φ are nilpotent. □

Terminology: In the situation of the theorem, we will call $\mathcal{L}'$ the ***spectral line bundle*** on Y, because in terms of the filtered sheaf we have

$$\mathcal{E}_0 = f_*(\mathcal{L}').$$

There are two main difficulties in the theoretical situation, both occasioned by the fact that the wobbly divisor $D = \mathrm{Wob}$ generally does not have normal-crossings singularities:

- How to calculate effectively the parabolic c_2 or the parabolic Bogomolov Δ-invariant?
- How to recognize if the Higgs field Φ will correspond to a genuinely logarithmic Higgs field on a normal crossings resolution?

Over the degree 0 moduli space X_0, the crude parabolic structure is going to be trivial. The divisor Wob then represents the location of singularities of the Higgs field. In that case, the strategy will be to follow the prescription in Theorem 3.13 to make a resolution of singularities of the divisor Wob into a normal crossings divisor, and look for a parabolic extension whose parabolic Chern class vanishes. In this case, the parabolic structures will have more than two jumps so one needs to consider filtrations of bigger length.

For the degree 1 moduli space X_1, we will be using a crude parabolic structure with parabolic weight $\alpha = 1/2$. This was found partly by computation and partly by guessing. Our strategy for the degree 1 case consists of passing to a ramified covering in order to remove the singularities of $D = \mathrm{Wob}$ or transform them to normal crossings. Although the pullback of a parabolic bundle with arbitrary weight under a ramified covering may be difficult to compute, in the case $\alpha = 1/2$ we can give a description.

Proposition 3.15. *Suppose $\underline{\mathcal{V}} = (\mathcal{V}, \mathcal{V}', \alpha, \Phi)$ is a crude parabolic pre-logarithmic Higgs bundle on (X, D) with parabolic weight $\alpha = 1/2$. Suppose $g : Z \to X$ is a Galois covering* *from a smooth variety, such that g is ramified of order 2 along $B^1 = g^{-1}(D^\circ)^{\mathrm{red}}$ over D°. Then the pullback $g^*(\underline{\mathcal{V}})$ has trivial parabolic structure, so it corresponds to a reflexive sheaf $\mathcal{V}_Z$ with logarithmic Higgs field that is described in the following way. Let $\mathcal{U} := \mathcal{V}/\mathcal{V}'$, let $\mathcal{U}_Z := g^*(\mathcal{U})$, and use a superscript $()^\circ$ to denote the restriction over X°. Let $\mathcal{U}_{B^\circ}$ be the restriction of $\mathcal{U}^\circ$ to B° seen as a coherent sheaf on Z°, also equal to $\mathcal{U}_Z^\circ/\mathcal{I}_{B^\circ \subset Z^\circ}\mathcal{U}_Z^\circ$. We have an exact sequence*

$$0 \to \mathcal{V}_Z^\circ \to g^*\mathcal{V}^\circ \to \mathcal{U}_{B^\circ} \to 0$$

and $\mathcal{V}_Z$ is the reflexive extension of $\mathcal{V}_Z^\circ$ from Z° to Z. The logarithmic Higgs field Φ is induced from that of $g^*(\mathcal{V})$.

Proof. Follows immediately from the definition of a pullback of parabolic structures. □

3.11 Pushforward statement

In order to calculate the Hecke transforms of the local systems that we are going to construct, we need a Dolbeault method for calculating higher direct images. This is provided by the theory of [DPS16], which is based in turn on the theory of twistor $\mathscr{D}$-modules of Sabbah and Mochizuki [Moc07a, Moc07b, Sab05].

We will need to extend the discussion of [DPS16] in order to apply it to the specific situations encountered for the Hecke transform. Luckily, the fibers of the Hecke correspondence are 1-dimensional, since we are dealing with moduli spaces of rank 2 bundles in this paper. In order to identify the higher direct image local systems, it suffices to restrict to curves in the target space, and to make things easy we can even use lines. Thus, the direct image calculations are for maps from a surface to a curve, the main setup of [DPS16]. The present situation differs in that the horizontal divisor typically has simple ramification points, whereas the hypothesis of [DPS16] was to have an etale horizontal divisor. Thus, some work needs to be done in the direction of looking at a ramified cover of the base.

The full discussion will be deferred to Chapter 12. In this subsection we summarize the basic knowledge to come out of that, as needed to treat the Hecke transforms. The setup we discuss here is therefore closely tailored to the situations that are encountered in the applications.

Suppose $f : X \to S$ is a projective morphism from a smooth surface to a smooth curve. Suppose $D \subset X$ is a reduced divisor, each of whose components dominates S. Classify the points of (X, D) among the following kinds:

3.11.1. Types of points of (X, D)

(a) Points in $X - D$ where f is smooth;

(b) Points in $X - D$ where the fiber of f has a simple normal crossing;

(c) Points on D where D is etale over S;

(d) Points on D where f is smooth and D is smooth with simple ramification over S;

(e) Points on D where f is smooth and D has a normal crossing such that both branches are etale over S;

(f) Other points.

Suppose we are given a normal variety Σ that we will call (under a small abuse of notation) the ***spectral variety***, with a map $\varpi : \Sigma \to X$, together with a line bundle $\mathcal{L}$ that we will call the ***spectral line bundle*** and a 1-form α on the smooth loculs of Σ that we will call the ***tautological form***.

Set $E_0 := \varpi_*(\mathcal{L})$. We assume that α leads to a map $\Sigma \to T^\vee(X, \log D)$, implying that tautological form provides E_0 with a Higgs field $\theta : E_0 \to E_0 \otimes \Omega^1_X(\log D)$.

Notice here that the image of Σ in the logarithmic cotangent bundle might not be normal, but we look at the normalization Σ and call that the spectral variety, whence the abuse of notation mentioned above.

We will consider two cases: the ***nilpotent case***, where the parabolic structure is trivial and the Higgs field has a nilpotent residue along D coming from the ramification of Σ; and the ***parabolic case*** where the parabolic weights are $0, 1/2$. In this case, we suppose $E_{1/2} = \varpi_*(\mathcal{L}(R))$ where $R \subset \Sigma$ is the part of the ramification divisor of $\Sigma \to X$ that sits over D. Note that Σ will in general have other ramification that is not included in R, the hypothesis is that all the ramification over D contributes to the parabolic structure.

The Higgs field is given by the spectral 1-form $\alpha \in H^0(\Sigma, \Omega^1_\Sigma)$ We assume that α is nonzero on normal vectors to R at general points. This implies, in the nilpotent case, that the nilpotent residues of the Higgs field are nonzero, giving a size 2 Jordan block at each point of R over a general point of D.

Define the ***upper critical locus***

$$\widetilde{\mathrm{Crit}}(X/S, E_\bullet, \theta) \subset \Sigma$$

to be the closure of the zero-locus of the projection of α to a section of relative differentials on X/S, from the open subset where Σ is smooth. The ***lower critical locus*** $\mathrm{Crit}(X/S, E_\bullet, \theta)$ is its image in X.

Let G denote the normalization of $\widetilde{\mathrm{Crit}}(X/S, E_\bullet, \theta)$, and let $g : G \to S$ be the induced morphism.

Hypothesis 3.16. We make the following hypotheses, in addition to the basic assumptions above.

- The lower critical locus $\mathrm{Crit}(X/S, E_\bullet, \theta)$ does not contain any points of type (3.11.1(f));
- The spectral variety Σ is smooth, except that each point above a point of type (3.11.1(e)) is an ordinary double point;
- If there are points of type (3.11.1(e)) then we are in the nilpotent case;
- The upper critical locus is smooth except over points of type (3.11.1(e)), in particular its normalization G is a smooth curve.
- The restriction of the spectral 1-form α to the vertical direction in Σ over a point of type (3.11.1(d)) is nonzero.
- At each ordinary double point over a point of type (3.11.1(e)), the curve G has two smooth branches whose tangent vectors are distinct.

Let $Q := (G \cap R)$ be the divisor in G that is the intersection with R. The hypothesis that α is nonzero in the vertical direction at points of type (3.11.1(d)) implies that G is transverse to R at those points, so Q is reduced at such points (see Lemma 12.9).

In practice, G will only meet R over points of type (3.11.1(d)) or (3.11.1(e)): in the nilpotent case that comes from constancy of the Jordan form of the residue of the Higgs field along D, while in the parabolic case it is a property of our setup, see Lemma 9.6. We can however state the pushforward property without including such an hypothesis.

Theorem 3.17. *Assume the basic hypotheses explained at the start, assume that we are either in the parabolic or nilpotent cases as described above, and assume the hypotheses 3.16. Then, the parabolic Higgs bundle $(F_\bullet, \Phi)$ on S corresponding to the middle direct image of the perverse sheaf associated to $(E_\bullet, \theta)$ has the following descriptions depending on the case.*
—In the parabolic case, $F_\bullet$ has trivial parabolic structure and

$$F_0 = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \otimes \mathcal{O}_G(Q) \right).$$

—In the nilpotent case, $F_\bullet$ has a parabolic structure with weights $0, 1/2$ and contains a parabolic subsheaf $F'_\bullet \hookrightarrow F_\bullet$, that being an isomorphism away from the images of points of type (3.11.1(e)), with

$$F'_0 = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \right)$$

and

$$F'_{-1/2} = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \otimes \mathcal{O}_G(-Q) \right).$$

In particular, if $F'_\bullet$ has parabolic degree 0 then $F'_\bullet = F_\bullet$ and these give expressions for $F_\bullet$. In both cases, the Higgs field on F_0 comes from the differential form on G obtained by restricting the tautological differential α from Σ.

3.12 Chern classes

For reference below, we recall here the required facts about Chern classes. We'll almost always be interested in what happens up to codimension two, so by convention—unless otherwise specified—our formulas will be truncated at codimension two. Thus for example we write

$$\mathrm{ch}(E) = r(E) + c_1(E) + (c_1(E)^2/2 - c_2(E).$$

In the other direction,

$$c_2 = c_1^2/2 - \mathrm{ch}_2.$$

An important invariant is Bogomolov's discriminant [Bog78, Bog94]

$$\Delta = \frac{1}{2r} c_1^2 - ch_2 = c_2 - \frac{r-1}{2r} c_1^2,$$

having the property that $\Delta(E \otimes L) = \Delta(E)$ for a line bundle E.

The same hold for parabolic Chern classes.

Suppose $\pi : Y \to X$ is a map. Then the Grothendieck-Riemann-Roch formula says that for a coherent sheaf L on Y we have

$$\mathrm{ch}(\pi_*(L)) = \pi_*(\mathrm{td}(Y/X) \cdot \mathrm{ch}(L)),$$

where the relative Todd class is

$$\mathrm{td}(Y/X) = \mathrm{td}(TY)\pi^*\mathrm{td}(TX)^{-1}$$

with

$$\mathrm{td}(TY) = 1 + \frac{c_1(TY)}{2} + \frac{c_1(TY)^2 + c_2(TY)}{12}$$

and similarly for $\mathrm{td}(TX)$.

We will specialize to the main cases that arise in our study: projective space, the intersection of two quadrics in $\mathbb{P}^5$, and blow-ups of abelian varieties.

Projective spaces: Let us consider the case of projective space. Let H denote the hyperplane class on $\mathbb{P}^n$, and as stated above by convention we truncate to codimension 2. The Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T\mathbb{P}^n \to 0$$

gives

$$\mathrm{ch}(T\mathbb{P}^n) = n + (n+1)H + \frac{n+1}{2}H^2$$

so

$$c_1(T\mathbb{P}^n) = (n+1)H, \quad c_2(T\mathbb{P}^n) = \frac{n^2+n}{2}H^2.$$

Thus

$$\begin{aligned}\mathrm{td}(T\mathbb{P}^n) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\ &= 1 + \frac{n+1}{2}H + \frac{3n^2+5n+2}{24}H^2.\end{aligned}$$

For $X = \mathbb{P}^3$ this gives

$$\mathrm{td}(TX) = 1 + 2H + \frac{11}{6}H^2.$$

Intersection of quadrics: Suppose $X \subset \mathbb{P}^5$ a complete intersection of two quadrics. Let H also denote the restriction of the hyperplane class of $\mathbb{P}^5$ to X. The normal bundle of X is $N_{X/\mathbb{P}^5} \cong \mathcal{O}_X(2)^2$ so the exact sequence

$$0 \to TX \to T\mathbb{P}^5|_X \to N_{X/\mathbb{P}^5} \to 0$$

gives

$$\begin{aligned}\mathrm{ch}(TX) &= \mathrm{ch}(T\mathbb{P}^5) - 2\exp(2H)\\ &= 5 + 6H + 3H^2 - 2 - 4H - 4H^2\\ &= 3 + 2H - H^2.\end{aligned}$$

We get

$$c_1(TX) = 2H, \quad c_2(TX) = 3H^2.$$

Thus,

$$\begin{aligned}\mathrm{td}(TX) &= 1 + c_1/2 + (c_1^2 + c_2)/12\\ &= 1 + H + 7H^2/12.\end{aligned}$$

Blow-ups of Pryms: Suppose $\mathcal{P}$ is a 3-dimensional abelian variety, so its tangent sheaf has trivial Chern classes, and we let $\boldsymbol{\varepsilon} : Y \to \mathcal{P}$ be the blow-up along a smooth subvariety $A \subset \mathcal{P}$. Let $E \subset Y$ denote the exceptional divisor. There are two cases of interest.

Suppose first A is a disjoint collection of a points. Then E is a disjoint collection of a planes $\mathbb{P}^2$. A point $y \in E$ corresponds to a tangent direction at the corresponding image point $\boldsymbol{\varepsilon}(y) \in \mathcal{P}$, and the tangent vectors on Y at y are vectors that are constrained to map into this tangent direction in $T_{\boldsymbol{\varepsilon}(y)}\mathcal{P}$. Over E there is the universal subbundle $\mathcal{O}_E(-1) = \mathcal{O}_E(E)$, fitting into an exact sequence

$$0 \to \mathcal{O}_E(E) \to \boldsymbol{\varepsilon}^* T\mathcal{P}|_E \to R \to 0$$

where the middle is a trivial bundle. Then, letting R_Y be R viewed as a coherent sheaf on Y supported on E, we then have an exact sequence

$$0 \to TY \to \boldsymbol{\varepsilon}^* T\mathcal{P} \to R_Y \to 0.$$

We have

$$\mathrm{ch}(R_Y) = \mathrm{ch}(\boldsymbol{\varepsilon}^* T\mathcal{P})\mathrm{ch}(\mathcal{O}_E) - \mathrm{ch}(\mathcal{O}_Y(E))\mathrm{ch}(\mathcal{O}_E).$$

Now $\mathrm{ch}(\boldsymbol{\varepsilon}^* T\mathcal{P}) = 3$ since the tangent sheaf of $\mathcal{P}$ is trivial of rank 3. The exact sequence

$$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$$

gives

$$\mathrm{ch}(\mathcal{O}_E) = E - E^2/2,$$

so we get

$$\begin{aligned}\operatorname{ch}(R_Y) &= (3-(1+E+E^2/2))(E-E^2/2)\\ &= 2E-2E^2.\end{aligned}$$

Putting this into the other exact sequence we get

$$\operatorname{ch}(TY) = 3-2E+2E^2.$$

We note that the shape of this formula is independent of the number a of points that were blown up (but of course the full exceptional divisor E contains that information).

Turn now to the other case: when A is a smooth irreducible curve. The normal bundle $N_{A/\mathcal{P}}$ is a rank 2 vector bundle over A. Let δ denote the degree of the normal bundle, i.e. the degree of its determinant line bundle. A point $y \in E$ corresponds to a normal direction to A at $b(y)$, so as before the tangent vectors to Y at y are constrained to have image in $T_{b(y)}\mathcal{P}$ that maps into this normal direction under the map $T\mathcal{P}|_A \to N_{A/\mathcal{P}}$. Let $\mathcal{O}_E(-1)$ denote the tautological sub-bundle over E. It fits into an exact sequence

$$0 \to \mathcal{O}_E(-1) \to \boldsymbol{\varepsilon}^* N_{A/\mathcal{P}} \to R \to 0$$

and as before, if R_Y denotes the corresponding coherent sheaf on Y supported over E, then we have the exact sequence

$$0 \to TY \to \boldsymbol{\varepsilon}^* T\mathcal{P} \to R_Y \to 0.$$

The tautological subbundle is also the normal bundle to E in Y, thus $\mathcal{O}_E(-1) = \mathcal{O}_E(E)$ as in the previous case.

The rank 2 bundle $N_{A/\mathcal{P}}$ over the curve A is determined, up to rational numerical equivalence, by its degree that we'll call δ.

Suppose G is an ample divisor class on $\mathcal{P}$, then $G \cdot A$ is the degree of G restricted to A. Working numerically, we can write

$$\det N_{A/\mathcal{P}} \sim \frac{\delta}{G \cdot A} G|_A$$

so $\boldsymbol{\varepsilon}^* N_{A/\mathcal{P}}$, considered as a coherent sheaf on Y, is rationally numerically equivalent to the expression (with a fractional divisor) $(\boldsymbol{\varepsilon}^* \mathcal{O}_Y(\frac{\delta}{G \cdot A}G) \oplus \mathcal{O}_Y) \otimes \mathcal{O}_E$. This gives

$$\operatorname{ch}(\boldsymbol{\varepsilon}^* N_{A/\mathcal{P}}) = \left(2 + \frac{\delta}{G \cdot A} G\right) \operatorname{ch}(\mathcal{O}_E).$$

Here we can truncate the expression on the left at codimension 1 because it is multiplied by $\mathrm{ch}(\mathcal{O}_E)$ of rank 0.

Recall as above that $\mathrm{ch}(\mathcal{O}_E) = E - E^2/2$. Similarly $\mathrm{ch}(\mathcal{O}_E(E)) = E + E^2/2$ and from the exact sequence,

$$\begin{aligned}
\mathrm{ch}(R_Y) &= \left(2 + \frac{\delta}{G \cdot A} G\right) \cdot (E - E^2/2) - E - E^2/2 \\
&= E - \frac{3}{2}E^2 + \frac{\delta}{G \cdot A} G \cdot E.
\end{aligned}$$

From the other exact sequence and recalling that $\boldsymbol{\varepsilon}^* T\mathcal{P}$ is trivial of rank 3 we get

$$\mathrm{ch}(TY) = 3 - E + \frac{3}{2}E^2 - \frac{\delta}{G \cdot A} G \cdot E.$$

The last term could perhaps more easily be understood as a sum of fibers: let fib denote the numerical class of a fiber of the projection $E \to A$, viewed as a codimension 2 class on Y. We have (writing also G for its pullback to Y)

$$\mathsf{fib} = \frac{1}{G \cdot A} G \cdot E$$

so we can also write

$$\mathrm{ch}(TY) = 3 - E + \frac{3}{2}E^2 - \delta \mathsf{fib}.$$

we get

$$\begin{aligned}
c_1(TY) &= -E, \\
c_2(TY) &= \delta \mathsf{fib} - E^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{td}(TY) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\
&= 1 - E/2 + \delta \mathrm{fib}/12.
\end{aligned}$$

4 The degree one moduli space

For odd degree we have fixed the line bundle $\mathcal{O}_C(\mathbf{p})$ to be the determinant, for the chosen Weierstrass point $\mathbf{p} \in C$. For purposes of the present section, $X := X_1$ is the moduli space of stable bundles E with $\det(E) \cong \mathcal{O}_C(\mathbf{p})$. Similarly, $\mathrm{Wob} := \mathrm{Wob}_1$ denotes the wobbly locus. Recall that Higgs_1 denotes the moduli space of Higgs bundles with determinant $\mathcal{O}_C(\mathbf{p})$ and trace of the Higgs field equal to zero.

Recall from Proposition 3.1 that the $2:1$ spectral covering $\pi : \widetilde{C} \to C$ has four branch points $x, x', y, y' \in C$ such that x and x' are opposite under the hyperelliptic involution, and

y and y' are opposite under the hyperelliptic involution. For a general point in the Hitchin base, the resulting two points of the hyperelliptic $\mathbb{P}^1$ are general with respect to the choice of C.

In this chapter, various notations such as X, Y, Wob etc. are used without the subscript 1 since we are talking about the degree 1 moduli space. For insertion into the rest of the paper one should read X_1, Y_1, Wob_1 and so on.

4.1 Geometry of the wobbly locus in degree one

We use the notations of subsections 3.1 and 3.7. In particular $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}}$ denotes the fixed point locus of the $\mathbb{C}^\times$-action on Higgs_1 and $\mathcal{P}^{\mathrm{unss}}$ denotes the unstable locus in $\mathcal{P}$ which by Theorem 3.6 coincides with the intersection of $\mathfrak{Q} \cap \mathcal{P}$ of $\mathcal{P}$ with the incoming variety $\mathfrak{Q}$ defined in subsection 3.7. The geometric description of these loci was already given in subsections 3.1 and 3.7 but we record it again in the following Lemma for ease of reference.

Lemma 4.1.

- $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}}$ *is equal to the curve* $\overline{C} \subset \mathsf{Higgs}_1$, *the* 16*-sheeted etale covering of* C *defined in subsection 3.1.*
- *If* $\mathcal{P}$ *is a generic Hitchin fiber corresponding to spectral curve* $\widetilde{C}$, *then* $\mathcal{P}^{\mathrm{unss}} = \mathfrak{Q} \cap \mathcal{P} \subset \mathcal{P}$ *is a smooth curve* $\widehat{C}$ *that can be expressed as a fiber product*

$$\widehat{C} = \widetilde{C} \times_C \overline{C}.$$

In particular, $\widehat{C}$ *is a* 16*-sheeted étale covering of* $\widetilde{C}$.

These curves have genera respectively

$$g_C = 2, \quad g_{\widetilde{C}} = 5, \quad g_{\overline{C}} = 17, \quad g_{\widehat{C}} = 65.$$

Proof. The spectral curve $\widetilde{C}$ has genus 5 and is a smooth double cover $\widetilde{C} \to C$ ramified over 4 points in two conjugate pairs. This follows from our running genericity assumption requiring that $\widetilde{C}$ is not ramified over any of the Weierstrass points of C. The Hitchin fiber $\mathcal{P} \subset \mathsf{Higgs}_1$ corresponding to such $\widetilde{C}$ is the Prym variety $\mathcal{P} = \mathcal{P}_3$ of line bundles on $\widetilde{C}$ whose direct image down to C is a rank 2 bundle of determinant $\mathcal{O}_C(\mathbf{p})$.

The statement that $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}} = \overline{C}$ was already proven in Proposition 3.3 but we recall the argument here for completeness. A fixed point that is in $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}}$ (i.e. such that the underlying vector bundle is not stable) is a Higgs bundle of the form $(A \oplus A^\vee(\mathbf{p}), \theta)$, such that $\theta : A^\vee(\mathbf{p}) \to A \otimes \omega_C$. Stability of this Higgs bundle means $\deg(A) \leq 0$ and existence of the map implies that $\deg(A) = 0$. As $\omega_C = \mathcal{O}_C(2\mathbf{p})$ this means that $A^{\otimes 2} \cong \mathcal{O}_C(t - \mathbf{p})$ for the point $t \in C$ where θ vanishes. This description provides an isomorphism $\mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}} \cong \overline{C}$.

The incoming variety $\mathfrak{Q} \subset \mathsf{Higgs}_1$ consists of Higgs bundles (E, θ) such that $\lim_{t\to 0}(E, t\theta) \in \mathsf{Higgs}_1^{\mathbb{C}^\times,\mathrm{nu}} \cong \overline{C}$. In subsection 3.6 we saw that the limiting map $\mathfrak{Q} \to \overline{C}$ is well defined and has fibers isomorphic to $\mathbb{A}^3$. We claim that

$$\mathcal{P}^{\mathrm{unss}} = \mathfrak{Q} \cap \mathcal{P}$$

has the structure

$$\widehat{C} = \widetilde{C} \times_C \overline{C}.$$

In particular, it is seen to be a genus 65 curve.

For a general Hitchin fiber the intersection $\mathcal{P}^{\mathrm{unss}}$ is smooth of dimension 1. A point of $\mathcal{P}^{\mathrm{unss}}$ is by definition a line bundle U of degree 3 on $\widetilde{C}$ such that π_*U has determinant $\mathcal{O}_C(\mathbf{p})$ but is unstable. Instability means that there is a sub-line bundle of degree 1 that we will denote $A^\vee(\mathbf{p}) \hookrightarrow \pi_*U$. By adjunction we have $\pi^*(A^\vee(\mathbf{p})) \hookrightarrow U$, and $\pi^*(A^\vee(\mathbf{p}))$ has degree 2. Thus, there is a unique point $\tilde{t} \in \widetilde{C}$ such that $U = \pi^*(A^\vee(\mathbf{p}))(\tilde{t})$. If t denotes the image of $\tilde{t}$ in C, the determinant condition says that

$$A^{\otimes -2}(2\mathbf{p} + t) \otimes \omega_C^{-1} \cong \mathcal{O}_C(\mathbf{p}).$$

Thus, A solves $A^{\otimes 2} \cong \mathcal{O}_C(t - \mathbf{p})$ and (A, t) is the corresponding point of $\overline{C}$. This identifies $\mathcal{P}^{\mathrm{unss}}$ with the image of $\widehat{C} = \widetilde{C} \times_C \overline{C}$ under the map (4). But by Lemma 3.2 this map is a closed embedding, and so we have $\mathcal{P}^{\mathrm{unss}} = \widehat{C}$. $\square$

Remark 4.2. One can notice that the genera of the curves involved here are of the form $4^k + 1$. We do not know if that is significant or not.

Lemma 4.3. *Let D denote the $\mathbb{P}^1$-bundle over $\overline{C}$ projectivization of the bundle of outgoing directions. Then the map sending a point on $\overline{C}$ together with an outgoing direction, to the limit in X of the resulting $\mathbb{C}^\times$-orbit, provides a map $D \to \mathrm{Wob}$ that is finite and generically injective. Thus, D is the normalization of* Wob.

Proof. Given a $\mathbb{C}^\times$-fixed Higgs bundle $(A \oplus A^\vee(\mathbf{p}), \theta)$ corresponding to $(A, t) \in \overline{C}$, the space of outgoing directions is $\mathbb{P}\mathrm{Ext}^1(A^\vee(\mathbf{p}), A)$ which is by definition the fiber of D over (A, t).

As we saw in Lemma 3.9**(b)** and we will see again by a different argument in Proposition 4.5 below, $D \cong \overline{C} \times \mathbb{P}^1$. Each $\mathbb{P}^1$ fiber of $D \to \overline{C}$ maps to the space of extensions which is the general form of a line in $X = X_1$. Both the lines and the horizontal $\overline{C}$'s map to positive degree curves, and an effective divisor on the product D is a positive sum of vertical and horizontal ones, so any effective divisor on D maps to a positive degree curve in X, in particular it is not contracted. For generic injectivity, we note that a general wobbly bundle E has a one dimensional space of nilpotent Higgs fields and if we θ is any non-zero nilpotent Higgs field, we get a well defined line subbundle $A = \ker \theta E$ and a realization of E as an extension of $A^\vee(\mathbf{p})$ by A. This determines a unque point in D which shows that the map $D \to \mathrm{Wob}$ has a rational section. Since D is irreducible this implies that $D \to \mathrm{Wob}$ is a birational morphism. Alternatively, we can see the generic injectivity from the geometry of the quadric line complex elucidated in section 2.5. Recall that at a general point w of Wob the four lines in X through that point include one with multiplicity 2. That line corresponds to the point $A \in \overline{C} \subset \mathrm{Jac}^0(C)$, and the position of w on it corresponds to the extension class, so we can recover the data of a point of D. It follows that D is the normalization of Wob. ∎

Lemma 4.4. *The divisor class of the wobbly locus is* $[\mathrm{Wob}] = 8H$.

Proof. We will consider lines $\ell \subset X$ and show that ℓ intersects W in 8 points. In the moduli point of view on X a line corresponds to fixing a line bundle L of degree 0 and looking at the set of bundles E fitting into an extension

$$0 \to L \to E \to L^\vee(\mathbf{p}) \to 0.$$

Such an extension is wobbly if there is $A \to E$ and $t \in C$ with $A^{\otimes 2} = \mathcal{O}(t - \mathbf{p})$. Assuming that L does not satisfy this condition, it means that $A = L^\vee(\mathbf{p} - y)$, so $L^{\otimes 2}(2y - 2\mathbf{p}) = \mathcal{O}(\mathbf{p} - t)$ in other words $L^{\otimes 2} = \mathcal{O}(3\mathbf{p} - t - 2y)$, and the pullback of the extension to A splits. For a choice of (y, t) there will be a unique extension up to scalars, i.e. a unique point on the line, such that the pullback extension splits, so we need to look at the number of solutions of the equation $L^{\otimes -2}(3\mathbf{p}) = \mathcal{O}(2y + t)$. This is the number of branch points of a trigonal map $C \to \mathbb{P}^1$, which is 8. Thus $\ell \cap \mathrm{Wob}$ has 8 points for a general line ℓ. We will discuss the trigonal covers more in subsection 4.5 below. ∎

The basic structure of Wob is captured by the following proposition.

Proposition 4.5. *We have $D \cong \overline{C} \times \mathbb{P}^1$ and the factor $\mathbb{P}^1$ is naturally identified with the base of the hyperelliptic covering $\mathbb{P}^1 = C/\boldsymbol{\iota}_C$. The map $D \to$ Wob identifies together pairs of points on 6 curves of the form $\overline{C} \times \{\mathsf{x}_i\}$, yielding a nodal locus in* Wob*. Furthermore, on the graph of the natural projection $\overline{C} \to \mathbb{P}^1$ the map $D \to X$ is non-immersive, yielding a cuspidal locus in* Wob*.*

Proof. Recall that $\mathsf{x}_1, \dots, \mathsf{x}_6 \in \mathbb{P}^1$ denoted the branch points for the hyperelliptic map $\mathsf{h}_C : C \to \mathbb{P}^1$, i.e. the images of the Weierstrass points $\mathsf{p}_1, \dots, \mathsf{p}_6 \in C$; also we chose $\mathsf{p} := \mathsf{p}_1$ to fix the determinant $\det(E) \cong \mathcal{O}_C(\mathsf{p})$ of our stable vector bundles.

The statement that the $\mathbb{P}^1$-fiber of D is naturally identified with the hyperelliptic $\mathbb{P}^1$ was already checked in Lemma 3.9**(b)** but we recall the argument here for convenience. A point in $\overline{C}$ is a line bundle L such that $L^{\otimes 2}(\mathsf{p})$ is effective, or equivalently such that $L^{\otimes 2} \cong \mathcal{O}_C(t - \mathsf{p})$ for some point $t \in C$. Thus L defines uniquely the point t.

A point in D is a pair (L, η) where $L \in \overline{C}$ and $\eta \in \mathbb{P}\operatorname{Ext}^1(L^\vee(\mathsf{p}), L)$, defining the extension

$$0 \to L \to E \to L^\vee(\mathsf{p}) \to 0.$$

We note that

$$\operatorname{Ext}^1(L^\vee(\mathsf{p}), L) = H^1(L^{\otimes 2}(-\mathsf{p})) \cong H^1(\mathcal{O}_C(t - 2\mathsf{p})) \cong H^1(\mathcal{O}_C(-t'))$$

where t' is the image of t under the hyperelliptic involution. The map $\mathcal{O}_C(-t') \to \mathcal{O}_C$ induces an isomorphism on H^1 so we can write

$$\operatorname{Ext}^1(L^\vee(\mathsf{p}), L) \cong H^1(\mathcal{O}_C).$$

The scalar multiplying this isomorphism is not canonical, it depends on something and indeed that leads (as we have discussed) to the degree 128 of the normal bundle of $\widehat{C}$ in $\mathcal{P}$. However, it does give a canonical isomorphism

$$\mathbb{P}\operatorname{Ext}^1(L^\vee(\mathsf{p}), L) \cong \mathbb{P}H^1(\mathcal{O}_C) = \mathbb{P}^1.$$

We note that if $y \in C$ then the map

$$H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(y))$$

is a quotient of rank 1 corresponding to the inage $\mathsf{h}_C(y) \in \mathbb{P}^1$ of y. This is the identification between the $\mathbb{P}^1$ factor of D, and the base of the hyperelliptic projection of C.

We would like to understand when two points of D correspond to the same bundle. Suppose given $(L, \eta) \in D$, corresponding to an extension as above. We look for a new line subbundle $L_1 \subset E$ such that $L_1 \in \overline{C}$. We note first of all that if $L \neq L_1$ as subbundles in E, then there is an inclusion

$$L_1 \hookrightarrow L^\vee(\mathbf{p}),$$

so we may write

$$L_1 = L^\vee(\mathbf{p} - y).$$

We have

$$L_1^{\otimes 2} = L^{\otimes -2}(2\mathbf{p} - 2y) = \mathcal{O}_C(\mathbf{p} - t)(2\mathbf{p} - 2y) = \mathcal{O}_C(3\mathbf{p} - t - 2y)$$

so the condition $L_1 \in \overline{C}$ becomes

$$\mathcal{O}_C(3\mathbf{p} - t - 2y) \cong \mathcal{O}_C(t_1 - \mathbf{p})$$

for some point $t_1 \in C$. In other words,

$$4\mathbf{p} \sim t + t_1 + 2y.$$

One may notice that for our hyperelliptic curve, any time that $a+b+c+d \sim 2\omega_C = 4\mathbf{p}$, we have $a+b+c+d = x+x'+y+y'$ such that x, x' (respectively y, y') are paired under the hyperelliptic involution $\boldsymbol{\iota}_C$.

Given the form $t + t_1 + 2y$, there are only two possibilities:

(i) either $t = t_1$, and t is $\boldsymbol{\iota}_C$-paired with y,

(ii) or (t, t_1) are $\boldsymbol{\iota}_C$-paired, and $y = \mathbf{p}_i$ is a Weierstrass point on C.

Case (i): If we are in the first case we get

$$L_1 = L^\vee(\mathbf{p} - y) = L^\vee(\mathbf{p} - (2\mathbf{p} - t)) = L^\vee(t - \mathbf{p})$$

and so we must have $L_1 \cong L$, that is L and L_1 correspond to two different embedings of the same line bundle in E. However, for $L_1 \cong L$ the space $\mathrm{Ext}^1(L_1, L)$ of extensions of L_1 by L has dimension 2 (it is $H^1(\mathcal{O}_C)$. Furthermore, the map on extension sapces

$$\mathrm{Ext}^1(L^\vee(\mathbf{p}), L) \to \mathrm{Ext}^1(L_1, L)$$

induced by the inclusion $L_1 \hookrightarrow L^\vee(\mathbf{p})$, is identified with the map on cohomology

$$H^1(L^{\otimes 2}(-\mathbf{p})) = H^1(\mathcal{O}_C(t - 2\mathbf{p})) = H^1(\mathcal{O}_C(-t')) \to H^1(\mathcal{O}_C)$$

induced by the inclusion $\mathcal{O}_C(-t') \hookrightarrow \mathcal{O}_C$, which is an isomorphism. Hence if $L_1 \cong L$, then no choice of bundle E admits a lifting from L_1 into E. Therefore, case (i) does not lead to any identification.

As we will see below, it will however lead to an identification infinitesimally, yielding the cuspidal locus. We ignore case (i) for now.

Case (ii): In this case, we have $y = p_i$ and $t_1 = t'$. This gives a linear equivalence $2\mathbf{p} \sim 2y$ and we get

$$L_1 = L^\vee(\mathbf{p} - p_i) = L^\vee \otimes \mathfrak{a}_i$$

where $\mathfrak{a}_i = \mathcal{O}_C(\mathbf{p} - \mathbf{p}_i)$ is a 2-torsion point. There are six possibilities in this case. Let us look at what is the extension η such that L_1 lifts into E. The extension η should be in the kernel of the map

$$\mathrm{Ext}^1(L^\vee(\mathbf{p}), L) \to \mathrm{Ext}^1(L_1, L)$$

i.e. the map

$$H^1(\mathcal{O}(-t')) \to H^1(\mathcal{O}(-t' + \mathbf{p}_i)).$$

We note that this map is isomorphic to the map

$$H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(\mathbf{p}_i))$$

so it means that η should be the extension corresponding to the branch point $\mathbf{x}_i \in \mathbb{P}^1$.

We obtain 6 glueings, one for each point $\mathbf{x}_i$, with the point $(L, \mathbf{x}_i)$ being glued to $(L \otimes \mathfrak{a}, \mathbf{x}_i)$. In other words the surface D is glued to itself along each of the 6 curves $\overline{C} \times \{\mathbf{x}_i\}$, $i = 1, \ldots, 6$. On each curve $\overline{C} \times \{\mathbf{x}_i\}$, the glueing map is the automorphism of $\overline{C}$ given by tensoring with the 2-torsion point $\mathfrak{a}_i$. This is the first part of the statement in the proposition.

The analysis of the cuspidal locus, to conclude the proof of the proposition, will be done later in subsection 4.4. □

4.2 Computations in degree one

By convention all answers to Chern calculations are truncated to the terms of degree ≤ 2.

By Lemma 3.2) and Theorem 3.6 the curve $\widehat{C}$ embeds in $\mathcal{P}$ and the blow-up $\boldsymbol{\varepsilon} : Y \to \mathcal{P}$ of $\mathcal{P}$ along $\widehat{C}$ resolves the natural rational map $\pi_*(-) : \mathcal{P} \dashrightarrow X$ producing a finite morphism $f : Y \to X$. Let $\boldsymbol{E} \subset Y$ denote the exceptional locus. It is a $\mathbb{P}^1$-bundle over $\widehat{C}$, namely $\boldsymbol{E} \cong \mathbb{P}(N)$ (this being the projective bundle of subspaces), where $N := N_{\widehat{C}/\mathcal{P}}$. In fact by Lemma 3.9**(b)** we have $N = H^1(\mathcal{O}_C) \otimes \widehat{\mathsf{sq}}^* \pi^* \omega_C$ and so $\boldsymbol{E} = \widehat{C} \times \mathbb{P}^1$ where the second factor is the hyperelliptic $\mathbb{P}^1$ for C.

Remark 4.6. The images in X of the $\mathbb{P}^1$'s that are fibers of $b_{\boldsymbol{E}} : \boldsymbol{E} \to \widehat{C}$ are lines in X. Indeed, these fibers are isomorphic to the projectivized bundle of outgoing directions along $\overline{C}$ which map to lines in Wob, see Lemma 4.3.

Let H denote the hyperplane class on X. Since $X \subset \mathbb{P}^5$ is a degree 4 subvariety of dimension 3, we get $H^3 = 4$.

Let $\boldsymbol{F} = f^*H$ be the inverse image of the class H on Y. We know that for a generic Hitchin fiber the Neron-Severi group of $\mathcal{P}$ is $\mathbb{Z}$, so the Neron-Severi group of Y is generated rationally by the classes $\boldsymbol{E}$ and $\boldsymbol{F}$.

Proposition 4.7. *The Todd classes for the degree* 1 *moduli spaces are:*

$$\mathrm{td}(TX) = 1 + H + 7H^2/12,$$

$$\pi^*\mathrm{td}(TX)^{-1} = 1 - \boldsymbol{F} + 5\boldsymbol{F}^2/12,$$
$$\mathrm{td}(TY) = 1 - \boldsymbol{E}/2 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9,$$
$$\mathrm{td}(Y/X) = (1 - \boldsymbol{F} + 5\boldsymbol{F}^2/12)(1 - \boldsymbol{E}/2 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9).$$

Proof. Recall from Section 3.12, the Chern class calculations for the intersection of two quadrics X say

$$c_1(TX) = 2H, \quad c_2(TX) = 3H^2,$$

and

$$\begin{aligned}\mathrm{td}(TX) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\ &= 1 + H + 7H^2/12.\end{aligned}$$

A computation shows that

$$(1 + H + 7H^2/12)(1 - H + 5H^2/12) = 1,$$

thus

$$\mathrm{td}(TX)^{-1} = 1 - H + 5H^2/12$$

which pulls back to the same formula with $\boldsymbol{F}$ on Y.

For the blow-up Y of an abelian threefold $\mathcal{P}$ along the curve $\widehat{C}$ we need to know the degree of the normal bundle. This degree is $2g_{\widehat{C}} - 2$ for the embedding of a curve in an abelian variety, thus by either the explicit description of N in Lemma 3.9**(b)** or by the genus calculation of Lemma 4.1 we have

$$\delta = \deg(N) = 128.$$

This could also be seen from the intersection number calculations above, one can see that it is the same as $-\boldsymbol{E}^3$.

Again from the calculations of Section 3.12 we have

$$c_1(TY) = -\boldsymbol{E},$$

$$c_2(TY) = \delta \mathsf{fib} - \boldsymbol{E}^2,$$

and

$$\begin{aligned}\mathrm{td}(TY) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\ &= 1 - \boldsymbol{E}/2 + \delta\mathsf{fib}/12.\end{aligned}$$

We note that $(\boldsymbol{E} + \boldsymbol{F})|_{\boldsymbol{E}}$ is a divisor on $\boldsymbol{E}$ whose intersection with a fiber is zero, indeed a fiber intersect $\boldsymbol{E}$ is -1 and it intersects $\boldsymbol{F}$ in 1 point since the image of a fiber is a line contained in X (Remark 4.6). Therefore $(\boldsymbol{E} + \boldsymbol{F}) \cdot \boldsymbol{E}$ is a sum of fibers. The number may be calculated as $(\boldsymbol{E} + \boldsymbol{F}) \cdot \boldsymbol{E} \cdot \boldsymbol{F}$ again from Remark 4.6, this gives

$$(\boldsymbol{E} + \boldsymbol{F}) \cdot \boldsymbol{E} = 96\mathsf{fib}.$$

Therefore $\delta\mathrm{fib} = (128/96)(\boldsymbol{E} + \boldsymbol{F}) \cdot \boldsymbol{E} = 4(\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/3$. Our formulas become

$$c_2(TY) = 4(\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/3 - \boldsymbol{E}^2 = (\boldsymbol{E}^2 + 4\boldsymbol{E}\boldsymbol{F})/3,$$

and

$$\mathrm{td}(TY) = 1 - \boldsymbol{E}/2 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9$$

which completes the calculation. □

Next we compute the triple intersections of divisor classes on Y.

Proposition 4.8. *The triple intersections of divisor classes on the degree* 1 *modular spectral covering* Y *are:*

$$\boldsymbol{F}^3 = 32, \quad \boldsymbol{E}\boldsymbol{F}^2 = 64, \quad \boldsymbol{E}^2\boldsymbol{F} = 32, \quad \boldsymbol{E}^3 = -128.$$

Recall that H denotes the divisor class of the hyperplane section on X, and $\boldsymbol{F} := f^*(H)$. We have

$$H^{1,1}(Y) \cap H^2(Y, \mathbb{Q}) = \langle \boldsymbol{E}, \boldsymbol{F} \rangle, \text{ and}$$
$$H^{2,2}(Y) \cap H^4(Y, \mathbb{Q}) = \langle \boldsymbol{E}\boldsymbol{F}, \boldsymbol{F}^2 \rangle.$$

The corresponding groups on X are generated (over $\mathbb{Q}$) by H and H^2 respectively.

Lemma 4.9. *The map* $f : Y \to X$ *has degree* 8*.*

Proof. Since over the very stable locus the map $\pi_*(-) : \mathcal{P} \dashrightarrow X$ is a proper morphism (see e.g. [PP21a, Zel20, PN20] it suffices to compute the number of preimages of a very stable point in X under $f : Y \to X$. Choose a general hence very stable point $E \in X$. The fiber of the cotangent bundle $T^\vee_E X$ is the space $H^0(\mathrm{End}_0(E) \otimes \omega_C)$ of ω_C-twisted endomorphisms. The Hitchin base is the 3-dimensional space $H^0(C, \omega_C^{\otimes 2}) \cong \mathbb{A}^3$ of quadratic differentials on C. The map $T^\vee_E X \to \mathbb{A}^3$ is the restriction of the Hitchin map and is thus given by three quadratic forms, so the inverse image of a general point $\mathsf{b} \in \mathbb{A}^3$ is the intersection of three quadrics: it has 8 points. As $\mathcal{P} = \boldsymbol{h}^{-1}(\mathsf{b})$, and the exceptional divisor $\boldsymbol{E}$ maps to the wobbly locus, we have that the intersection of $Y \cap T^\vee_E X$ is equal to $\mathcal{P} \cap T^\vee_E X$ and so is this set of 8 points. Thus for a very stable E and a generic $\widetilde{C}$ the inverse image $f^{-1}(E) \subset Y$ consists of 8 points. $\square$

Remark 4.10. Note that the proof of Lemma 4.9 repeats verbatim to show that in the degree 0 case the map $f_0 : Y_0 \to X_0$ has degree 8 as well.

We have $H^3 = 4$ since $X \subset \mathbb{P}^5$ is a degree 4 subvariety. Hence,

$$\boldsymbol{F}^3 = 32.$$

We also checked that $\mathcal{O}_Y(\boldsymbol{E}+\boldsymbol{F})|_{\boldsymbol{E}}$ is a sum of 96 fibers of the map $\boldsymbol{E} \to \widehat{C}$. In particular, the divisor $(\boldsymbol{E}+\boldsymbol{F})$ restricted to $\boldsymbol{E}$, has trivial self-intersection. We get the formula

$$(\boldsymbol{E}+\boldsymbol{F}) \cdot \boldsymbol{E} \cdot (\boldsymbol{E}+\boldsymbol{F}) = 0,$$

or

$$\boldsymbol{E}^3 + 2\boldsymbol{E}^2\boldsymbol{F} + \boldsymbol{E}\boldsymbol{F}^2 = 0.$$

Lemma 4.11. *The wobbly locus is the image* $\mathrm{Wob} = f(\boldsymbol{E}) \subset X$*. The intersection of* Wob *with a line* ℓ *in* X *(in other words a subvariety that is a line in* $\mathbb{P}^5$*) has* 8 *points. The map* $f_{|\boldsymbol{E}} : \boldsymbol{E} \to \mathrm{Wob}$ *is generically* 2 *to* 1*, and if* D *denotes the normalization of* Wob *(see Lemma 4.3 and Proposition 4.5 above) this map factors through a* $2:1$ *map* $\boldsymbol{E} \to D$*.*

Proof. Recall from Lemma 4.3 that D can be described as the space of downward or outgoing directions to the curve $\overline{C} \subset \mathsf{Higgs}_1$. As we saw in subsections 3.6 and 3.7 the incoming flow gives a $2:1$ map $\widehat{C} \to \overline{C}$ and a normal direction to a general point of $\widehat{C}$ maps to a downward direction at the image point in $\overline{C}$. This gives the map $\boldsymbol{E} \to D$. In the present case, downward flow lines are broken only once and the broken flow line depends only on the first normal derivative to the point of $\widehat{C}$ in the Hitchin fiber, so the closure of the downward flow map restricts on $\boldsymbol{E}$ to the composition $\boldsymbol{E} \to D \to \mathrm{Wob} \subset X_1$. □

Proof of Proposition 4.8. We saw above that $\boldsymbol{F}^3 = 32$.

A line ℓ lying on X satisfies $\ell \cap H = 1$, but also $(H^2) \cap H = 4$; hence in the group $H^{2,2}(Y) \cap H^4(Y, \mathbb{Q})$ we have

$$H^2 = 4\ell.$$

This gives $H^2 \cap \mathrm{Wob} = 32$.

From the $2:1$ covering property we get

$$\boldsymbol{F}^2 \cap \boldsymbol{E} = H^2 \cap 2\mathrm{Wob}$$

so

$$\boldsymbol{E}\boldsymbol{F}^2 = 64.$$

Let us now calculate $\boldsymbol{E}^3$. It is the self-intersection of the divisor class $c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E}))$ on the surface $\boldsymbol{E}$. By the usual blowup picture the line bundle $\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E})$ is the universal subbundle for the projectivization $\mathbb{P}(N) = \boldsymbol{E}$ so the universal quotient bundle is

$$UQ = b_{\boldsymbol{E}}^*(N)/\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E}).$$

In terms of divisor classes on $\boldsymbol{E}$ this gives

$$c_1(UQ) = b_{\boldsymbol{E}}^* c_1(N) - c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E}).$$

The relative tangent bundle $T(\boldsymbol{E}/\widehat{C})$ of the map $b_{\boldsymbol{E}} : \boldsymbol{E} \to \widehat{C}$ is given by

$$T(\boldsymbol{E}/\widehat{C}) = \operatorname{Hom}(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E}), UQ),$$

which gives

$$c_1(T(\boldsymbol{E}/\widehat{C})) = c_1(UQ) - c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E})) = b_{\boldsymbol{E}}^* c_1(N) - 2c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E}))$$

on the level of divisor classes.

Since $\boldsymbol{E} \cong \widehat{C} \times \mathbb{P}^1$, the Neron-Severi group of $\boldsymbol{E}$ is freely generated by two classes $\hat{\mathfrak{c}}$ and fib, where $\hat{\mathfrak{c}}$ is the class of $\widehat{C} \times \mathrm{pt}$ and as before fib is the class of $\mathrm{pt} \times \mathbb{P}^1$. In terms of these classes we have

$$b_{\boldsymbol{E}}^* c_1(N) = 128\mathsf{fib} \qquad \text{and} \qquad c_1(T(\boldsymbol{E}/\widehat{C})) = 2\hat{\mathfrak{c}}.$$

Therefore we get

$$c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E})) = \frac{1}{2}\left(b_{\boldsymbol{E}}^* c_1(N) - c_1(T(\boldsymbol{E}/\widehat{C}))\right) = 64\mathsf{fib} - \hat{\mathfrak{c}},$$

and so taking into account that $\hat{\mathfrak{c}}^2 = \mathsf{fib}^2 = 0$ and $\hat{\mathfrak{c}} \cdot \mathsf{fib} = 1$ we get

$$c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E})) \cdot c_1(\mathcal{O}_{\boldsymbol{E}}(\boldsymbol{E})) = (64\mathsf{fib} - \hat{\mathfrak{c}})^2 = -128.$$

That was the self-intersection on $\boldsymbol{E}$, so it gives on Y, $\boldsymbol{E}^3 = -128$.

Now from the formula $\boldsymbol{E}^3 + 2\boldsymbol{E}^2\boldsymbol{F} + \boldsymbol{E}\boldsymbol{F}^2 = 0$ we get

$$(-128) + 2\boldsymbol{E}^2\boldsymbol{F} + (64) = 0, \text{ so } \boldsymbol{E}^2\boldsymbol{F} = (128 - 64)/2 = 32.$$

This completes our list of intersection numbers on Y. $\square$

4.3 Main construction

Let $\mathcal{L}_0$ be the pullback to Y of a degree zero line bundle on $\mathcal{P}$, and set

$$\mathcal{L}_{a,b} := \mathcal{L}_0(a\boldsymbol{F} + (b+1)\boldsymbol{E}) \text{ and } \mathcal{V}_{a,b} := f_*(\mathcal{L}_{a,b}).$$

Following the notation we adopted in the definition of crude parabolic structures in section 3.10 we will also set

$$\mathcal{L}'_{a,b} := \mathcal{L}_0(a\boldsymbol{F} + b\boldsymbol{E}) \ \subset \ \mathcal{L}_{a,b} \text{ and } \mathcal{V}'_{a,b} = f_*(\mathcal{L}'_{a,b}) \ \subset \ \mathcal{V}_{a,b}.$$

The notations are chosen so that the spectral line bundle defined in section 3.10 is precisely $\mathcal{L}' = \mathcal{L}_0(a\boldsymbol{F} + b\boldsymbol{E})$. Notice that $\mathcal{V}'_{a,b} = \mathcal{V}_{a,b-1}$.

The next step is to look at the calculations of the Chern characters of the direct image $\mathcal{V}_{a,b} = f_*(\mathcal{L}_{a,b})$. The Chern character of $\mathcal{L}_{a,b}$ is

$$\mathrm{ch}(\mathcal{L}_{a,b}) = 1 + (a\boldsymbol{F} + (b+1)\boldsymbol{E}) + (a\boldsymbol{F} + (b+1)\boldsymbol{E})^2/2.$$

Putting the Todd class formula of Proposition 4.7 into the Grothendieck-Riemann-Roch formula gives

$$\begin{aligned}
\mathrm{ch}(\mathcal{V}_{a,b}) &= \mathrm{ch}(f_*(\mathcal{L}_{a,b})) \\
&= f_*(1 + (a\boldsymbol{F} + (b+1)\boldsymbol{E}) \\
&\qquad + (a\boldsymbol{F} + (b+1)\boldsymbol{E})^2/2)(1 - \boldsymbol{F} + 5\boldsymbol{F}^2/12)(1 - \boldsymbol{E}/2 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9) \\
&= f_*\left[1 + ((a-1)\boldsymbol{F} + (b+1/2)\boldsymbol{E}) + (a\boldsymbol{F} + (b+1)\boldsymbol{E})^2/2 \right. \\
&\qquad \left. + 5\boldsymbol{F}^2/12 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9 + \boldsymbol{E}\boldsymbol{F}/2 - (\boldsymbol{F} + \boldsymbol{E}/2)(a\boldsymbol{F} + (b+1)\boldsymbol{E})\right].
\end{aligned}$$

The rational cohomology of X_1 is generated by H in degree 2 and H^2 in degree 4, so if $a_1 \in H^2(Y)$ and $a_2 \in H^4(Y)$ are classes on Y we can write

$$f_*(1 + a_1 + a_2) = 8 + b_1 H + b_2 H^2,$$

for some rational numbers $b_1, b_2 \in \mathbb{Q}$. Then, using $H^3 = 4$ we get $4b_1 = H^2 \cdot f_*(a_1) = \boldsymbol{F}^2 a_1$ and $4b_2 = H \cdot f_*(a_2) = \boldsymbol{F} a_2$.

We get, using the calculations of Proposition 4.8:

$$H^2 \cdot \mathrm{ch}_1(\mathcal{V}_{a,b}) = \boldsymbol{F}^2((a-1)\boldsymbol{F} + (b+1/2)\boldsymbol{E}) = 32(a-1) + 64(b+1/2) = 32a + 64b$$

so

$$\mathrm{ch}_1(\mathcal{V}_{a,b}) = (8a + 16b)H.$$

And

$$\begin{aligned} H \cdot \mathrm{ch}_2(\mathcal{V}_{a,b}) &= \boldsymbol{F}\left[(a\boldsymbol{F} + (b+1)\boldsymbol{E})^2/2 \right. \\ &\qquad \left. +5\boldsymbol{F}^2/12 + (\boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{F})/9 + \boldsymbol{E}\boldsymbol{F}/2 - (\boldsymbol{F} + \boldsymbol{E}/2)(a\boldsymbol{F} + (b+1)\boldsymbol{E})\right] \\ &= (a^2/2 + 5/12 - a)\boldsymbol{F}^3 + (a(b+1) + 1/9 + 1/2 - (b+1+a/2))\boldsymbol{E}\boldsymbol{F}^2 \\ &\qquad + ((b+1)^2/2 + 1/9 - (b+1)/2)\boldsymbol{E}^2\boldsymbol{F} \\ &= 16a^2 - 32a + 32a + 64ab + 16b^2 + -64b + 16b + 40/3 + 64/9 - 32 + 32/9 \\ &= 16a^2 + 16b^2 + 64ab - 48b - 8. \end{aligned}$$

We get

$$\mathrm{ch}_2(\mathcal{V}_{a,b}) = 4a^2 + 16ab + 4b^2 - 12b - 2.$$

Together these prove the following:

Proposition 4.12. *For $\mathcal{L}_{a,b}$ in the numerical class of $\mathcal{O}_Y(a\boldsymbol{F} + (b+1)\boldsymbol{E})$ the direct image $\mathcal{V}_{a,b} = \pi_*(\mathcal{L}_{a,b})$ has Chern character (truncated to codimension 2 as usual)*

$$\mathrm{ch}(\mathcal{V}_{a,b}) = 8 + (8a + 16b)H + (4a^2 + 16ab + 4b^2 - 12b - 2)H^2.$$

Later we will consider a parabolic modification of $\mathcal{L}_{a,b}$ over the divisor $\boldsymbol{E}$, giving a new smooth parabolic structure defined locally upstairs over the modular spectral cover or somewhat equivalently over the projectivization of the bundle.

4.4 The cusp locus of the wobbly divisor

In this subsection we prove the part of Proposition 4.5 about the cuspidal locus. Specifically we compute the locus $D_{\mathrm{cusp}} \subset D$ on which the map $\mathbf{f} : D \to X$ is not immersive. Recall that this map is induced from the map $f : Y \to X$. More precisely, the restriction $f_{|\boldsymbol{E}} : \boldsymbol{E} \to \mathrm{Wob} \subset X$ factors through the double cover $\boldsymbol{E} = \widehat{C} \times \mathbb{P}^1 \to \overline{C} \times \mathbb{P}^1 = D$ and $\mathbf{f} : D \to X$ is just the induced map.

Let $(L, \eta) \in D = \overline{C} \times \mathbb{P}^1$ be a point, and let $E = \mathbf{f}(L, \eta) \in X$ be the corresponding rank two bundle. Our goal is to understand when the differential

$$d\mathbf{f}_{(L,\eta)} : T_{D,(L,\eta)} \to T_{X,E}$$

has a non-trivial kernel.

Recall that $L \in \overline{C} \subset \mathrm{Jac}^0(C)$ means that $L^{\otimes 2}(\mathbf{p})$ is effective, i.e. that $L^{\otimes 2} \cong \mathcal{O}_C(t-\mathbf{p})$ for some point $t \in C$. Thus $L^{\otimes 2}(-\mathbf{p}) \cong \mathcal{O}_C(t-2\mathbf{p}) = \mathcal{O}_C(-t')$, where $t' \in C$ is the point corresponding to t under the hyperelliptic involution.

Hence we have

$$\mathrm{Ext}^1(L^\vee(\mathbf{p}), L) = H^1(C, L^{\otimes 2}(-\mathbf{p})) = H^1(C, \mathcal{O}_C(-t')).$$

From the long exact sequence in cohomology associated with the short exact sequence $0 \to \mathcal{O}_C(-t') \to \mathcal{O}_C \to \mathcal{O}_{t'} \to 0$ it follows that the natural map

$$\iota_{t'} : \qquad H^1(C, \mathcal{O}_C(-t')) \xrightarrow{\cong} H^1(C, \mathcal{O}_C)$$

is an isomorphism, and hence we can view η as an element in $\iota_{t'}^{-1}(\eta) \in \mathrm{Ext}^1(L^\vee(\mathbf{p}), L)$. The rank two vector bundle $E = f(L, \eta)$ is the extension

$$0 \to L \to E \to L^\vee(\mathbf{p}) \to 0$$

given by the extension class $\iota_{t'}^{-1}(\eta)$.

We have $T_{D,(L,\eta)} = T_{\overline{C},L} \oplus T_{\mathbb{P}^1,\eta}$ and so to understand the map $df_{(L,\eta)}$ it is enough to understand its restrictions to the two coordinate lines $T_{\overline{C},L} \oplus \{0\}$ and $\{0\} \oplus T_{\mathbb{P}^1,\eta}$ in $T_{D,(L,\eta)}$. These restrictions which will abbreviate as

$$\begin{aligned} d\mathbf{f}_{(L,\eta)|T_{\overline{C},L}} : &\qquad T_{\overline{C},L} \longrightarrow T_{X,E} \\ d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}} : &\qquad T_{\mathbb{P}^1,\eta} \longrightarrow T_{X,E} \end{aligned}$$

have a natural modular interpretation.

4.4.1 Interpretation of $d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}}$

Fix $L \in \overline{C}$ and hence the points $t, t' \in C$. The extensions of $L^\vee(\mathbf{p})$ by L corresponding to a varying point $\eta \in \mathbb{P}(H^1(C, \mathcal{O}_C))$ fit in a natural universal family parametrized by $\mathbb{P}^1$. Indeed, consider the surface $C \times \mathbb{P}^1$. By Künneth we have

$$\begin{aligned} \mathrm{Ext}^1_{C\times\mathbb{P}^1}(p_C^* L^\vee(\mathbf{p}), p_C^* L \otimes p_{\mathbb{P}^1}^* \mathcal{O}(1)) &= H^1(C \times \mathbb{P}^1, p_C^* \mathcal{O}(-t') \otimes p_{\mathbb{P}^1}^* \mathcal{O}(1)) \\ &= H^1(C, \mathcal{O}(-t')) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)) \\ &\cong H^1(C, \mathcal{O}) \otimes H^1(C, \mathcal{O})^\vee. \end{aligned}$$

Here in the last step we used $\iota_{t'}$ to identify $H^1(C, \mathcal{O}(-t'))$ with $H^1(C, \mathcal{O})$ and we used $\mathbb{P}^1 = \mathbb{P}(H^1(C, \mathcal{O}))$ to identify $H^0(\mathbb{P}^1, \mathcal{O}(1))$ with $H^1(C, \mathcal{O})^\vee$.

The extension

$$0 \to p_C^* L \otimes p_{\mathbb{P}^1}^* \mathcal{O}(1) \to \mathscr{E} \to p_C^* L^\vee(\mathbf{p}) \to 0$$

corresponding to the identity element in $H^1(C, \mathcal{O}) \otimes H^1(C, \mathcal{O})^\vee$ is a rank two bundle $\mathscr{E}$ on $C \times \mathbb{P}^1$ whose restrictiion to $C \times \{\xi\}$ is the extension

$$0 \to L \to \mathscr{E}_\xi \to L^\vee(\mathbf{p}) \to 0$$

given by the class $\iota_{t'}^{-1}(\xi)$. In particular we have $\mathscr{E}_\xi = E$.

Therefore $\mathbf{f}_{|\{L\} \times \mathbb{P}^1} : \mathbb{P}^1 \to X$ can be viewed as the classifying map for the bundle $\mathscr{E} \to C \times \mathbb{P}^1$. Let $\mathrm{End}_0(E)$ denote the bundle of traceless endomorphisms of E. By deformation theory we have $T_{X,E} = H^1(C, \mathrm{End}_0(E))$ and we can identify the restricted map

$$d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}} : T_{\mathbb{P}^1,\eta} \to H^1(C, \mathrm{End}_0(E))$$

with the Kodaira-Spencer class of the family $\mathscr{E} \to C \times \mathbb{P}^1$ at η. But tensoring a family with a fixed line bundle does not change the Kodaira-Spencer class and so equivalently we can compute the Kodaira-Spencer class for the family $\mathscr{F} = \mathscr{E} \otimes p_C^* L(-\mathbf{p}) \to C \times \mathbb{P}^1$. By definition $\mathscr{F}$ is the extension

$$0 \to \mathcal{O}(-t') \boxtimes \mathcal{O}(1) \to \mathscr{F} \to \mathcal{O}_{C \times \mathbb{P}^1} \to 0 \tag{19}$$

corresponding to the element

$$\iota_{t'}^{-1} \in H^1(C, \mathcal{O}(-t')) \otimes H^1(C, \mathcal{O}))^\vee = \mathrm{Ext}^1_{C \times \mathbb{P}^1}(\mathcal{O}, \mathcal{O}(-t') \boxtimes \mathcal{O}(1)),$$

and so is the universal family of extensions

$$0 \to \mathcal{O}_C(-t') \to \mathscr{F}_\xi \to \mathcal{O}_C \to 0,$$

corresponding to points $\xi \in \mathbb{P}^1$. To simplify notation we will denote $\mathscr{F}_\eta$ by F.

Now from the exact sequence (19) it follows that $\mathrm{End}_0(\mathscr{F}) = \mathrm{End}_0(\mathscr{E})$ maps naturally onto $\mathrm{Hom}(\mathcal{O}(-t') \boxtimes \mathcal{O}(1), \mathscr{F})$ and so we have an exact sequence

$$0 \to \mathcal{O}(-t') \boxtimes \mathcal{O}(1) \to \mathrm{End}_0(\mathscr{F}) \to \mathscr{F} \otimes (\mathcal{O}(t') \boxtimes \mathcal{O}(-1)) \to 0$$

which for each $\xi \in \mathbb{P}^1$ specializes to a short exact sequence

$$0 \to \mathcal{O}_C(-t') \to \mathrm{End}_0(\mathscr{E}_\xi) \to \mathscr{F}_\xi(t') \to 0. \tag{20}$$

Since the family $\mathscr{E}$ encodes the variation of the bundle $\mathscr{E}_\xi$ as the extension class $\xi \in H^1(C, \mathcal{O}))$ varies, the Kodaira-Spencer class $\mathsf{ks}_\xi^{\mathscr{E}} : T_{\mathbb{P}^1,\xi} \to H^1(C, \mathrm{End}_0(\mathscr{E}_\xi))$ is captured by the maps between cohomology of the terms in (20). Indeed, taking into account that $H^0(C, \mathcal{O}(-t')) = 0$ and $H^0(C, \mathrm{End}_0(\mathscr{E}_\xi)) = 0$, the long exact sequence of cohomology associated with (20) reads

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^0(C, \mathscr{F}(t')) \arrow[dlll, rounded corners] \\ H^1(C, \mathcal{O}(-t')) \arrow[r] & H^1(C, \mathrm{End}_0(\mathscr{E}_\xi)) \arrow[r] & H^1(C, \mathscr{F}_\xi(t')) \arrow[r] & 0 \end{tikzcd} \tag{21}$$

Also, in the long exact sequence associated with

$$0 \to \mathcal{O}_C \to \mathscr{F}_\xi(t') \to \mathcal{O}_C(t') \to 0$$

the first edge homomorphism $H^0(C, \mathcal{O}(t')) \to H^1(C, \mathcal{O})$ is given simply by cup product with $\iota_{t'}^{-1}(\xi) \in H^1(C, \mathcal{O}(-t'))$. Since ξ is not zero and $H^0(C, \mathcal{O}(t')) \cong \mathbb{C}$ this implies that this edge homomorphism is injective and so the natural map $H^0(C, \mathcal{O}) \to H^0(C, \mathscr{F}_\xi(t'))$ is an isomorphism. If we identify $H^0(C, \mathscr{F}_\xi(t'))$ with $H^0(C, \mathcal{O})$ via this isomorphism, then the first edge homomorphism in (21) becomes the cup product

$$\iota_{q'}^{-1}(\xi)\cup\ : H^0(C, \mathcal{O}) \to H^1(C, \mathcal{O}(-t')).$$

Using $\iota_{t'}$ to identify $H^1(C, \mathcal{O}(-t'))$ with $H^1(C, \mathcal{O})$ we then see that the long exact sequence (21) reduces to a short exact sequence

$$0 \to H^1(C, \mathcal{O})/\mathbb{C} \cdot \xi \to H^1(C, \mathrm{End}_0(\mathscr{E}_\xi)) \to H^1(C, \mathscr{F}_\xi(t')) \to 0.$$

But $H^1(C, \mathcal{O})/\mathbb{C} \cdot \xi = T_{\mathbb{P}^1,\xi}$ and the first map in this sequence by construction sends an infinitesimal deformation of the extension ξ to an infinitesimal deformation of $\mathscr{E}_\xi$. Therefore this map is the Kodaira-Spencer map and we can rewrite the previous sequence as

$$0 \longrightarrow T_{\mathbb{P}^1,\xi} \xrightarrow{\mathsf{ks}_\xi^{\mathscr{E}}} H^1(C, \mathrm{End}_0(\mathscr{E}_\xi)) \longrightarrow H^1(C, \mathscr{F}_\xi(t')) \longrightarrow 0.$$

or in the special case $\xi = \eta$ as

$$0 \longrightarrow T_{\mathbb{P}^1,\eta} \xrightarrow{d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}}} H^1(C, \mathrm{End}_0(E)) \longrightarrow H^1(C, F(t')) \longrightarrow 0.$$

This gives the desired modular interpretation of $d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}}$, and confirms that $\mathbf{f}$ restricted to any ruling $\{L\} \times \mathbb{P}^1$ is an immersion.

4.4.2 Interpretation of $d\mathbf{f}_{(L,\eta)|T_{\overline{C},L}}$

Fix $\eta \in \mathbb{P}^1 = \mathbb{P}(H^1(C,\mathcal{O}))$. The extensions of $L^\vee(\mathbf{p})$ by L corresponding to the fixed extension class η and a variable $L \in \overline{C}$ fit in a natural family parametrized by $\overline{C}$. Indeed, consider the surface $C \times C$ with its diagonal divisor $\Delta \subset C \times C$. Pushing down the short exact sequence

$$0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

via the projection $p_2 : C \times C \to C$ onto the second factor gives a long exact sequence of direct images:

$$\begin{tikzcd} 0 \arrow[r] & p_{2*}\mathcal{O}(-\Delta) \arrow[r] & p_{2*}\mathcal{O} \arrow[r] & p_{2*}\mathcal{O}_\Delta \arrow[dll, rounded corners] & \\ & R^1p_{2*}\mathcal{O}(-\Delta) \arrow[r] & R^1p_{2*}\mathcal{O} \arrow[r] & R^1p_{2*}\mathcal{O}_\Delta \arrow[r] & 0 \end{tikzcd}$$

We have $p_{2*}\mathcal{O}(-\Delta) = 0$ since $\mathcal{O}(-\Delta)$ has negative degree along the fibers of p_2. Since $p_2 : \Delta \to C$ is an isomorphism we also have $p_{2*}\mathcal{O}_\Delta = \mathcal{O}_C$ and $R^1p_{2*}\mathcal{O}_\Delta = 0$. Finally by Künneth we have $p_{2*}\mathcal{O} = \mathcal{O}_C$ and $R^1p_{2*}\mathcal{O} = H^1(C,\mathcal{O}) \otimes \mathcal{O}_C$. Substituting these in the long exact sequence gives

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{O}_C \arrow[dll, rounded corners] & \\ & R^1p_{2*}\mathcal{O}(-\Delta) \arrow[r] & H^1(C,\mathcal{O}) \otimes \mathcal{O}_C \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Therefore we have

$$p_{2*}\mathcal{O}(-\Delta) = 0, \qquad R^1p_{2*}\mathcal{O}(-\Delta) = H^1(C,\mathcal{O}) \otimes \mathcal{O}_C$$

With this in mind we compute

$$\begin{aligned} \mathrm{Ext}^1_{C\times C}(\mathcal{O},\mathcal{O}(-\Delta)) &= H^1(C \times C, \mathcal{O}(-\Delta)) \\ &= H^0(C, R^1p_{1*}\mathcal{O}(-\Delta)) \\ &= H^1(C,\mathcal{O}) \otimes H^0(C,\mathcal{O}) \end{aligned}$$

where at the second step we used the Leray spectral sequence for p_2.

Therefore the class $\eta \otimes 1 \in H^1(C,\mathcal{O}) \otimes H^0(C,\mathcal{O})$ corresponds to a rank two bundle $\mathbb{F}$ on $C \times C$ which is an extension

$$0 \to \mathcal{O}_{C\times C}(-\Delta) \to \mathbb{F} \to \mathcal{O}_{C\times C} \to 0.$$

By construction, for every point $x \in C$, the restriction $\mathbb{F}_x = \mathbb{F}_{|C \times \{x\}}$ is the extension

$$0 \to \mathcal{O}_C(-x) \to \mathbb{F}_x \to \mathcal{O}_C \to 0$$

given by the class $\iota_x^{-1}(\eta) \in H^1(C, \mathcal{O}(-x))$. In particular we have $\mathbb{F}_{t'} = F$.

Now consider the degree 16 map $C \times \overline{C} \to C \times C$ which is the identity on the first factor and on the second factor is the composition of the natural map $\overline{C} \to C$ with the hyperelliptic involution[5]. Let $\mathscr{F} \to C \times \overline{C}$ be the pull back of $\mathbb{F}$ by this map, and let $\mathscr{L} \to C \times \overline{C}$ be the Poincaré line bundle. Then the bundle $\mathscr{E} = \mathscr{F} \otimes \mathscr{L}^\vee(\{\mathbf{p}\} \times \overline{C})$ is an extension

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^\vee(\{\mathbf{p}\} \times \overline{C}) \to 0.$$

For each $M \in \overline{C}$ restricts to the bundle $\mathscr{E}_M := \mathscr{E}_{|C \times \{M\}}$ which is the extension

$$0 \to M \to \mathscr{E}_M \to M^\vee(\mathbf{p}) \to 0$$

corresponding to the class η. In particular $\mathscr{E}_L = E$ and the map $\mathbf{f}_{|\overline{C} \times \{\eta\}} : \overline{C} \to X$ is the classifying map for the family $\mathscr{E} \to C \times \overline{C}$.

This identifies the restricted differential $d\mathbf{f}_{(L,\eta)|T_{\overline{C},L}}$ with the Kodaira-Spencer class

$$\mathsf{ks}_L^{\mathscr{E}} : T_{\overline{C},L} \to H^1(C, \mathrm{End}_0(E))$$

of the family $\mathscr{E}$ computed $L \in \overline{C}$.

The family $\mathscr{E}$ differs from the family of bundles $\mathscr{F}$ by a tensoring with a family of line bundles. So, even though $\mathscr{F}$ is a family of bundles with varying determinant, the traceless parts of the Kodaira-Spencer classes for $\mathscr{E}$ and $\mathscr{F}$ are equal. Given a family of rank two bundles $\mathcal{F}$ on C let us write $\kappa^{\mathcal{F}} = \mathsf{ks}^{\mathcal{F}} - (\mathrm{tr}(\mathsf{ks}^{\mathcal{F}})/2) \cdot \mathrm{id}$ for the traceless part of the Kodaira-Spencer class of $\mathcal{F}$. With this notation we then have

$$d\mathbf{f}_{(L,\eta)|T_{\overline{C},L}} = \mathsf{ks}_L^{\mathscr{E}} = \kappa_L^{\mathscr{E}} = \kappa_L^{\mathscr{F}}.$$

As a last simplification, note that the map $\overline{C} \to C$, $L \mapsto t'$ is étale and so its differential is an isomorphism $T_{\overline{C},L} \xrightarrow{\cong} T_{C,t'}$ of tangent spaces. Taking into account that the family $\mathscr{F}$ is the pullback of the family $\mathbb{F}$ we conclude that modulo the isomorphism $T_{\overline{C},L} \xrightarrow{\cong} T_{C,t'}$ the traceless Kodaira-Spencer class $\kappa_L^{\mathscr{F}}$ equals the traceless Kodaira-Spencer class $\kappa_{t'}^{\mathbb{F}}$.

[5]The canonical map from $\overline{C}$ to C assigns to each L the unique point $t \in C$ such that $L^{\otimes 2} = \mathcal{O}_C(t)$. The map we consider here sends L not to t but to its image under the hyperelliptic involution t'

4.4.3 The kernel of the differential

After these preliminaries we are now ready to understand where the map $\mathbf{f} : D \to X$ is not immersive.

Proposition 4.13. *Let $(L, \eta) \in D = \overline{C} \times \mathbb{P}^1$ be a point, and let $E = \mathbf{f}(L, \eta) \in X$ be the corresponding rank two bundle. Then the differential*

$$d\mathbf{f}_{(L,\eta)} : T_{D,(L,\eta)} \to T_{X,E}$$

has a non-trivial kernel if and only if $L \in \overline{C}$ maps to $\eta \in \mathbb{P}^1 = \mathbb{P}(H^1(C, \mathcal{O}_C))$ under the natural degree 32 map $\overline{C} \to C \to \mathbb{P}^1$.

Proof. Since $d\mathbf{f}_{(L,\eta)}$ is determined by its restricions to $T_{\overline{C},L}$ and $T_{\mathbb{P}^1,\eta}$ and since we saw that $d\mathbf{f}_{(L,\eta)|T_{\mathbb{P}^1,\eta}}$ is always injective, it suffices to characterize all points $(L, \eta) \in D$ for which the image $d\mathbf{f}_{(L,\eta)}(T_{\overline{C},L})$ is contained in the line $d\mathbf{f}_{(L,\eta)}(T_{\mathbb{P}^1,\eta}) \subset T_{X,E} = H^1(C, \mathrm{End}_0(E)) = H^1(C, \mathrm{End}_0(F))$.

In view of the modular interpretations of the restricted differentials in sections 4.4.1 and 4.4.2 this question is equivalent to the problem of characterizing all points $(t', \eta) \in C \times \mathbb{P}^1$ such that

$$\kappa^{\mathbb{F}}_{t'}(T_{C,t'}) \subset \ker\left[H^1(C, \mathrm{End}_0(F)) \to H^1(C, F(t'))\right]. \tag{22}$$

Here as before $\mathbb{F}$ is the extension

$$0 \longrightarrow \mathcal{O}_{C\times C}(-\Delta) \xrightarrow{a} \mathbb{F} \xrightarrow{b} \mathcal{O}_{C\times C} \to 0, \tag{23}$$

corresponding to η and $F = \mathbb{F}_{t'} := \mathbb{F}_{|C\times\{t'\}}$.

More invariantly the condition (22) can be rewritten as follows. The short exact sequence (23) induces a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-\Delta)) \arrow[r, "a\circ - \circ b"] \arrow[d, equal] & \mathrm{End}_0(\mathbb{F}) \arrow[r, "-\circ a"] & \mathrm{Hom}(\mathcal{O}(-\Delta), \mathbb{F}) \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}(-\Delta) & & \mathbb{F}(\Delta) & \end{tikzcd} \tag{24}$$

Pushing this down the map $-\circ a : \mathrm{End}_0(\mathbb{F}) \to \mathbb{F}(\Delta)$ via the second projection $p_2 : C\times C \to C$ gives us a map of coherent sheaves

$$R^1p_{2*}(-\circ a) : R^1p_{2*}\mathrm{End}_0(\mathbb{F}) \to R^1p_{2*}\mathbb{F}(\Delta).$$

We also have the universal traceless Kodaira-Spencer class of $\mathbb{F}$

$$\kappa^{\mathbb{F}} : T_C \to R^1p_{2*}\mathrm{End}_0(\mathbb{F}).$$

Composing these two maps we get a map

$$R^1 p_{2*}(- \circ a) \circ \kappa^{\mathbb{F}} : T_C \to R^1 p_{2*}\mathbb{F}(\Delta),$$

or equivalently a global section

$$R^1 p_{2*}(- \circ a) \circ \kappa^{\mathbb{F}} \in H^0\left(C, \Omega^1_C \otimes R^1 p_{2*}\mathbb{F}(\Delta)\right). \tag{25}$$

Note that the bundle $\mathbb{F}$ and the global section (25) depend only on the class $\eta \in \mathbb{P}(H^1(C, \mathcal{O}))$. The condition (22) is simply the condition that the point t' belongs to the zero locus of this section. Therefore we need to describe all pairs $(t', \eta) \in C \times \mathbb{P}^1$ for which the section (25) defined by η vanishes at the point t'.

To understand these pairs observe that $\mathcal{O}(-\Delta) \subset \mathbb{F}$ can be viewed as the family of filtered bundles $\mathcal{O}(-x) \subset \mathbb{F}_x$ parametrized by $x \in C$. In particular, the traceless Kodaira-Spencer class $\kappa^{\mathbb{F}}$ will preserve the filtration. In other words if ${}^{\text{filt}}\text{End}_0(\mathbb{F})$ denotes the bundle of traceless endomorphisms of $\mathbb{F}$ preserving the subbundle $\mathcal{O}(-\Delta)$, then the global traceless Kodaira-Spencer class $\kappa^{\mathbb{F}}$ is induced from the traceless part of the Kodaira-Spencer class

$${}^{\text{filt}}\kappa^{\mathbb{F}} : T_C \to R^1 p_{2*}{}^{\text{filt}}\text{End}_0(\mathbb{F})$$

classifying filtered deformations. This means that $\kappa^{\mathbb{F}}$ factors as

$$\begin{tikzcd} T_C \arrow[r, "{}^{\text{filt}}\kappa^{\mathbb{F}}"] \arrow[rr, bend right, "\kappa^{\mathbb{F}}"'] & R^1 p_{2*}{}^{\text{filt}}\text{End}_0(\mathbb{F}) \arrow[r] & R^1 p_{2*}\text{End}_0(\mathbb{F}). \end{tikzcd}$$

By definition

$${}^{\text{filt}}\text{End}_0(\mathbb{F}) = \{\varphi \in \text{End}(F) \mid b \circ \varphi \circ a = 0 \text{ and } \text{tr}(\varphi) = 0\},$$

and hence

$${}^{\text{filt}}\text{End}_0(\mathbb{F}) = \ker\left[\text{End}_0(\mathbb{F}) \xrightarrow{b \circ - \circ a} \mathcal{O}(\Delta)\right].$$

This implies that $R^1 p_{2*}(- \circ a) \circ \kappa^{\mathbb{F}}$ maps T_C into the subsheaf

$$\ker\left[R^1 p_{2*}\mathbb{F}(\Delta) \to R^1 p_{2*}\mathcal{O}(\Delta)\right] \subset R^1 p_{2*}\text{End}_0(\mathbb{F}.$$

But from the long exact sequence of p_2 direct images of the short exact sequence

$$0 \to \mathcal{O} \to \mathbb{F}(\Delta) \to \mathcal{O}(\Delta) \to 0$$

it follows immediately that

$$\ker\left[\mathrm{End}_0(\mathbb{F})\xrightarrow{b\circ -\circ a}\mathcal{O}(\Delta)\right] = (H^1(C,\mathcal{O})/\eta)\otimes\mathcal{O}_C.$$

This imlplies that the section $R^1p_{2*}(-\circ a)\circ\kappa^{\mathbb{F}}$ in $\omega_C\otimes R^1p_{2*}\mathbb{F}(\Delta)$ is in fact the tautological section in the natural subbundle

$$\omega_C\otimes\big((H^1(C,\mathcal{O})/\eta)\otimes\mathcal{O}_C)\big) = (H^1(C,\mathcal{O})/\eta)\otimes\omega_C,$$

i.e. is the element in

$$(H^1(C,\mathcal{O})/\eta)\otimes H^0(C,\omega_C) = (H^1(C,\mathcal{O})/\eta)\otimes H^1(C,\mathcal{O})^\vee$$

corresponding to the graph of the quotient map $H^1(C,\mathcal{O})\to H^1(C,\mathcal{O})/\eta$. This is precisely the section of ω_C that vanishes at the preimage of η under the hyperelliptic map $\mathsf{h}_C : C\to\mathbb{P}^1$. Therefore the set of pairs (t',η) we were seeking is just the graph of the hyperelliptic map and so the set of points (L,η) at which $\mathbf{f}$ is not immersive is the graph of the projection $\overline{C}\to C\to\mathbb{P}^1$. This completes the proof of the proposition. ☐

4.5 The trigonal cover of a general line

Recall from [New68, Theorem 2], [NR69, Theorem 5] that the variety of lines on X is isomorphic to the Jacobian $\mathrm{Jac}^0(C)$ of C. Taking up the theme used in the proof of Lemma 4.4 in more detail, consider a general line $\ell\subset X$. It corresponds to a degree 0 line bundle $A\in\mathrm{Jac}^0(C)$, and in terms of vector bundles the line is the set of isomorphism classes of nonzero extensions of the form

$$0\to A\to V\to A^\vee(\mathbf{p})\to 0.$$

A point corresponding to a given extension, will lie on a different line ℓ' corresponding to $A'\in\mathrm{Jac}^0(C)$ distinct from A, if and only if there is a map $A'\to A^\vee(\mathbf{p})$ such that the pullback extension splits. Such a map corresponds to a point $t\in C$ with $A' = A^\vee(\mathbf{p}-t)$.

The splitting of the extension may be analyzed as follows. We have

$$\mathrm{Ext}^1(A^\vee(\mathbf{p}),A)\cong H^1(C,A^{\otimes 2}(-\mathbf{p}))$$

which is dual to $H^0(C,A^{\otimes -2}(3\mathbf{p}))$. These spaces have dimension 2, so a nonzero element of $H^1(C,A^{\otimes 2}(-\mathbf{p}))$ modulo scalars also corresponds to a nonzero element of $H^0(C,A^{\otimes -2}(3\mathbf{p}))$ modulo scalars, the unique element that pairs with it to zero by the duality.

The extension splits when pulled back to $A^\vee(\mathbf{p}-t)$ if and only if the class maps to zero under the map

$$H^1(C, A^{\otimes 2}(-\mathbf{p})) \to H^1(C, A^{\otimes 2}(t-\mathbf{p})),$$

or equivalently if the dual element is in the image of the map

$$H^0(C, A^{\otimes -2}(3\mathbf{p}-t)) \to H^0(C, A^{\otimes -2}(3\mathbf{p})).$$

In turn, this is equivalent to saying that our dual element, viewed as a section of the line bundle $A^{\otimes -2}(3\mathbf{p})$, vanishes at t.

In conclusion, each extension corresponds to a class (up to scalar multiples) of nonzero sections of $A^{\otimes -2}(3\mathbf{p})$, and the other lines ℓ' passing through the point determined by the extension correspond to the points where this section vanishes.

We may view this as saying that the complete linear system associated to the line bundle $A^{\otimes -2}(3\mathbf{p})$ of degree 3 determines a trigonal map $C \to \ell \cong \mathbb{P}^1$, and the other three lines through a point on ℓ correspond to the fibers.

Proposition 4.14. *The trigonal cover ramifies over* 8 *points in* ℓ*. These are the* 8 *intersection points of* ℓ *with the wobbly locus. If* x *is such an intersection point, and if we write the divisor in* C *as* $2u+v$ *then* u *corresponds to the line through that point in the wobbly locus.*

Proof. If A is general then the line bundle $A^{\otimes -2}(3\mathbf{p})$ is general and hence has no base points. For such an A the trigonal map will be a morphism. The Hurwitz formula implies that there are 8 ramification points. A general fiber of the trigonal cover is a section of $A^{\otimes -2}(3\mathbf{p})$ that vanishes at three points $u, v, w \in C$. A simple ramification is a point u such that there is a section of $A^{\otimes -2}(3\mathbf{p})$ that vanishes at $2u+v$. We get $A^{\otimes -2}(3\mathbf{p}) \cong \mathcal{O}_C(2u+v)$. As discussed above, the extension corresponding to this point of ℓ splits on

$$A^{-1}(\mathbf{p}-u) \cong A(u+v-2\mathbf{p}) \cong A(u-v')$$

where v' is the hyperelliptic conjugate of v, and we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & E \arrow[r] & A^{-1}(\mathbf{p}) \arrow[r] & 0 \\
 & & & A(u-v') \arrow[ul] \arrow[u] &
\end{tikzcd}$$

and the bundle fits in an extension

$$0 \to A(u - v') \to E \to A^{-1}(3\mathbf{p} - u - v) \cong A(u) \to 0.$$

Now $A(u - v') \otimes \omega_C = A(2\mathbf{p} + u - v') = A(u + v)$ so we get a nonzero map

$$A(u) \to A(u + v) \cong A(u - v') \otimes \omega_C$$

yielding a nonzero nilpotent Higgs field. Therefore, the bundle E associated to the given extension is in the wobbly locus. □

Remark: We already argued synthetically in section 2.2.3 that the line ℓ goes through the cuspidal locus of the wobbly locus when the divisor in C has the form $3u$.

Choose a general line $\ell \subset X = X_1$. Over this line we have the trigonal cover $C \to \ell$ associating to each point $x \in \ell$ the set of three lines passing through x that are distinct from ℓ. Let u, v, w be the points of C over $x \in \ell$.

Let $Y_\ell := Y \times_X \ell$. It is a degree 8 ramified cover of ℓ.

Lemma 4.15. *Assume x general in ℓ. The fiber of $Y_\ell \to \ell$ over the point x is naturally isomorphic to the set of 8 points obtained by choosing $\tilde{u}, \tilde{v}, \tilde{w} \in \widetilde{C}$ over $u, v, w \in C$ respectively.*

Proof. Let L be the line bundle of degree 0 on C corresponding to the line ℓ, and let E be the rank 2 vector bundle corresponding to x. It fits into an exact sequence

$$0 \to L \to E \to L^{-1}(\mathbf{p}) \to 0$$

whose extension class corresponds to the point in ℓ. The fiber of Y over x is the set of line bundles U on $\widetilde{C}$, of degree 3, such that $\pi_*(U) \cong E$. The map $L \to \pi_*(U)$ corresponds by adjunction to a map $\pi^*(L) \to U$. We get

$$U \cong \pi^*(L) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{u} + \tilde{v} + \tilde{w}).$$

Let u, v, w be the images of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ in C. The determinant of $\pi_*(U)$ is $L^{\otimes 2} \otimes \mathcal{O}_C(u+v+w) \otimes \omega_C^{-1}$ so we get

$$L^{\otimes 2}(u + v + w) \cong \mathcal{O}_C(3\mathbf{p}).$$

It means that $u + v + w$ is in the linear system $|L^{\otimes -2}(3\mathbf{p})|$, in other words it is one of the fibers of the trigonal cover of ℓ. To show that it is the fiber over the point corresponding to the given extension defining E, we need to see that the extension splits when restricted to $L^{-1}(\mathbf{p}-u)$ (and the same for v, w). For this, in turn, it suffices to see that there is a nonzero map

$$\pi^*(L^{-1}(\mathbf{p}-u)) \to U.$$

Using the expression

$$L^{-1}(\mathbf{p}-u) = L \otimes (L^{\otimes -2}(3\mathbf{p})) \otimes \mathcal{O}_C(-2\mathbf{p}-u) = L \otimes \mathcal{O}_C(-2\mathbf{p}-u+u+v+w) = L(v+w-2\mathbf{p}),$$

we are looking for a map

$$\pi^*(L)(\tilde{v}+\tau\tilde{v}+\tilde{w}+\tau\tilde{w}) \to \pi^*(L \otimes \omega_C) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{u}+\tilde{v}+\tilde{w}). \tag{26}$$

Here, as usual, τ denotes the covering involution of the spectral cover $\pi : \widetilde{C} \to C$.

Having a map (26) is equivalent to asking for a section of

$$\pi^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{u}-\tau\tilde{v}-\tau\tilde{w}).$$

The ramified cover $\pi : \widetilde{C} \to C$ has 4 ramification points $a, a', b, b' \in \widetilde{C}$ such that $\pi(a), \pi(a')$ and $\pi(b), \pi(b')$ are opposite pairs under the hyperelliptic involution of C.

Recall now that $\widetilde{C}$ itself is hyperelliptic of genus 5, with the map $\mathsf{h}_{\widetilde{C}} : \widetilde{C} \to \mathbb{P}^1$ branched over the 12 inverse images in $\widetilde{C}$ of the 6 Weierstrass points of C. We have $\omega_{\widetilde{C}} = \mathsf{h}_{\widetilde{C}}^*\mathcal{O}_{\mathbb{P}^1}(4)$. The hyperelliptic involution σ of $\widetilde{C}$ is one of the lifts of the hyperelliptic involution of C, so a, a' and b, b' are also opposite pairs under the hyperelliptic involution of $\widetilde{C}$. It follows that $\mathcal{O}_{\widetilde{C}}(a+a') \cong \mathcal{O}_{\widetilde{C}}(b+b') \cong \mathsf{h}_{\widetilde{C}}^*\mathcal{O}_{\mathbb{P}^1}(1)$. We get

$$\pi^*(\omega_C) \cong \omega_{\widetilde{C}}(-a-a'-b-b') = \mathsf{h}_{\widetilde{C}}^*\mathcal{O}_{\mathbb{P}^1}(2).$$

In particular, $\pi^*(\omega_C)$ admits a section vanishing at the two points $\tau\tilde{v}, \tau\tilde{w}$. This gives the required section of $\pi^*(\omega_C)(\tilde{u}-\tau\tilde{v}-\tau\tilde{w})$.

This (together with the analogous arguments for v and w) completes the proof that the divisor $u+v+w$ is the fiber of the trigonal map over $x \in \ell$. Thus, our line bundle U corresponds to one of the choices of 8 liftings as in the lemma.

Running this argument backwards shows that each of the 8 liftings corresponds to a choice of U. This gives the isomorphism claimed in the lemma. $\square$

Let $Y_\ell := Y \times_X \ell$. It is a degree 8 ramified cover of ℓ whose general fiber is described by the lemma. From this, we can describe the ramification of Y_ℓ/ℓ:

Corollary 4.16. *Suppose ℓ is general. The branch locus of Y_ℓ/ℓ consists two disjoint groups of points: the $x \in \ell$ of the branch locus of C/ℓ, plus the points that are images of the four points $y \in C$ of the branch locus of $\widetilde{C}/C$. The former class of points is the intersection of ℓ with the wobbly locus. As ℓ is general, the second group consists of four distinct points.*

Proof. The line ℓ gives a trigonal map $C \to \mathbb{P}^1$, and by Lemma 4.15, the fiber of Y_ℓ over a point $x \in \ell$ consists of the 8 liftings to $\widetilde{C}$ of the three points u, v, w in the trigonal fiber over x.

When x is a branch point of C/ℓ, as we have seen in Proposition 4.14 it means $x \in \ell \cap \mathrm{Wob}$, and two of the three points come together, let's say $u = v$. The monodromy around such a point interchanges u and v. We may assume that $u = v$ is general in C, in particular it isn't a ramification point of $\widetilde{C}/C$ so we may identify the two sheets of $\widetilde{C}$ labeled with $0, 1$ near this point, similarly near w. With these identifications we may write the 8 points as (a, b, c) where $a, b, c \in \{0, 1\}$. The monodromy sends (a, b, c) to (b, a, c). This has two transpositions $(1, 0, c) \leftrightarrow (0, 1, c)$ for $c = 0, 1$.

Suppose x is the image of one of the branch points $z \in C$ of $\widetilde{C}/C$. For ℓ general we may assume that x isn't wobbly. The trigonal fiber is u, v, w with, say, u the branch point. As we move around such a point, the set of choices of lifting of u undergoes a transposition; there are four such transpositions corresponding to the various liftings of v, w.

The last statement, that was also mentioned in the previous paragraph, is that two different branch points of $\widetilde{C}/C$ are not contained in the same trigonal fiber. Indeed, the branch points consist of two general pairs of opposite points under the hyperelliptic involution. If the trigonal map identified two such points, they would have to be two opposite points, but that would mean that the trigonal map identifies hyperelliptically opposite points, hence that it factors through the hyperelliptic map, which isn't the case. $\square$

Lemma 4.17. *Suppose V is a vector bundle corresponding to a general point in X. Let $A_1, \ldots, A_4$ be the points in $\mathrm{Jac}^0(C)$ corresponding to the four lines in X through V. Then the sum of these points in $\mathrm{Jac}^0(C)$ is the origin of $\mathrm{Jac}^0(C)$.*

Proof. Let $A = A_1$ be one of the line bundles, so we have

$$0 \to A \to V \to A^\vee(\mathbf{p}) \to 0.$$

As we have seen above, the other A_i are of the form $A_i = A^\vee(\mathbf{p} - t_i)$ where $t_2 + t_3 + t_4$ is a divisor in the linear system $A^{\otimes -2}(3\mathbf{p})$. This yields the equation in $\mathrm{Jac}(C)$

$$t_2 + t_3 + t_4 = [A^{\otimes -2}(3\mathbf{p})] = 3\mathbf{p} - 2[A].$$

Now, the sum of all the points is the sum in $\mathrm{Jac}(C)$

$$\begin{aligned}\text{sum} &= [A] + [A^\vee(\mathbf{p} - q_2)] + [A^\vee(\mathbf{p} - t_3)] + [A^\vee(\mathbf{p} - t_4)] \\ &= [A] + 3[A^\vee] + 3\mathbf{p} - (t_2 + t_3 + t_4) \\ &= [A] - 3[A] + 3\mathbf{p} - (3\mathbf{p} - 2[A]) = 0.\end{aligned}$$

$\square$

Given two of the four lines corresponding to line bundles A_1 and A_2, we obtain a point $q \in C$ such that $A_2 = (A_1)^\vee(\mathbf{p} - q)$, in other words $A_1 \otimes A_2 = \mathcal{O}_C(\mathbf{p} - q)$. In view of the lemma, if A_3 and A_4 are the other two points, we have $A_1 \otimes A_2 \otimes A_3 \otimes A_4 = \mathcal{O}_C$ so $A_3 \otimes A_4 = \mathcal{O}_C(q - \mathbf{p}) = \mathcal{O}_C(\mathbf{p} - q')$. Thus, the point of C corresponding to A_3, A_4 is q' the image of q by the hyperelliptic involution.

Remark 4.18. In particular, we obtain the conclusion that the point $V \in X_1$ yields a well-defined triple of points in $\mathbb{P}^1$. This gives a map $X_1 \to Sym^3(\mathbb{P}^1) \cong \mathbb{P}^3$ that will be useful in Section 13.5.

4.6 Local structure of the spectral cover along the cusp locus

Let $R^{\text{mov}} \subset Y$ be the movable locus of the ramification of the map $Y \to X$.

Recall that $c_1(\Omega^1_X) = 2H$ and $c_1(\Omega^1_Y) = \boldsymbol{E}$ since Y is obtained from the abelian variety $\mathcal{P}$ by blowing up a curve of codimension 2. Comparing these, we conclude that the ramification locus has class $\boldsymbol{E} + 2\boldsymbol{F}$.

Lemma 4.19. *The order of ramification of $Y \to X$ along $\boldsymbol{E}$ is two (i.e. it has simple ramification at a general point of $\boldsymbol{E}$).*

Proof. If we assume that the map has ramification of order a along $\boldsymbol{E}$, we get

$$2\boldsymbol{E} + \boldsymbol{F} = (a-1)\boldsymbol{E} + [R^{\text{mov}}].$$

On the other hand, R^{mov} is effective, so $2\boldsymbol{F} - (a-2)\boldsymbol{E}$ is effective. If $a > 2$ this says that $2\boldsymbol{F} - \boldsymbol{E}$ is effective. We note that $2\boldsymbol{F} - \boldsymbol{E}$ is not zero, since $\boldsymbol{E}$ is clearly not ample whereas $\boldsymbol{F}$ is ample (it is the pullback of an ample divisor by a finite map).

Since $\boldsymbol{F}$ is ample, if $2\boldsymbol{F} - \boldsymbol{E}$ is effective then $(2\boldsymbol{F} - \boldsymbol{E}) \cdot \boldsymbol{F}^2 > 0$. However, from Proposition 4.8 saying $\boldsymbol{F}^3 = 32$ and $\boldsymbol{E}\boldsymbol{F}^2 = 64$ it gives $(2\boldsymbol{F} - \boldsymbol{E}) \cdot \boldsymbol{F}^2 = 0$, contradicting the effectivity. $\square$

Proposition 4.20. *For generic choice of point in the Hitchin base, the movable ramification locus $R^{\rm mov}$ intersects $\boldsymbol{E}$ in a union of two constant sections of the fibration $\boldsymbol{E} \to \widehat{C}$ and 64 fibers. The constant sections correspond to the two points in $\mathbb{P}^1$ whose inverse images in C make up the 4 branch points of $\widetilde{C}/C$.*

Proof. From above, the full ramification locus has class $\boldsymbol{E} + 2\boldsymbol{F}$ and the fixed part has multiplicity 1 along $\boldsymbol{E}$. Thus, the class of $R^{\rm mov}$ is $2\boldsymbol{F}$.

We use the description of Corollary 4.16. We are interested here in points where the movable ramification meets the intersection with the wobbly locus.

We can describe more precisely, over points where ℓ meets the wobbly locus, the fiber of Y_ℓ. Suppose it is a general point of the wobbly locus, so the degree 3 divisor in C over $x \in \ell$ has the form $2u+v$. Suppose u and v are not in the branch locus of $\widetilde{C}/C$. To describe the liftings to points of Y_ℓ we need to consider liftings of nearby divisors into $\widetilde{C}$: the piece $2u$ splits into $u_1 + u_2$ and this has 4 liftings. The monodromy action permutes two of those liftings and leaves fixed the two other ones, depending on whether $\tilde{u}_1$ and $\tilde{u}_2$ are in different or the same sheets of $\widetilde{C}/C$. Then each of those configurations becomes doubled depending on the lifting of v. We obtain 2 branch points and 4 unbranched points of Y_ℓ/ℓ over $x \in \ell$. This is as expected.

The points of $\boldsymbol{E}_\ell := \boldsymbol{E} \cap Y_\ell$ are the two branch points.

On the other hand, the points of the movable ramification locus correspond to divisors $u_1 + u_2 + v$ such that one of those points is a branch point of $\widetilde{C}/C$. When that branch point is v it does not correspond to a point of $\boldsymbol{E}_\ell$. We would like to describe the cases where u is a branch point of $\widetilde{C}/C$.

Suppose given a divisor $2u + v$ in the linear system associated to $A^{\otimes -2}(3\mathbf{p})$. Then as we saw above, the other line bundle of degree 0 associated to u, that we'll now denote $L := A'$, is of the form

$$L = A^\vee(\mathbf{p} - u).$$

Our bundle V is an up-Hecke of $A \oplus L$ at the point u. We can write

$$0 \to L \to V \to L^\vee(\mathbf{p}) \to 0$$

and $A = L^\vee(\mathbf{p} - u)$. The extension is the one that vanishes when pulled back to A, or equivalently vanishing in the extension group $H^1(L^{\otimes 2}(u - \mathbf{p}))$.

We have

$$L^{\otimes 2}(\mathbf{p}) = A^{\otimes -2}(3\mathbf{p} - 2u)$$

but

$$A^{\otimes -2}(3\mathsf{p}) \cong \mathcal{O}_C(2u+v)$$

so

$$L^{\otimes 2}(\mathsf{p}) = \mathcal{O}_C(v).$$

This is the equation saying that the line associated to L is in the wobbly locus. Furthermore, the set of extension classes corresponding to points of this line is

$$H^1(L^{\otimes 2}(-\mathsf{p})) = H^1(\mathcal{O}_C(v-2\mathsf{p})) = H^1(\mathcal{O}_C(-v'))$$

which maps isomorphically to $H^1(\mathcal{O}_C)$. Here v' is the image of v by the hyperelliptic involution.

An extension vanishes at the point u if and only if it maps to zero in $H^1(\mathcal{O}_C(u))$, which is equivalent to saying that it corresponds to the point in $\mathbb{P}^1 = \mathbb{P}H^1(\mathcal{O}_C)$ that is the image of u under the hyperelliptic double cover.

We conclude that under the birational map from the wobbly locus to $\mathbb{P}^1$, the image of a point corresponding to the degree 3 divisor $2u+v$ with respect to the trigonal covering corresponding to the original line ℓ, it the image of u in $\mathbb{P}^1$ corresponding to the hyperelliptic map.

Now, we are interested in cases where u coincides with one of the branch points of $\widetilde{C}/C$. We know from Proposition 3.1 that there are 4 branch points in C that are the inverse images of two points in $\mathbb{P}^1$. Therefore, the points in question are ones where u maps to one of these two points in $\mathbb{P}^1$.

We conclude from this discussion that the points of $\boldsymbol{E}$ over the wobbly locus that map to one of these two points in $\mathbb{P}^1$ are in $R^{\mathrm{mov}} \cap \boldsymbol{E}$.

We have made some genericity assumptions in the above discussion. These hold for the points we have just described, since the two points in $\mathbb{P}^1$ are general because we chose a general spectral cover $\widetilde{C}/C$. This says that $R^{\mathrm{mov}} \cap \boldsymbol{E}$ contains at least two constant sections of the fibration $\boldsymbol{E} \to \widehat{C}$.

We now rely on the intersection number calculations to conclude. Indeed, we have seen that $\boldsymbol{F} \cap \boldsymbol{E}$ has the class of $\hat{\mathfrak{c}} + 32\mathsf{fib}$ where $\hat{\mathfrak{c}}$ is a constant section. This may be re-verified as follows: this class intersects the fiber in one point, since the fiber maps to a line in X that meets the hyperplane class in one point. Our class is thus of the form $\hat{\mathfrak{c}} + a \cdot \mathsf{fib}$. On the other hand, the self-intersection of this class on $\boldsymbol{E} \cong \mathbb{P}^1 \times \widehat{C}$ is equal to the intersection number $\boldsymbol{F}^2\boldsymbol{E}$ on Y; that is 64. The self intersection of $\hat{\mathfrak{c}} + a \cdot \mathsf{fib}$ is $2a$ giving $a = 32$.

This shows that $2\boldsymbol{F}\cap\boldsymbol{E}$ is a divisor in the class $2\hat{\mathsf{c}}+64\mathsf{fib}$. Since we have already exhibited a subset of this divisor consisting of two sections, we conclude that it has to be of the form two sections plus 64 fibers. This completes the proof of the proposition. □

For a point b in the Hitchin base, let $\mathrm{Branch}^{\mathrm{mov}}(\mathsf{b}) \subset X$ be the image in X of the ramification locus $R^{\mathrm{mov}} \subset Y$ for the point b. These form a family of closed subvarieties of X.

We define the *base locus* of this family to consist of the set of points $x \in X$ that lie in $\mathrm{Branch}^{\mathrm{mov}}(\mathsf{b})$ for b general with respect to x, or equivalently for all b.

Corollary 4.21. *If the base locus contains an irreducible subvariety of codimension ≤ 2 then it is one of the lines in the wobbly locus.*

Proof. If $x \in X$ is not on the wobbly locus then it is a very stable bundle, and the space of Higgs fields on that bundle maps properly to the Hitchin base. Thus, the intersection with a general Hitchin fiber over b is transverse, so x is not in the branch locus of the map $Y \to X$ for general b.

Suppose x is in the wobbly locus and let S be the space of Higgs fields on the corresponding bundle. It maps to the Hitchin base, by a map that is not proper. However, for a general point b in the Hitchin base, the intersection of the fiber over b with S is transverse. Therefore, for such a general b, if $y \in Y$ is a point over x that is not in $\boldsymbol{E}$, then it is in $S \cap \boldsymbol{h}^{-1}(\mathsf{b})$ hence not in R^{mov}. This shows that the points of R^{mov} lying over x are in $R^{\mathrm{mov}} \cap \boldsymbol{E}$.

We have seen in the proposition that for b general, the intersection of R^{mov} with $\boldsymbol{E}$ consists of two constant sections of the map $\boldsymbol{E} = \widehat{C} \times \mathbb{P}^1 \to \widehat{C}$, that move as a function of b, and of 64 fibers of the map $\boldsymbol{E} = \widehat{C} \times \mathbb{P}^1 \to \widehat{C}$. The images of these in the wobbly locus have the same description.

So, suppose x is a point of the wobbly locus contained in the image of R^{mov} for two different general values of b. Then it is in the intersection of two subsets of the form 2 sections plus images of 64 fibers; but the sections move, so x has to be contained in at most one of the 64 lines. This completes the proof of the corollary. □

Corollary 4.22. *A general point on the cuspidal locus of the wobbly locus is not contained in the movable part of the branch locus.*

Proof. This follows immediately from the previous corollary since the cuspidal locus has codimension 2 and is not a union of fibers. □

Corollary 4.23. *The "movable part" of the ramification locus does really move, in the sense that the base locus does not contain any divisors.*

Proof. The above corollary implies that the codimension of the base locus is ≤ 2. □

4.7 The step in the parabolic filtration

Recall that $\mathcal{V}_{a,b} = f_*(\mathcal{L}_{a,b})$ and consider the sheaf

$$\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1}$$

which is supported on Wob $\subset X$ by definition. Over X° this is the quotient that will be used to define the crude parabolic structure. For the computations, it will be necessary to have a refined definition that holds over all of X. Recall that $\mathbf{f} : D \to X$ denotes the map from the normalization D of Wob.

Lemma 4.24. *There is a quotient defined over D,*

$$\mathbf{f}^*(\mathcal{V}_{a,b}) \to \mathcal{U}_{a,b}$$

such that $\mathcal{U}_{a,b}$ is a rank 2 torsion-free sheaf on D, hence a bundle outside of a finite collection of points. Over X° (this restriction being denoted by a superscript as usual) we have $(\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1})\,|_{X^\circ} = \mathbf{f}_*\mathcal{U}^\circ_{a,b}$ as quotients of $\mathcal{V}^\circ_{a,b}$ and this characterizes $\mathcal{U}_{a,b}$.

Proof. Start by noting that the property of the lemma uniquely characterizes $\mathcal{U}_{a,b}$. Indeed, the condition at the end states that the kernel of the map $\mathbf{f}^*(\mathcal{V}_{a,b}) \to \mathcal{U}_{a,b}$ is, over X° (where $D^\circ = \text{Wob}^\circ \subset X$) the subsheaf of sections generated by $\mathcal{V}_{a,b-1}$. The condition that the quotient $\mathcal{U}_{a,b}$ is a torsion-free sheaf means that the kernel is a saturated subsheaf, so it is defined by its restriction to D°.

This discussion tells us how to construct $\mathcal{U}_{a,b}$, namely set

$$\mathcal{U}^\circ_{a,b} := \mathbf{f}^*\left((\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1})\,|_{X^\circ}\right) = (\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1})\,|_{X^\circ}$$

and then let $\mathcal{U}_{a,b}$ be the unique torsion-free quotient of $\mathbf{f}^*\mathcal{V}_{a,b}$ that restricts to $\mathcal{U}^\circ_{a,b}$ over D°. It is constructed by taking the saturated subsheaf extending the kernel and taking the quotient.

For this discussion, one should note that $\mathcal{V}_{a,b}$ is a reflexive sheaf on X, indeed it is the dirct image of a line bundle under a map from a normal (and indeed smooth) variety to X.

In particuler, $\mathcal{V}_{a,b}$ is a vector bundle outside of codimension 3, therefore $\mathbf{f}^*(\mathcal{V}_{a,b})$ is a vector bundle outside of codimension 2 on D.

The fact that $\mathcal{U}_{a,b}$ has rank 2 will become apparent from the next lemma. $\square$

Lemma 4.25. *Let $f_{\boldsymbol{E}/D} : \boldsymbol{E} \to D$ be the morphism induced by f. That is $f_{|\boldsymbol{E}} : \boldsymbol{E} \to X$ factors as*

$$\boldsymbol{E} \xrightarrow{f_{\boldsymbol{E}/D}} D \xrightarrow{\mathbf{f}} X.$$

Then we have

$$\mathcal{U}_{a,b} = f_{\boldsymbol{E}/D,*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}}).$$

Proof. Over X°, we have $D^\circ = \mathrm{Wob}^\circ \subset X^\circ$ and $\mathbf{f}_*\mathcal{U}^\circ_{a,b} = \mathcal{U}^\circ_{a,b} = f_*(\mathcal{L}^\circ_{a,b})/f_*(\mathcal{L}^\circ_{a,b}(-\boldsymbol{E}^\circ))$. This quotient is equal, in turn, to $f_*\left(i_{\boldsymbol{E}^\circ,*}\left(\mathcal{L}^\circ_{a,b}|_{\boldsymbol{E}^\circ}\right)\right)$, where $i_{\boldsymbol{E}^\circ} : \boldsymbol{E}^\circ \to Y$ is the natural inclusion. Over these smooth points $\mathbf{f}_*$ induces an equivalence between sheaves on D° and sheaves on X° supported on D°. Also, $\mathbf{f} \circ f_{\boldsymbol{E}/D} = f \circ i_{\boldsymbol{E}}$ so

$$\mathbf{f}_* f_{\boldsymbol{E}^\circ/D^\circ,*}(\mathcal{L}^\circ_{a,b}|_{\boldsymbol{E}^\circ}) = f_*(i_{\boldsymbol{E}^\circ,*}\mathcal{L}^\circ_{a,b}|_{\boldsymbol{E}^\circ}).$$

From the equivalence we conclude that

$$f_{\boldsymbol{E}^\circ/D^\circ,*}(\mathcal{L}^\circ_{a,b}|_{\boldsymbol{E}^\circ}) = \mathcal{U}^\circ_{a,b}.$$

This gives the claimed statement over D°.

Next, consider the commutative diagram

$$\begin{tikzcd}
\boldsymbol{E} \arrow[r] \arrow[dr] & Y \times_X D \arrow[r] \arrow[d] & Y \arrow[d] \\
& D \arrow[r] & X
\end{tikzcd}$$

we see that $\mathcal{V}_{a,b} = f_*(\mathcal{L}_{a,b})$ restricts over D to the direct image under the middle downward map, and this maps to $f_{\boldsymbol{E}/D,*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}})$. In other words, we have a map expressing the relationship in question. This map is an isomorphism over D° by the preceding discussion. Since $\boldsymbol{E}$ is smooth $f_{\boldsymbol{E}/D,*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}})$ is torsion-free—in fact it is a bundle, by "miracle flatness". Also, the map $\boldsymbol{E} \to Y \times_X D$ is a closed embedding so

$$f_*(\mathcal{L}_{a,b}) \to f_{\boldsymbol{E}/D,*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}})$$

is surjective. It follows that the right hand side is the unique torsion-free quotient sheaf extending the quotient $\mathcal{U}^\circ_{a,b}$ already known over D°. This completes the proof that the right hand side is $\mathcal{U}_{a,b}$. Notice that we obtain $\mathcal{U}_{a,b}$ as a torsion free sheaf of rank 2 because $\boldsymbol{E} \to D$ is a $2:1$ covering. $\square$

The next step is to use this characterization to calculate the Chern character of $\mathcal{U}_{a,b}$. This calculation may be truncated at codimension 1 on D since that corresponds to codimension 2 on X.

Proposition 4.26. *The Chern character of $\mathcal{U}_{a,b}$ truncated to codimension 1 on D is*

$$\mathrm{ch}(\mathcal{U}_{a,b}) = 2 + (2a+b)H_D - (3b+2)H_D^{\perp}$$

where $H_D^{\perp} = [\overline{C}] - 16[\mathbb{P}^1]$ is a class whose direct image to X vanishes.

Proof. Write

$$H_D := \mathbf{f}^*(H) = \alpha[\overline{C}] + \beta[\mathbb{P}^1].$$

Noting that $H_D \cdot [\mathbb{P}^1] = 1$ gives $\alpha = 1$, then noting that

$$H_D \cdot H_D = H^2 \cdot \mathrm{Wob} = 32$$

(see Lemma 4.4), we get $2\alpha\beta = 32$ so $\beta = 16$. Thus

$$H_D = \mathbf{f}^*(H) = [\overline{C}] + 16[\mathbb{P}^1].$$

We have

$$\mathbf{f}_*[\overline{C}] \cdot H = [\overline{C}] \cdot H_D = 16$$

so (recalling $H^3 = 4$) we get $\mathbf{f}_*[\overline{C}] = 4H^2$. We have that $\mathbf{f}_*[\mathbb{P}^1]$ is the class of a line in X, this is $H^2/4$. We get

$$\mathbf{f}_*(H_D) = 8H^2.$$

The class $H_D^{\perp} := [\overline{C}] - 16[\mathbb{P}^1]$ has the property that $\mathbf{f}_*(H_D^{\perp}) = 0$.

Turn to the question of the proposition. Consider the diagram

$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "i_{\boldsymbol{E}}"] \arrow[d, "f_{\boldsymbol{E}/D}"'] & Y \arrow[d, "f"] \\ D \arrow[r, "\mathbf{f}"'] & X. \end{tikzcd}$$

We have

$$i_{\boldsymbol{E},*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}}) = \mathcal{L}_{a,b}/\mathcal{L}_{a,b-1}.$$

Applying f_* to this gives $\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1}$.

Recall that $\mathcal{U}_{a,b} := f_{\boldsymbol{E}/D,*}(\mathcal{L}_{a,b}|_{\boldsymbol{E}})$. From the diagram,

$$\mathbf{f}_*(\mathcal{U}_{a,b}) \cong \mathcal{V}_{a,b}/\mathcal{V}_{a,b-1}.$$

Therefore, in order to calculate $\mathrm{ch}(\mathcal{U}_{a,b})$ we would like to know the relationship between it and $\mathrm{ch}(\mathbf{f}_*(\mathcal{U}_{a,b})$. Some further information will be needed due to the fact that the codimension 1 cycle class group of $D \cong \overline{C} \times \mathbb{P}^1$ does not inject into that of X.

First apply the Grothendieck-Riemann-Roch formula for $\mathbf{f}$. Recall that

$$\mathrm{td}(TX)^{-1} = 1 - H + 5H^2/12.$$

Truncated to codimension 1 we get $\mathbf{f}^*\mathrm{td}(TX)^{-1} = 1 - H_D$. On the other hand, $2g_{\overline{C}} - 2 = 32$ so $c_1(TD) = 2[\overline{C}] - 32[\mathbb{P}^1]$, and

$$\mathrm{td}(TD) = 1 + [\overline{C}] - 16[\mathbb{P}^1].$$

The Grothendieck-Riemann-Roch formula says

$$\begin{aligned}\mathrm{ch}(\mathbf{f}_*(\mathcal{U}_{a,b}) &= \mathbf{f}_*(2 + c_1(\mathcal{U}_{a,b}))(1 + [\overline{C}] - 16[\mathbb{P}^1])(1 - H_D)\\ &= \mathbf{f}_*\left(2 + c_1(\mathcal{U}_{a,b}) + 2[\overline{C}] - 32[\mathbb{P}^1] - 2[\overline{C}] - 32[\mathbb{P}^1]\right).\end{aligned}$$

Noting that $\mathbf{f}_*(2[D]) = 2[\mathrm{Wob}]$ and recalling from Lemma 4.4 that $[\mathrm{Wob}] = 8H$, then also using $\mathbf{f}_*[\mathbb{P}^1] = H^2/4$, we get

$$\mathrm{ch}(\mathbf{f}_*(\mathcal{U}_{a,b}) = 16H + i_*c_1(\mathcal{U}_{a,b}) - 16H^2.$$

Denote by Δ_b the operation on a function f, $\Delta_b(f) := f(a,b) - f(a,b-1)$. Thus for example $\Delta_b(b) = 1$ and $\Delta_b(b^2) = 2b - 1$ With this notation,

$$\begin{aligned}\mathrm{ch}(\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1}) &= \Delta_b\left[8 + (8a + 16b)H + (4a^2 + 16ab + 4b^2 - 12b - 2)H^2\right]\\ &= 16H + (16a + 8b - 16)H^2.\end{aligned}$$

The comparison $\mathrm{ch}(i_*(\mathcal{U}_{a,b}) = \mathrm{ch}(\mathcal{V}_{a,b}/\mathcal{V}_{a,b-1})$ yields

$$\mathbf{f}_*c_1(\mathcal{U}_{a,b}) = (16a + 8b)H^2.$$

In particular, it follows that

$$c_1(\mathcal{U}_{a,b}) = (2a+b)H_D + \xi H_D^{\perp}$$

for some ξ.

On the other hand, we can note that the inverse image in $\boldsymbol{E}$ of a line $\mathbb{P}^1 \subset D$ consists of two fibers in $\boldsymbol{E}$. A fiber maps to a line in X so its intersection with $\boldsymbol{F}$ has one point. On the other hand, $\boldsymbol{E}$ intersected with a fiber is -1. Thus, the bundle $\mathcal{L}_{a,b}$ restricted to a fiber has class $\mathcal{O}_{\text{fib}}(a-b-1)$ so its direct image to the line $\mathbb{P}^1 \subset D$ has $c_1 = 2c_1[\mathcal{O}_{\mathbb{P}^1}(a-b-1)]$. This tells us that

$$c_1(\mathcal{U}_{a,b}|_{\mathbb{P}^1}) = 2a - 2b - 2.$$

Thus

$$((2a+b)H_D + \xi H_D^{\perp}).[\mathbb{P}^1] = 2a - 2b - 2.$$

This gives

$$(2a+b) + \xi = 2a - 2b - 2$$

so $\xi = -2 - 3b$ and we get

$$c_1(\mathcal{U}_{a,b}) = (2a+b)H_D - (3b+2)H_D^{\perp}.$$

□

From this calculation we immediately get the following

Corollary 4.27. *Suppose $S \subset D$ is a curve in the class of kD_H, and take $2a+b=0$. Then $\mathcal{U}_{a,b}|_S$ is a rank 2 vector bundle of degree 0 on S.*

4.8 Hyperplane section

Let us now intersect the wobbly locus $\mathrm{Wob} \subset X$ with a general hyperplane section X_H, this denoting a smooth divisor in the class of H on X.

The class of the divisor H on D is H_D, described above as

$$H_D = [\overline{C}] + 16[\mathbb{P}^1].$$

One has $(H_D)^2 = 32$. We note also that H_D is the trace of X_H on D, in particular its image in X_H is $\mathrm{Wob} \cap X_H$. That is a curve Wob_H in X_H whose normalization is isomorphic to $\overline{C}$ since the projection $H_D \to \overline{C}$ is an isomorphism.

We would now like to count the double points of the curve Wob_H. The intersection of the curve with the 6 divisors

$$\overline{C} \times \{\mathbf{p}_i\}$$

are the points in the normalization $\overline{C}$ that get identified in pairs to form the double points of the image curve $\mathrm{Wob}_H = \mathrm{Wob} \cap X_H$. The number of points in the intersection of H_D with these 6 horizontal divisors is $6 \cdot 16 = 96$, so there are 96 points lying over the double points. We conclude:

- There are 48 double points on the curve Wob_H.

We now consider the genus of the normalization of Wob_H. The class of Wob_H is in $8H$ on X_H, so

$$(\mathrm{Wob}_H)^2 = (8H)^2_{X_H} = 64H^3 = 64 \cdot 4 = 256.$$

The genus of a smooth curve S in this class is given as follows: we recall that $K_{X_H} = -H$ so

$$K_S = K_{X_H} \cdot S + \mathcal{O}_S(S) = -H \cdot (8H)_{X_H} + (8H) \cdot (8H)_{X_H} = 56H^3 = 224.$$

This gives $2g_S - 2 = 224$ so $g_S = 113$.

If there were no singularities except the 48 double points, we would get the genus of the normalization of Wob_H to be $113 - 48 = 65$. However, the normalization is $\overline{C}$ that has genus 17. Hence, there are additional singularities accounting for a drop in the genus of $65 - 17 = 48$. We first guessed then proved:

- The curve Wob_H has its 48 nodes, plus 48 cusps.

The count of cusps comes from the structural result on the cuspidal locus in Proposition 4.5, proven in Subsection 4.4. The number of cusps of is obtained as follows. The structural result says that the cuspidal locus of Wob is the graph of a map $\overline{C} \to \mathbb{P}^1$ of degree 32. It intersects $[\mathbb{P}^1]$ in 1 point and $[\overline{C}]$ in 32 points so it intersects $H_D = [\overline{C}] + 16[\mathbb{P}^1]$ in 48 points.

4.9 Summary of properties of our given sheaves

We are defining a vector bundle $\mathcal{V}_{a,b}$ on X by taking the direct image of the line bundle $\mathcal{L}_0(a\boldsymbol{F} + (b+1)\boldsymbol{E})$ for a line bundle $\mathcal{L}_0$ of degree 0. We have calculated its Chern character in Proposition 4.12,

$$\mathrm{ch}(\mathcal{V}_{a,b}) = 8 + (8a + 16b)H + (4a^2 + 16ab + 4b^2 - 12b - 2)H^2.$$

It is instructive to calculate the Bogomolov Δ-invariant. Notice that modifying a corresponds to tensoring with a line bundle pulled back from X, so we may assume $a = 0$. Then

$$\begin{aligned}\Delta(\mathcal{V}_{0,b}) &= \frac{1}{2r}c_1^2 - \mathrm{ch}_2 \\ &= (16b^2 - (4b^2 - 12b - 2))H^2 \\ &= (12b^2 + 12b + 2)H^2.\end{aligned}$$

For integer values of b, the minimum of $2H^2$ is attained at both values $b = -1$ and $b = 0$. Symmetry considerations suggest that the desired parabolic weight should be $1/2$. with the parabolic structure creating an interpolation from $\mathcal{V}_{0,-1}$ to $\mathcal{V}_{0,0}$.

One can see, by the way, that it is not correct to just imagine a fractional value of b, indeed the above formula at $b = -1/2$ would give $\Delta = -H^2$ contradicting the Bogomolov-Gieseker inequality. Nonetheless, the symmetry consideration leads us to try using a parabolic weight of $1/2$ which will turn out to work, once the second Chern class is adjusted appropriately at the singularities.

Over D we have a morphism

$$\mathbf{f}^*(\mathcal{V}_{a,b}) \to \mathcal{U}_{a,b} \to 0$$

and the Chern character of $\mathcal{U}_{a,b}$ on D is

$$\mathrm{ch}(\mathcal{U}_{a,b}) = 2 + (2a + b)H_D - (3b + 2)H_D^{\perp}$$

by Proposition 4.26.

We would like to use this to define a parabolic structure along Wob. It will be easier now to pass to the hyperplane section X_H and consider the parabolic structure along Wob_H. Let D_H denote the normalization of Wob_H, and H_{D_H} the restriction of H to here.

Note that $H_D^{\perp}$ restricts to zero on the hyperplane section D_H, so

$$\mathrm{ch}(\mathcal{U}_{a,b})|_{D_H} = 2 + (2a + b)H_{D_H}.$$

It is not at all easy to understand how to calculate the contributions of cusps to the parabolic c_2. Taher's papers [Tah10, Tah13] consider a multiple point, rather than a cusp, and in any case the procedure—which could theoretically be implemented also for the cusp, or even imported by going to a covering and transforming the cusp to a triple point—is rather complicated.

Therefore, we shall adopt a shortcut: we will just look at the calculation for the parabolic weight $\alpha = 1/2$, and we will do that by going to a cover that is ramified with ramification index of 2 over the curve Wob_H. Then, on the cover rather than defining a parabolic structure we will use the quotient $\mathcal{U}$ to make a modification of the bundle. The construction follows the technique described in Proposition 3.15.

4.10 First parabolic Chern class

The first parabolic Chern class may be described by a simplified integral formula as just the average of $\mathrm{ch}_1(\mathcal{E}_a)$ over $a \in [0,1]$. In our situation of a crude parabolic structure with sheaves $\mathcal{V} = \mathcal{V}_{a,b}$ and $\mathcal{V}' = \mathcal{V}_{a,b-1}$ and parabolic weight $\alpha = 1/2$ the formula becomes

$$\begin{aligned}
\mathrm{ch}_1^{\mathrm{par}}(\mathcal{E}_\cdot) &= \int_{a=0}^{1} \mathrm{ch}_1(\mathcal{E}_a)da \\
&= \frac{\mathrm{ch}_1(\mathcal{V}') + \mathrm{ch}_1(\mathcal{V})}{2} \\
&= \frac{(8a + 16(b-1))H + (8a + 16b)H}{2} \\
&= (8a + 16b - 8)H.
\end{aligned}$$

We should choose $a = 1$ and $b = 0$ to give $\mathrm{ch}_1^{\mathrm{par}} = 0$. This is motivated by the symmetry consideration described above and the fact that we are looking for a parabolic structure with vanishing first Chern class.

However, recall that modifying a amounts to tensoring with a line bundle on $X = X_1$. We can do that later. For simplicity we will therefore assume $a = 0, b = 0$.

4.11 Going to a cover of X_H

In order to put this strategy into play, let us suppose we are given a finite ramified covering $g : Z \to X_H$ together with a ramification locus $R \subset Z$, such that $g^{-1}(\mathrm{Wob}_H) = 2R$, in other words g is ramified with ramification of order 2 along R over Wob_H.

We recall that the cusp is the downstairs branch locus of the quotient

$$\mathbb{C}^2 \to \mathbb{C}^2/S_3 = \mathbb{C}^2.$$

Upstairs, the branch locus consists of 3 crossed lines. Therefore, we may assume that this provides the local model for our covering $Z \to X_H$ over cusps of Wob_H. This local model is a covering of degree 6.

On the other hand, we will suppose that the local model for the covering over a double point, is just the product of two ramified coverings with ramification two. This local model is a covering of degree 4.

The covering Z may be constructed globally by the Kawamata covering trick [Kaw88]. There may be ramification elsewhere but that does not need to worry us.

Let d denote the degree of the covering $Z \to X_H$.

- Over the 48 cusps of Wob_H we will therefore have $48d/6 = 8d$ triple points of R.
- Over the 48 double points of Wob_H we will in turn have $48d/4 = 12d$ double points of R.

Let $\widetilde{R}$ denote the normalization of R. Let $\mathcal{V}_Z$ denote the restriction of $\mathcal{V} = \mathcal{V}_{a,b}$ to Z. We have a map, defined over $\widetilde{R}$:

$$\mathcal{V}_Z|_{\widetilde{R}} \to \mathcal{U}_{\widetilde{R}}.$$

Here $\mathcal{U}_{\widetilde{R}}$ is the pullback of $\mathcal{U}_{a,b}$ to $\widetilde{R}$. Let

$$i_{\widetilde{R}} : \widetilde{R} \to Z$$

denote the birational inclusion. We can define the "normal bundle" $N_{\widetilde{R}/Z}$ by

$$i^*_{\widetilde{R}} K_Z \otimes N_{\widetilde{R}/Z} \cong K_{\widetilde{R}}.$$

Let us calculate this in the following way. From the picture, a double point leads to two extra self-intersection points, whereas a triple point leads to 6 extra self-intersection points. Therefore, we get

$$N_{\widetilde{B}/Z} + 2(\text{double pts}) + 6(\text{triple pts}) = R^2.$$

Now R has class $4H$ so $R^2 = (4H)^2 = 16H^2$, which pulled back to Z together with $H^2 = 4$ on X_H, gives

$$R^2 = 64d.$$

Recall that we have $12d$ double points and $8d$ triple points, so our formula gives

$$\begin{aligned}\deg N_{\widetilde{R}/Z} &= 64d - 2 \cdot 12d - 6 \cdot 8d \\ &= 64d - 24d - 48d \\ &= -8d.\end{aligned}$$

On the other hand, let us recall the Grothendieck-Riemann-Roch formula (ignoring higher order terms)

$$\mathrm{ch}(i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}) = i_{\widetilde{R}*}\mathrm{ch}(\mathcal{U}_{\widetilde{R}}) - r_{\mathcal{U}_{\widetilde{R}}} i_{\widetilde{R}*}N_{\widetilde{R}/Z}/2.$$

Recall that $\mathcal{U}$ has rank 2. Thus the term $-r_{\mathcal{U}_{\widetilde{R}}} i_{\widetilde{R}*}N_{\widetilde{R}/Z}/2$ is equal to $-(-8d) = +8d$ and so we get

$$\mathrm{ch}(i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}) = 2R + \deg \mathcal{U}_{\widetilde{R}} + 8d.$$

Recall that $2R = g^*(\mathrm{Wob}_H)$ has class $8H$.

Let us now set $(a, b) = (0, 0)$. As explained in the previous subsections, this differs from the value we guess that it will be good to look at, by a modification of a that amounts to tensoring with a line bundle on X, and we will recover that later.

With this choice, $\mathcal{U}$ has degree 0 by Corollary 4.27, so we get

$$\mathrm{ch}(i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}) = 8H + 8d.$$

We now note that the map

$$V_Z \to i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}$$

is not going to be surjective.

Here is a guess as to what the image is going to be. The proof will be a consequence of the Bogomolov-Gieseker inequality.

A little like the principle that the two quotients are independent at the double points, it looks likely that the quotient over the double point is something that is "moving", corresponding to a smooth curve in the Grassmanian of the total bundle above the cusp. The guess is that the image of the above map corresponds then, not to the structure sheaf of the embedded plane curve R, but rather to the version of R that has a triple point embedded as a space curve. Let us call this curve $R^\sharp$, it is obtained by not glueing together the points lying over double points, but by glueing together the points lying over a triple point in as small a way as possible. We get

$$0 \to \mathcal{O}_{R^\sharp} \to \mathcal{O}_{\widetilde{R}} \to \mathcal{S} \to 0$$

where $\mathcal{S}$ is a skyscraper sheaf that has length 2 over each triple point. (To get to the plane curve R one would use a skyscraper sheaf of length 3).

Lemma 4.28. *The image of the map*

$$V_Z \to i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}$$

is the subsheaf

$$\mathcal{U}_{R^\sharp} \subset i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}.$$

The proof will be deferred to below, since it is motivated by the following computations.

With this hypothesis, we have

$$0 \to \mathcal{U}_{R^\sharp} \to i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}} \to \mathcal{S}_U \to 0$$

where $\mathcal{S}_U$ has length 4 over each triple point (4 because U has rank 2).

Then, the hypothesis is that we have a surjection

$$V_Z \to \mathcal{U}_{R^\sharp} \to 0.$$

Let V'_Z be the kernel, this will be our vector bundle over the covering Z corresponding to the "pullback of the parabolic bundle to Z".

Using the fact that there are $8d$ double points, we get

$$\begin{aligned} \mathrm{ch}(\mathcal{U}_{R^\sharp}) &= \mathrm{ch}(i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}) - 8d \cdot 4 \\ &= 8H + 8d - 32d = 8H - 24d. \end{aligned}$$

Let us put this together with the Chern character of V_Z. Recall that we are setting $(a,b) = (0,0)$ so $\mathrm{ch}_2(\mathcal{V}_{0,0}) = -2H^2$ on X_H. Recall that H^2 has 4 points on X_H, and then we pullback by the covering g of degree d, so we get

$$\mathrm{ch}_2(V_Z) = -8d$$

(here measuring in terms of numbers of points) on Z.

Recall also that at $(a,b) = (0,0)$ we have $\mathrm{ch}_1(\mathcal{V}_{0,0}) = 0$.

Altogether:

$$\begin{aligned} \mathrm{ch}(V'_Z) &= \mathrm{ch}(V_Z) - \mathrm{ch}(\mathcal{U}_{R^\sharp}) \\ &= 8 - 8H - 8d + 24d \\ &= 8 - 8H + 16d \\ &= 8(1 - H + 2d). \end{aligned}$$

We now note that $H^2/2 = 2d$ points on Z so, the term $2d$ above that is counted in terms of points on Z, may be written as $H^2/2$. We get the formula:

$$\mathrm{ch}(V'_Z) = 8(1 - H + H^2/2).$$

This clearly expresses the bundle V'_Z as equivalent to 8 times a line bundle since $1 - H + H^2/2 = e^{-H}$. Therefore, just by inspection, its Δ-invariant vanishes and we have a projectively flat bundle.

If we then tensor with $\mathcal{O}_X(H)$ we obtain a Higgs bundle with vanishing Chern classes.

4.11.1 Proof of the lemma

Following through what would happen in the above calculations if the image were different, we can now give the proof of the lemma.

Proof of Lemma 4.28. We note that the map $V_Z \to i_{\widetilde{R}*}\mathcal{U}_{\widetilde{R}}$ sends sections of V_Z to sections of $\mathcal{U}_{\widetilde{R}}$ that agree over the point where the three branches come together. Thus, the morphism factors through a map $V_Z \to \mathcal{U}_{R^\sharp}$. The question is to show that it is surjective. Suppose not. Let $\mathcal{U}'$ denote the image, and let ℓ denote the total length of the quotient sheaf $\mathcal{U}'/\mathcal{U}_{R^\sharp}$ (take the sum over all the cuspidal points).

In the above discussion, let V'_Z still denote the kernel of the map $V_Z \to \mathcal{U}'$, so we have a left exact sequence

$$0 \to V'_Z \to V_Z \to \mathcal{U}' \to 0.$$

Now $\mathrm{ch}(\mathcal{U}') = \mathrm{ch}(\mathcal{U}_{R^\sharp}) - \ell = 8H - 24d - \ell$, and following the same calculation as before but with the extra term ℓ gives

$$\begin{aligned}
\mathrm{ch}(V'_Z) &= \mathrm{ch}(V_Z) - \mathrm{ch}(\mathcal{U}') \\
&= 8 - 8H - 8d + 24d + \ell \\
&= 8(1 - H + 2d) + \ell \\
&= 8(1 - H + H^2/2) + \ell.
\end{aligned}$$

Recall that the Bogomolov-Gieseker inequality says that c_2 can only go up from the flat case, hence ch_2 can only go down. The bundle V'_Z would be a stable Higgs bundle on Z, and if $\ell > 0$ this would contradict Bogomolov-Gieseker. Thus, we conclude $\ell = 0$ which completes the proof of the lemma. $\square$

4.12 Degree one case—conclusion

Theorem 4.29. *Suppose $\mathcal{L}_0 \in \mathrm{Pic}(Y)$ is the pullback of a flat line bundle on the abelian variety $\mathcal{P}$, and set $\mathcal{L} := \mathcal{L}_0 \otimes \mathcal{O}_Y(\boldsymbol{E} + \boldsymbol{F})$. Let*

$$\mathcal{V} := f_*(\mathcal{L}) = \mathcal{V}_{0,0} \otimes \mathcal{O}_X(H)$$

provided with its natural crude parabolic structure with weight $\alpha = 1/2$ (tensoring the one defined above with quotient $\mathcal{U}_{0,0}$, by $\mathcal{O}_X(H)$). Then this is a crude parabolic logarithmic Higgs bundle over $(X^\circ, \mathrm{Wob}^\circ)$ that admits an extension to a purely imaginary twistor $\mathscr{D}$-module as in Theorem 3.13. The associated local system on $X - \mathrm{Wob}$ has rank 8 and its monodromy around Wob is semisimple with eigenvalues 1 of multiplicity 6 and -1 of multiplicity 2.

The spectral line bundle on Y is $\mathcal{L}' = \mathcal{L}_0 \otimes \mathcal{O}_Y(\boldsymbol{F})$.

Proof. We saw in the previous subsections that choosing $\mathcal{V}_{a,b}$ for $(a,b) = (0,0)$ yields a projectively flat solution whose (truncated) parabolic Chern character is $8(1 - H + H^2/2)$.

Tensoring this by $\mathcal{O}_{X_1}(H)$ corresponds to choosing $(a,b) = (1,0)$, taking the crude parabolic structure $\mathcal{V}_{1,0}$ with subsheaf $\mathcal{V}'_{1,0} = \mathcal{V}_{1,-1}$, and this yields a parabolic Higgs bundle with vanishing parabolic Chern classes. The rank is 8 and the parabolic structure along Wob has parabolic weight $1/2$ with multiplicity equal to the rank of $\mathcal{U}$ which is 2, completed by parabolic weight 0 with multiplicity 6. This gives the stated eigenvalues of the monodromy.

We note that the spectral line bundle, by definition the one whose direct image from Y to X gives $\mathcal{E}_0$, is $\mathcal{L}'_{a,b} = \mathcal{L}_{a,b-1} = \mathcal{L}_0 \otimes \mathcal{O}_Y(\boldsymbol{F})$. □

5 The degree zero moduli space

In this section X denotes the degree 0 moduli space and Y denotes its modular spectral covering of degree 8 that is the blow-up of the Prym variety $\mathcal{P} = \mathcal{P}_2$ in 16 points. Similarly Wob means Wob_0 and so forth.

Narasimhan and Ramanan show that $X \cong \mathbb{P}^3$ [NR69]. There is a ***Kummer surface*** of degree 4 with 16 nodes that we will denote by $\mathsf{Kum} \subset X$. This is the moduli space of bundles that are semistable but not stable, up to S-equivalence. The S-equivalence class corresponding to a point of Kum is represented by a polystable bundle of the form $L \oplus L^\vee$ where $L \in \mathrm{Jac}^0(C)$ is a line bundle of degree 0.

We have

$$\mathsf{Kum} \cong \mathrm{Jac}^0(C)/\pm 1,$$

a point on Kum corresponding to a pair (L, L^{-1}) of degree zero line bundles on X, up to interchangeing L and L^{-1}. The corresponding polystable bundle is $E = L \oplus L^{-1}$. When $L = L^{-1}$, that is to say at the 16 points of order two on $\mathrm{Jac}^0(C)$, we get a singular point of Kum.

Let Higgs_0 be the corresponding Hitchin moduli space of Higgs bundles with determinant $\mathcal{O}_C$, the Higgs field having zero trace.

5.1 Geometry of the wobbly locus in degree zero

The Higgs bundle moduli space has a locus analogous to the Kummer surface.

Lemma 5.1. *The Hitchin moduli space is singular along a codimension two subvariety* $\mathsf{Higgs}_0^{\mathrm{sing}}$*, and this subvariety consists of the set of polystable but not stable Higgs bundles. We have*

$$\mathsf{Higgs}_0^{\mathrm{sing}} \cong \frac{\mathrm{Jac}^0(C) \times H^0(C, K_C)}{\pm 1},$$

a point here parametrizing a Higgs bundle of the form $(L, \phi) \oplus (L^{-1}, -\phi)$*. The set of* $\mathbb{C}^\times$ *fixed points of* $\mathsf{Higgs}_0^{\mathrm{sing}}$ *is equal to the Kummer surface* $\mathsf{Kum} \subset X$*. Outside of the* 16 *singular points of* Kum*, the transverse local structure of* Higgs_0 *along* $\mathsf{Higgs}_0^{\mathrm{sing}}$ *is that of a simple double point.*

Proof. A stable Higgs bundle $\mathbb{E}$ has only scalar endomorphisms, and by duality the trace-free H^2, that is to say $H^2_{\mathrm{Dol}}(End_0(\mathbb{E}))$, vanishes. This is the obstruction space to deforming $\mathbb{E}$. Thus, a stable Higgs bundle is a smooth point of Higgs_0 (this is well-known, of course). Thus, a singular point must be a strictly semistable point represented by a polystable Higgs bundle. Since we are in rank 2, it is a direct sum of line bundles, and the condition that the determinant is trivial means that it must have the stated form $(L, \phi) \oplus (L^{-1}, -\phi)$. The structure is that of the moduli space of (L, ϕ) modulo the involution $(L, \phi) \mapsto (L^{-1}, -\phi)$.

A Higgs bundle of this form is fixed by $\mathbb{C}^\times$ if and only if $\phi = 0$, the fixed points are therefore inside the moduli space of bundles $X \subset \mathsf{Higgs}_0$ and are in the Kummer surface.

Suppose we are at a point $\mathbb{L} \in \mathsf{Kum}$ corresponding to $(L, \phi = 0)$ with $L \not\cong L^{-1}$. In other words $\mathbb{L} = (L, 0) \oplus (L^{-1}, 0)$, $H^2_{\mathrm{Dol}}(End^0(\mathbb{L})) = \mathbb{C}$, and

$$H^1_{\mathrm{Dol}}(End_0(\mathbb{L})) = H^1_{\mathrm{Dol}}(\mathcal{O}) \oplus H^1_{\mathrm{Dol}}(L^{\otimes 2}) \oplus H^1_{\mathrm{Dol}}(L^{\otimes -2})$$

with the Kuranishi map being given by the trace of the cup-product (recall that the Goldman-Millson deformation theory is formal [GM88, Sim92]). The cup product vanishes on $H^1_{\mathrm{Dol}}(\mathcal{O})$, which is the unobstructed deformation space of $(L,0)$. On $H^1_{\mathrm{Dol}}(L^{\otimes 2}) \oplus H^1_{\mathrm{DOL}^{\mathrm{par}}_{L^2}}(L^{\otimes -2})$ it is the same as the Poincaré duality form. Thus, as a quadratic form it defines a simple double point in the direction transverse to the deformation space of $(L,0)$ i.e. the tangent space of $\mathsf{Higgs}_0^{\mathrm{sing}}$. We see in particular that Higgs_0 is indeed singular along this locus. $\square$

In Proposition 3.3**(a)** we showed that the set $\mathsf{Higgs}_0^{\mathbb{C}^\times,\mathrm{nu}}$ of $\mathbb{C}^\times$-fixed points on Higgs_0 that are not on X, is a disjoint union of 16 points. These points cprrespond to the 16 Higgs bundles

$$\mathbb{E}_\kappa = (E_\kappa, \theta_\kappa) = \left(\kappa \oplus \kappa^{-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad \kappa \in \mathsf{Spin}(C).$$

From this picture, we can now describe the wobbly locus in X completely.

Proposition 5.2. *The wobbly locus* $\mathrm{Wob} \subset X$ *decomposes as*

$$\mathrm{Wob} = \mathsf{Kum} \cup \left(\bigcup_{\kappa \in \mathsf{Spin}(C)} \mathsf{Trope}_\kappa \right)$$

where each $\mathsf{Trope}_\kappa \cong \mathbb{P}^2 \subset X = \mathbb{P}^3$ *is a plane corresponding to the fixed point* $\mathbb{E}_\kappa$*. Outside of its* 16 *nodes* Kum *is simply tangent to each* Trope_κ *along a smooth conic* $\mathfrak{C}_\kappa \subset \mathsf{Trope}_\kappa \cong \mathbb{P}^2$*, and any two trope planes intersect transversally.*

Proof. We already described the components of the wobbly locus in the proof of Theorem 3.6 but for completeness we recall the relevant arguments here. A wobbly bundle E is either strictly semistable, in which case it corresponds to a point of Kum, or it is stable but has a nonzero nilpotent Higgs field. In the second case, the upward limit of the $\mathbb{C}^\times$ orbit corresponding to that Higgs field is a fixed point not in X, so by Proposition 3.3**(a)** the limiting Higgs bundle has the form $\kappa \oplus \kappa^{-1}$ where κ is a square-root of ω_C. In terms of the bundle E, it means that there is an exact sequence

$$0 \to \kappa^{-1} \to E \to \kappa \to 0.$$

Thus E corresponds to an extension class $\xi \in H^1(\kappa^{-2}) = H^1(\omega_C^{-1}) \cong H^0(\omega_C^{\otimes 2})^\vee$. The set of bundles corresponding to extension classes like that is a $\mathbb{P}^2$. Narasimhan-Ramanan's discussion [NR69, Proposition 6.1, Theorem 2] shows that the map from the projectivized space of extension classes into $X = \mathbb{P}^3$ is linear, thus we obtain a linear $\mathsf{Trope}_\kappa \cong \mathbb{P}^2 \subset \mathbb{P}^3$.

These define the 16 divisors Trope_κ. They are different since the fixed points are different. The intersections $\mathsf{Trope}_\kappa \cap \mathsf{Trope}_{\kappa'}$ are therefore transverse. Note also that, as we will see in Corollary 5.5 below, three distinct trope planes can not meet along a line, so the intersections between trope planes are normal crossings in X-codimension 2.

We consider next the condition that an extension of the above form lies in the Kummer surface. It means that there is a line bundle U of degree 0 with a map $U \to \kappa$ such that the extension splits. We may write $U = \kappa(-t)$ for a point $t \in C$. Splitting the extension means that the image of ξ under the map

$$H^1(\omega_C^{-1}) \to H^1(\omega_C^{-1}(t)) \cong H^1(\mathcal{O}_C(-t'))$$

should vanish, where t' is the hyperelliptic conjugate of t. This determines an extension ξ uniquely up to scalars. Furthermore, the extension determined by t' yields an S-equivalent bundle since

$$\kappa(-t) \otimes \kappa(-t') = \omega_C(-t-t') = \mathcal{O}_C.$$

Thus, the map from C to the space of extensions factors through a map from $\mathbb{P}^1$. Notice that the map from C to Kum comes by sending t to the line bundle $\kappa(-t) \in \mathrm{Jac}^0(C)$ and then projecting from the Jacobian to the Kummer.

Let us consider a different plane given by the space of extensions of the form

$$0 \to V^{-1} \to E \to V \to 0$$

for V some general line bundle of degree 1. This plane is the space of bundles such that $H^0(V \otimes E) \neq 0$. If that holds for E then it holds for the polystable bundle in the S-equivalence class of E, so it means that we look for the condition $H^0(V \otimes \kappa(-t)) \neq 0$. We note that the degree 2 bundle $V \otimes \kappa$ is general, so it has a single section with two zeros. If t is one of these zeros, then we get a solution. The case of a hyperelliptically conjugate point is when the other piece $H^0(V \otimes \kappa(-t')) \neq 0$. There are two points in $\mathbb{P}^1$ that are the images of the zeros of the section of $V \otimes \kappa$, corresponding to the intersection with this plane. We conclude that the rational curve image of $\mathbb{P}^1$ is a conic.

Thus, $\mathsf{Kum} \cap \mathsf{Trope}_\kappa$ has reduced scheme equal to a conic, so it is twice a conic in the plane $\mathsf{Trope}_\kappa \cong \mathbb{P}^2$. $\square$

Remark 5.3. In the classical studies of the geometry of the Kummer surface (see [Kle70, GH94, NR69, Bea96, Keu97, Dol20, Hud05]), these 16 planes were known as the *trope planes*.

Each trope plane passes through 6 of the 16 singular points of the Kummer surface, and each singular point is contained in 6 of the 16 trope planes. This configuration was known as the ***Kummer*** 16_6 ***configuration***. The conics $\mathfrak{C}_\kappa$ are called ***trope conics***. We have adopted the 'trope' terminology.

Proposition 5.4. *Inside one of the trope plane* Trope_κ*, the trope conic may be identified with the hyperelliptic* $\mathbb{P}^1$*. The six branch points in the hyperelliptic* $\mathbb{P}^1$ *correspond to the* 6 *nodes of* Kum *contained in that trope plane. The* 15 *lines of intersection with the other trope planes, consist of all lines passing through pairs of branch points (i.e. nodes of* Kum *on the trope conic.*

Proof. In modular terms, a trope plane corresponds to a choice of a square-root κ of ω_C, with the corresponding $\mathsf{Trope}_\kappa \subset X \cong \mathbb{P}^3$ consisting of all the bundles that are extensions of the form

$$0 \to \kappa^{-1} \to E \to \kappa \to 0.$$

Such a bundle E belogs to Kum if it contains a degree 0 line bundle; such would be of the form $L = \kappa(-t)$ with the dual $L^{-1} = \kappa(-t')$ where t' is the hyperelliptic conjugate of t. The bundle determines in this way, and is determined by, a point on the hyperelliptic $\mathbb{P}^1$. The nodes of the Kummer surface occur when $L^{-1} \cong L$, thus $t' = t$ in other words these correspond to branch points in the hyperelliptic $\mathbb{P}^1$. Now, given two different branch points $\mathbf{x}_1$ and $\mathbf{x}_2$ we get $(\kappa')^{-1} := \kappa(-\mathbf{x}_1 - \mathbf{x}_2)$. This is the dual of another square-root κ' of ω_C. The set of bundles E that admit non-zero maps from $(\kappa')^{-1}$ is the intersection of Trope_κ and $\mathsf{Trope}_{\kappa'}$. This set contains in particular the bundles containing $\kappa(-\mathbf{x}_1)$ respectively $\kappa(-\mathbf{x}_2)$, i.e. this intersection line passes through the two branch points $\mathbf{x}_1$ and $\mathbf{x}_2$ on the trope conic. We have identified a set of 15 lines that is on the one hand the set of intersection lines between Trope_κ and all the other planes, and on the other hand is the set of lines through pairs of the 6 branch points. □

Corollary 5.5. *No three trope planes pass through the same line.*

Proof. Recall that we are assuming genericity of the set of 6 branch points of the hyperelliptic curve. If there were three trope planes passing through the same line, then another trope plane would contain a configuration of three lines of the form $\mathsf{L}_{\kappa\kappa'} = \mathsf{Trope}_\kappa \cap \mathsf{Trope}_{\kappa'}$ as in the proof of the proposition, that pass through the same point. However, noting that all smooth plane conics are projectively equivalent to a given fixed one, our set of 6 branch points is general on the conic. If three of the lines $L_{\kappa\kappa'}$ were to pass through the same point, we could jiggle one of the branch points to move one of the three lines away from that intersection point, so this can not happen under our genericity hypothesis. $\square$

Let $\mathcal{P} \subset \mathsf{Higgs}_0$ be the fiber of the Hitchin fibration corresponding to a spectral curve $\pi : \widetilde{C} \to C$. Notice that, although the Hitchin fibrations are different for degree 0 and 1, the base is still the space of quadratic differentials and the spectral curves are the same. Thus, Proposition 3.1 applies, and $\widetilde{C}/C$ is branched over two pairs of hyperelliptically conjugate points a, a', b, b'.

Then, as we saw in Theorem 3.6 we need to blow up $\mathcal{P} = \mathcal{P}_2$ at the 16 points $\{\pi^*\kappa\}_{\kappa \in \mathsf{Spin}(C)}$ to get Y with a map $f : Y \to X$. Let H be the hyperplane class on X, let $\boldsymbol{F}$ be its pullback to Y, and let $\boldsymbol{E} = \sum_{\kappa \in \mathsf{Spin}(C)} \boldsymbol{E}_\kappa$ be the exceptional divisor (union of 16 disjoint $\mathbb{P}^2$'s).

Lemma 5.6. *Under the finite morphism $f : Y \to X$ each plane $\boldsymbol{E}_\kappa$ maps isomorphically to the corresponding plane Trope_κ in X, the map from Y to X being ramified with index 2 there, and this accounts for the ramification over general points of Trope_κ.*

Proof. We have seen in the proof of Theorem 3.6 and in Proposition 5.2 that the plane Trope_κ is the projectivized set of downward directions at the fixed point, which is isomorphic to the projectivized normal bundle of the upward subset emanating from the fixed point. This in turn is isomorphic to the projectivized normal bundle of the intersection point of the upward set with $\mathcal{P}$, which is the exceptional divisor $\boldsymbol{E}_\kappa$.

Let us recall why the upward subset intersects the Prym in a single point (this explains why there are 16 points to blow up, as stated above). Our fixed point corresponds to a nilpotent Higgs bundle with underlying bundle of the form $\kappa \oplus \kappa^{-1}$ where κ is a square-root of ω_C. The upward subset is the set of Higgs bundles whose underlying vector bundle is unstable with detabilizing subbundle L. To intersect with the Prym, we look for line bundles U on $\widetilde{C}$ of degree 2 whose norm is ω_C, such that there is a map $\kappa \to \pi_*(U)$ or equivalently $\pi^*\kappa \to U$. But $\pi^*\kappa$ also has degree 2 on $\widetilde{C}$ so $U = \pi^*\kappa$ is the unique solution.

There is an involution of $\mathcal{P}$ given by applying the hyperelliptic involution on $\widetilde{C}$ (which covers the hyperelliptic involution on C). Up to translation by ω_C, this amounts to doing -1 on the abelian variety, and the 16 points being blown up are the fixed points. Normal directions to these points are also fixed, so the $\boldsymbol{E}_\kappa$ are fixed points of the induced involution of Y.

The pullback of a semistable degree 0 bundle by the hyperelliptic involution on C is S-equivalent to itself. This is known for local systems in the work of Goldman and Heu-Loray [Gol97, HL19], in particular it applies to unitary local systems and hence to polystable bundles.

The map $f : Y \to X$ therefore factors through the quotient by the involution. But the involution acts by -1 on the normal directions, so it acts by -1 on the normal bundle of $\boldsymbol{E}_\kappa$, giving a simple ramification divisor of the map from Y to its quotient by this involution along $\boldsymbol{E}_\kappa$.

We claim that this accounts for the ramification of the map $f : Y \to X$ along $\boldsymbol{E}_\kappa$. Suppose that a point of $\boldsymbol{E}_\kappa$ counts for m points in the fiber of $f : Y \to X$ that has degree 8 (Lemma 4.9). We are claiming that $m = 2$, and have shown in the preceding paragraph that $m \geq 2$. A monodromy argument (when moving everything around, the planes $\boldsymbol{E}_\kappa$ get interchanged transitively) tells us that this coefficient m is the same for all $\boldsymbol{E}_\kappa$. Now, Trope_κ is a plane inside $X = \mathbb{P}^3$, so a different component $\mathsf{Trope}_{\kappa'}$ intersects it in a line. Consider yet a third component $\mathsf{Trope}_{\kappa''}$ that intersects this line in at least one point. This is not a smooth point of the Kummer surface, since the trope planes are tangent planes to the Kummer surface at points on the trope conics that are different from the 16 singular points, and the tangent plane is unique so in that case the three planes would be the same. The point might be one of the 16 singular points of the Kummer surface. However, that does not always happen. Indeed, the other trope planes intersect Trope_κ in 15 lines, and the various intersection points of these lines can not be limited to only the singular points of the Kummer surface (of which there are 6 corresponding to the Weierstrass points inside the $\mathbb{P}^1$ of the trope conic). So we can choose $\mathsf{Trope}_{\kappa'}$ and $\mathsf{Trope}_{\kappa''}$ to correspond to lines that intersect somewhere in Trope_κ other than a singular point.

Now, the exceptional divisors upstairs in Y are three disjoint planes $\boldsymbol{E}_\kappa, \boldsymbol{E}_{\kappa'}, \boldsymbol{E}_{\kappa''}$. In particular, lying over the intersection point inside X, there are at least three different points in the $\boldsymbol{E}_\kappa, \boldsymbol{E}_{\kappa'}, \boldsymbol{E}_{\kappa''}$. This would give $3m \leq 8$, from which we conclude $m = 2$.

We now know that in a fiber over a general point of Trope_κ, there is a ramification point in $\boldsymbol{E}_\kappa$ counting for 2 points in the degree 8 fiber, plus 6 other points on Y that are not in the exceptional divisor, in other words they are in the locus of Higgs_0 whose underlying bundle is stable. An argument similar to the proof of Lemma 5.13 below shows that for a general point in the Hitchin base, and over a general point of Trope_κ, the other 6 points are unramified. ∎

5.2 Computations in degree zero

Recall that by convention we drop terms of degree ≥ 3 in all expressions.

Proposition 5.7. *The Todd classes for the degree* 0 *moduli spaces are:*

$$\begin{aligned} \mathrm{td}(TX) &= 1 + 2H + 11H^2/6, \\ f^*\mathrm{td}(TX)^{-1} &= 1 - 2\boldsymbol{F} + 13\boldsymbol{F}^2/6, \\ \mathrm{td}(TY) &= 1 - \boldsymbol{E} + \boldsymbol{E}^2/3, \\ \mathrm{td}(Y/X) &= (1 - 2\boldsymbol{F} + 13\boldsymbol{F}^2/6)(1 - \boldsymbol{E} + \boldsymbol{E}^2/3). \end{aligned}$$

Proof. For Y which is the blow-up of an abelian variety at 16 points we saw

$$\mathrm{ch}(TY) = 3 - 2\boldsymbol{E} + 2\boldsymbol{E}^2.$$

In particular $c_2(TY) = 0$. Thus

$$\begin{aligned} \mathrm{td}(TY) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\ &= 1 - \boldsymbol{E} + \boldsymbol{E}^2/3. \end{aligned}$$

Next,

$$\begin{aligned} \mathrm{td}(T\mathbb{P}^n) &= 1 + c_1/2 + (c_1^2 + c_2)/12 \\ &= 1 + \frac{n+1}{2}H + \frac{3n^2 + 5n + 2}{24}H^2. \end{aligned}$$

For $X = \mathbb{P}^3$ this gives

$$\mathrm{td}(TX) = 1 + 2H + 11H^2/6,$$

hence

$$\mathrm{td}(TX)^{-1} = 1 - 2H + 13H^2/6.$$

Thus

$$(f^*\mathrm{td}(X))^{-1} = 1 - 2\boldsymbol{F} + 13\boldsymbol{F}^2/6.$$

∎

Proposition 5.8. *The triple intersections of divisor classes on the degree* 0 *modular spectral covering* Y *are:*

$$\boldsymbol{F}^3 = 8, \qquad \boldsymbol{E}\boldsymbol{F}^2 = 16, \qquad \boldsymbol{E}^2\boldsymbol{F} = -16, \qquad \boldsymbol{E}^3 = 16.$$

Proof. The degree of $f : Y \to X$ is 8 and we have $H^3 = 1$ in $X = \mathbb{P}^3$, so $\boldsymbol{F}^3 = 8$. Next, $\boldsymbol{F}^2$ is the pullback of a line that intersects the 16 trope planes in 16 points. As the map π identifies the trope planes with the corresponding components of $\boldsymbol{E}$ we have $\boldsymbol{E}\boldsymbol{F}^2 = 16$.

Next, for each $\boldsymbol{E}_\kappa$ the self-intersection $\boldsymbol{E}_\kappa^2$ is the class on $\boldsymbol{E}_\kappa$ of the normal bundle. The normal bundle of $\boldsymbol{E}_\kappa$ in Y is $\mathcal{O}_{\boldsymbol{E}_\kappa}(-1)$ so it is minus the class of a line in $\boldsymbol{E}_\kappa$. We get that $\boldsymbol{E}_\kappa^3$ is the intersection of two of these together, so it is 1. Adding 16 of these together gives $\boldsymbol{E}^3 = 16$. The intersection of the line with $\boldsymbol{F}$ is the same as the intersection of its image in X, which is again a line in the trope plane, with H. This is 1 so with the minus sign and 16 planes we get $\boldsymbol{E}^2\boldsymbol{F} = -16$. □

Proposition 5.9. *Suppose* $\mathcal{L} = \mathcal{O}_Y(a\boldsymbol{F} + b\boldsymbol{E})$ *is a line bundle on* Y *and let* $\mathcal{E} := f_*\mathcal{L}$. *Then*

$$\mathrm{ch}_1(\mathcal{E}) = (8(a-2) + 16(b-1))H$$

and

$$\mathrm{ch}_2(\mathcal{E}) = 4(a^2 + 4ab - 2b^2 - 8a - 4b + 11)H^2.$$

Proof. The Grothendieck-Riemann-Roch formula says

$$\begin{aligned}\mathrm{ch}(\mathcal{E}) &= f_*\left(\mathrm{td}(Y/X)e^{\mathcal{L}}\right)\\ &= f_*\left[(1 - \boldsymbol{E} + \boldsymbol{E}^2/3)(1 - 2\boldsymbol{F} + 13\boldsymbol{F}^2/6)(1 + (a\boldsymbol{F} + b\boldsymbol{E}) + (a\boldsymbol{F} + b\boldsymbol{E})^2/2)\right].\end{aligned}$$

Thus

$$H^2 \cdot \mathrm{ch}_1(\mathcal{E}) = \boldsymbol{F}^2((a-2)\boldsymbol{F} + (b-1)\boldsymbol{E}) = 8(a-2) + 16(b-1).$$

And

$$\begin{aligned}H \cdot \mathrm{ch}_1(\mathcal{E}) &= \boldsymbol{F}(\boldsymbol{E}^2/3 + 13\boldsymbol{F}^2/6 + (a\boldsymbol{F} + b\boldsymbol{E})^2/2 + 2\boldsymbol{E}\boldsymbol{F} - (a\boldsymbol{F} + b\boldsymbol{E})(\boldsymbol{E} + 2\boldsymbol{F}))\\ &= \boldsymbol{E}^2\boldsymbol{F}(1/3 + b^2/2 - b) + \boldsymbol{E}\boldsymbol{F}^2(ab + 2 - a - 2b) + \boldsymbol{F}^3(13/6 + a^2/2 - 2a)\end{aligned}$$

and using Proposition 5.8 this becomes

$$\begin{aligned}&4a^2 + 16ab - 8b^2 - 32a - 16b + (52/3 + 32 - 16/3)\\ &= 4(a^2 + 4ab - 2b^2 - 8a - 4b + 11).\end{aligned}$$

Noting that $H^3 = 1$ on $\mathbb{P}^3$ this gives the required formula. □

Corollary 5.10. *If we impose the condition* $\mathrm{ch}_1(\mathcal{E}) = 0$ *by setting*

$$\mathcal{L} = \mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F} + m(\boldsymbol{E} - 2\boldsymbol{F})) \quad \textit{with } m \in \mathbb{Z}$$

then

$$\mathrm{ch}_2(\mathcal{E}) = (-24m^2 + 4)H^2.$$

Proof. Indeed we then have $a = 2(1-m)$ and $b = (m+1)$ so the formula becomes

$$\begin{aligned} 4(4(1-m)^2 &+ 8(1+m)(1-m) - 2(m+1)^2 - 16(1-m) - 4(m+1) + 11) \\ &= 4(4m^2 - 8m + 4 + 8 - 8m^2 - 2m^2 - 4m - 2 - 16 + 16m - 4m - 4 + 11) \\ &= 4(-6m^2 + 1) = (-24m^2 + 4). \end{aligned}$$

□

Corollary 5.11. *The extremal value for* $\mathrm{ch}_2(\mathcal{E})$ *is at* $m = 0$ *in other words* $\mathcal{L} = \mathcal{O}_Y(\boldsymbol{E}+2\boldsymbol{F})$ *and then*

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(\pi_*(\mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F})) = 4H^2.$$

Note that the extremal value for $\mathcal{L}$ is the same as $-\mathrm{td}_1$. This is a general phenomenon. At this value, $\mathrm{ch}_2(\mathcal{E}) = 4H^2$ contradicts the Bogomolov-Gieseker inequality. That indicates, on the one hand, that $\mathcal{E}$ is not a stable bundle. On the other hand, it is stable as a bundle with meromorphic Higgs field. The Higgs field is logarithmic along the smooth points of the wobbly divisor, but the contradiction to the Bogomolov-Gieseker inequality indicates that if we pull back to a resolution making the wobbly divisor into a normal crossings divisor, the resulting Higgs field there is not going to be logarithmic. We will see that in more detail in the next sections, in which we investigate what parabolic structure can be put over the resolution of singularities in order to get a logarithmic Higgs field with maximal value of ch_2. We will see that the maximal value is then 0 yielding a flat bundle.

We may note here a general principle: the corrections to ch_2 coming from the required parabolic structures at the singularities, over a planar slice (so the singularities are tacnodes), are local and do not depend on $\mathcal{L}$. Therefore, whatever they are, the extremal value has to be obtained when the ch_2 calculated above is extremal. This means that $\mathcal{L}$ has to be numerically equivalent to $\mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F})$.

5.3 Ramification and Riemann-Roch

Recall that $f : Y \to X$ is a finite covering of degree 8. We would like to understand the ramification. First is the ramification over the wobbly locus.

Proposition 5.12. *The ramification of the map $f : Y \to X$ over general points of the components of the wobbly locus is as follows. Over a general point of the Kummer surface $\mathsf{Kum} \subset X$, the covering f is fully ramified, breaking into four pieces with simple ramification. Over each of the 16 trope planes, the covering f breaks generically into a simply ramified map along the corresponding component of $\boldsymbol{E}$ plus a degree 6 etale cover.*

Proof. The part about ramification over the trope planes was shown in Lemma 5.6. We need to describe the ramification over a general point of the Kummer surface.

Let us describe the points of Y lying over a general point of Kum. If $E \in \mathsf{Kum}$ is such a point, there is a line bundle L of degree 0 on C with an inclusion $L \hookrightarrow E$. If $E = \pi_*(U)$ for a degree 2 line bundle U on $\widetilde{C}$, by adjunction it means that there is a nonzero map $\pi^*(L) \to U$, hence

$$U = \pi^*(L) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{u} + \tilde{v})$$

for two points $\tilde{u}, \tilde{v} \in \widetilde{C}$. Let u, v denote their images in C. The determinant of $\pi_*(U)$ is $L^{\otimes 2} \otimes \omega_C^{-1}(u + v)$ so the trivial determinant condition $\det E = \mathcal{O}_C$ says

$$\mathcal{O}_C(u + v) = \omega_C \otimes L^{\otimes -2}.$$

This determines the points u, v as the zeros of the unique section of $\omega_C \otimes L^{\otimes -2}$ (for generic L). Then, $\tilde{u}$ and $\tilde{v}$ are liftings of these points to $\widetilde{C}$. There are 4 combinations of liftings. Conversely, each one determines a bundle U such that there exists a map $L \hookrightarrow \pi_*(U)$.

We claim that $\pi_*(U)$ is polystable. This may be seen by noting that as we move around in Kum, the line bundles L and L^{-1} interchange, so if there is an inclusion from L at a general point there has to be an inclusion from L^{-1} too. It may also be seen as follows. Recall that $\widetilde{C}$ is hyperelliptic too, and let σ denote its hyperelliptic involution, that covers the hyperelliptic involution $\boldsymbol{\iota}_C$ of C. Thus, $\sigma^*\pi^*(L) = \pi^*(L^{-1})$ since L^{-1} is the pullback of L by the hyperelliptic involution on C. We obtain a map $L^{-1} \to \pi_*(\sigma^*(U)) = \boldsymbol{\iota}_C^*(E)$. As noted in the proof of Lemma 5.6, $\boldsymbol{\iota}_C^*(E) \cong E$ so we get a map $L^{-1} \hookrightarrow E$. For L general, this gives an isomorphism $L \oplus L^{-1} \to E$.

Using the fact that $\pi_*(U)$ are polystable, we can calculate the fiber using either one of the degree 0 line bundles contained in E. The previous calculation shows that there are 4 points.

As $L \oplus L^{-1}$ moves around in Kum, the points u, v move around in C and this induces a transitive action on the set of 4 liftings. Thus, the ramification degree of Y/X at the different points must be the same, and since $\deg(Y/X) = 8$ we conclude that the 4 points each have ramification degree 2, in other words simple ramification. This completes the proof. $\square$

Next we calculate the class of the ramification divisor of the map $f : Y \to X$. We have $\omega_X = \mathcal{O}_X(-4H)$ so $f^*\omega_X^{-1} = \mathcal{O}_Y(4\boldsymbol{F})$. From the formula for the canonical class of the blow-up, we have $\omega_Y = \mathcal{O}_Y(2\boldsymbol{E})$. This gives

$$\mathcal{O}_Y(R) = \omega_Y \otimes f^*\omega_X^{-1} = \mathcal{O}_Y(2\boldsymbol{E} + 4\boldsymbol{F}).$$

The ramification divisor includes a copy of the reduced divisor of the inverse image of the Kummer surface Kum since π is fully ramified over Kum. Let us call this part R^{Kum}. As $\mathcal{O}_X(\text{Kum}) = \mathcal{O}_X(4H)$, but the pullback of Kum is twice R^{Kum} we get

$$R^{\text{Kum}} = 2\boldsymbol{F}.$$

Let $R^{\text{Tro}} = \boldsymbol{E}$ be the part of the ramification over the trope planes. Let R^{Mov} denote the novable part $R^{\text{Mov}} := R - R^{\text{Kum}} - R^{\text{Tro}}$ of the ramification divisor. We therefore have

$$R^{\text{Mov}} \in \left|\mathcal{O}_Y\left(2\boldsymbol{F} + \boldsymbol{E}\right)\right|.$$

Lemma 5.13. *The ramification R^{Mov} is movable in the following sense: all components of the image of this divisor in X move as a function of the point in the Hitchin base.*

Proof. Let $B^{\text{Mov}} \subset X$ be the image of R^{Mov}. This does not contain any component of the wobbly locus, indeed we have seen in Proposition 5.12 that the only ramifications over the various components of the wobbly locus are those given by R^{Kum} and R^{Tro}. Suppose that some component was fixed. That means that we would have a divisor $B' \subset B^{\text{Mov}}$ remaining fixed as our point in the Hitchin base moves. In particular, for a general point $x \in B'$, which is very stable, the well-defined fiber $\mathbb{A}^3$ of the projection $\mathsf{Higgs}_0 \to X$ over x would have a ramification point inside every Hitchin fiber that it intersects. This would give a 3-dimensional family of ramification points, which would have to be all of $\mathbb{A}^3$. In that case, the projection map from Higgs_0 to X (over the very stable open subset) would be non-smooth there, but the general Hitchin fiber is smooth so that can not be the case. This contradiction shows that there are no non-movable components in B^{Mov}. $\square$

Suppose $\boldsymbol{E}_\kappa$ is one of the components of $\boldsymbol{E}$. It follows from the lemma and the previous proposition, that R^{Mov} does not include $\boldsymbol{E}_\kappa$. Therefore $R^{\mathrm{Mov}} \cap \boldsymbol{E}_\kappa$ is a transverse intersection. Recall that $\boldsymbol{E}_\kappa \cong \mathbb{P}^2$. The map $\boldsymbol{E}_\kappa \to X$ is an isomorphism to a trope plane in $X \cong \mathbb{P}^3$, so $\mathcal{O}_{\boldsymbol{E}_\kappa}(\boldsymbol{F} \cap \boldsymbol{E}_\kappa) = \mathcal{O}_{\boldsymbol{E}_\kappa}(1)$. On the other hand, $\mathcal{O}_{\boldsymbol{E}_\kappa}(\boldsymbol{E}) \cong \mathcal{O}_{\boldsymbol{E}_\kappa}(-1)$ since $\boldsymbol{E}_\kappa$ is the exceptional divisor from blowing up $\mathcal{P}$ at a point. We conclude that

$$\mathcal{O}_{\boldsymbol{E}_\kappa}(R^{\mathrm{Mov}} \cap \boldsymbol{E}_\kappa) \cong \mathcal{O}_{\boldsymbol{E}_\kappa}(1).$$

This proves the following

Corollary 5.14. *The divisor $R^{\mathrm{Mov}} \cap \boldsymbol{E}_\kappa \subset \boldsymbol{E}_\kappa$ is a line in the plane $\boldsymbol{E}_\kappa$. In particular, it does not contain the trope conic which is the inverse image in $\boldsymbol{E}_\kappa$ of the intersection of the trope plane with the Kummer surface.*

Consider a general point $y \in R^{\mathrm{Kum}} \cap R^{\mathrm{Tro}}$. From the previous corollary, the ramification of the map f in a neighborhood of y is only the ramification due to the two pieces R^{Kum} and R^{Tro}.

Corollary 5.15. *In the neighborhood of such a point y, the map f has a piece of degree 4 given in local coordinates (u, v, z) of Y and (x, y, z) of X by*

$$x = u^2 + v, \qquad y = v^2.$$

There remain two pieces of degree 2 each that are simply ramified over the Kummer surface only.

Proof. The picture is local and occurs at a general point of the trope conic consisting of a one-dimensional family of tacnodes, so we can use the third coordinate z in both coordinate systems. The trope conic is given by $x = y = 0$ in this picture. Transverse to the z direction the Kummer surface is given by $y = 0$ and the trope plane is given by $y = x^2$. Our coordinates are, of course, not those of $X = \mathbb{P}^3$.

The local fundamental group of this singularity is generated by loops around these two pieces, and from the ramification picture we know that they act on the covering by a single transposition (for the loop around $y = x^2$) and a product of four distinct transpositions (for the loop around $y = 0$). The fact that the moving part of the ramification does not meet our general point of the trope conic means that there is no other ramification and Y is a covering of X given by the action of these two elements on a set of 8 elements.

There are only two possibilities: either the single transposition is equal to one of the four, or else it connects two of them. In the first case, the resulting covering is singular, but we know that Y is smooth, so we must be in the second case. In particular, the covering breaks into two pieces consisting of ramification along $y = 0$ only (for the two out of four transpositions that do not touch the single one) and a piece of degree 4 with two transpositions over $y = 0$ and one transposition over $y = x^2$. Furthermore, any local model that has this type of ramification and with smooth total space must be isomorphic locally to our covering.

We can construct such a model in the following way: first take a degree 2 cover ramified along $y = 0$ given by $y = v^2$, and look at the inverse image in here of $x^2 - x$. It breaks into two irreducible components as $x^2 - v^2 = (x + v)(x - v)$. Choose one, let's say $(v - x)$, and take a covering of degree 2 ramified a long there, so $u^2 = x - v$. We have our coordinate system (u, v) with the equations as stated. Let us verify the Jacobian:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 0 & 2v \end{vmatrix} = 4uv$$

so the ramification locus upstairs is the union of $u = 0$ and $v = 0$ and these project to $y = x^2$ and $y = 0$ respectively. $\square$

5.4 Blowing up the tacnodes

Cut with a general plane in $X_0 = \mathbb{P}^3$. Locally using the coordinates above, we may assume that our plane is $z = 0$ so it has local coordinates (x, y) and the degree 4 piece of the covering is given in coordinates as described in Corollary 5.15. We are going to use the description as a composition of two covers of degree 2 to describe this piece of Y_0.

In what follows, we'll change temporarily the meaning of the notations X, Y: assume that we have cut X_0 down to the general plane that will be denoted by X, and Y_0 is cut down to the corresponding covering, denoted by Y for simplicity of notations. We will also focus on a single tacnode, noting that altogether there are 32 of them in the plane.

Blow up twice to resolve the tacnode. Let $\widehat{X}$ denote the blown-up variety and let $\widehat{Y}$ denote the local degree 4 piece of the blown-up covering, normalized, with map $\widehat{f} : \widehat{Y} \to \widehat{X}$.

Let D^{Tro} and D^{Kum} be the trope and Kummer divisors in X, locally near our point.

When we blow up the first time, the strict transforms of D^{Tro} and D^{Kum} form, together with the exceptional divisor, an ordinary triple point. Then blow that up again to get $\widehat{X}$. Let A be the strict transform in $\widehat{X}$ of the first exceptional divisor, and let B be the exceptional divisor of the second blow-up. Let T and K denote the strict transforms of the original divisors D^{Tro} and D^{Kum} respectively. Let $\alpha : \widehat{X} \to X$ be the map. Then

$$\alpha^*(D^{\mathrm{Tro}}) = T + A + 2B$$

and

$$\alpha^*(D^{\mathrm{Kum}}) = K + A + 2B.$$

We can factorize our covering $\widehat{f}$ as a composition

$$\widehat{Y} \xrightarrow{\mu} \widehat{Z} \xrightarrow{\nu} \widehat{X}$$

where ν is a normalized double cover ramified over $\alpha^*(D^{\mathrm{Kum}})$ and μ is a double cover ramified over one half of the pullback divisor $\nu^*\alpha^*(D^{\mathrm{Tro}})$ as will be described below.

One description of $\widehat{Z}$ is to say that we take the smooth double cover Z of X ramified over D^{Kum}, then $\widehat{Z}$ is the normalization of $\widehat{X} \times_X Z$.

Another viewpoint is that ν is a double cover ramified over K and A, because normalization removes the double cover over $2B$. Since K and A are disjoint, the covering space $\widehat{Z}$ is smooth.

The inverse image of B in $\widehat{Z}$ is a double covering $B_Z \to B$ ramified over $A \cap B$ and $K \cap B$.

The divisor $\alpha^*(D^{\mathrm{Tro}})$ pulls back in $\widehat{Z}$ to a divisor that has multiplicity 2 along B_Z, has two disjoint pieces that compose $\nu^*(T)$, and includes $2A_Z$ where A_Z is the reduced inverse image of A in $\widehat{Z}$. Taking half of this will be one of the two pieces over T plus A_Z plus B_Z. The double covering μ therefore has simple ramification over A_Z, B_Z, and the one piece of $\nu^*(T)$ that we have chosen.

Let A', B' and T' denote the reduced inverse images of these pieces in $\widehat{Y}$. Let K' be the reduced inverse image of K in $\widehat{Y}$. It is a double covering of the ramification locus K_Z of $\widehat{Z}/\widehat{X}$ that lies over K.

We see that $\widehat{Y}$ has ordinary double points at $A' \cap B'$ and $T' \cap B'$.

The map $\widehat{f} : \widehat{Y} \to \widehat{X}$ has the following types of ramification: a cyclic covering of order 4 along A'; simple ramification along B' that maps to B by a generically 2-sheeted covering; simple ramification along T' that maps isomorphically to T, with the other part T'' of the inverse image of T being a generically 2-sheeted covering; and simple ramification along K' that maps by a generically 2-sheeted covering to K. We may write altogether

$$\begin{aligned}
\widehat{f}^*(A) &= 4A' \\
\widehat{f}^*(B) &= 2B' \\
\widehat{f}^*(T) &= 2T' + T'' \\
\widehat{f}^*(K) &= 2K'.
\end{aligned}$$

We can construct a parabolic Higgs bundle over $\widehat{X}$ in the following way: choose a line bundle $\widehat{\mathcal{L}}$ on $\widehat{Y}$, take its direct image, then use the natural filtrations over A, B, T, K to put a parabolic structure with some parabolic weights. The pullback of the tautological 1-form on Y is a 1-form on the smooth locus of $\widehat{Y}$ and this will lead to a logarithmic Higgs field on $\widehat{f}_*(\widehat{\mathcal{L}})$ away from $A \cap B$ and $T \cap B$. As $\widehat{f}_*(\widehat{\mathcal{L}})$ is a bundle over Y, this logarithmic Higgs field extends to a logarithmic Higgs field over all of Y. Furthermore, it will respect the filtrations if we choose them to be compatible with the covering.

On the other hand, if we let $\mathcal{L}$ denote the corresponding line bundle on Y (reflexive extension of the line bundle restricted to the complement of the points over the tacnodes) then $f_*(\mathcal{L})$ is a bundle on X, and it has a meromorphic Higgs field that is logarithmic along the smooth points of the divisor $D = D^{\mathrm{Kum}} + D^{\mathrm{Tro}}$. As we will see below, the Bogomolov-Gieseker inequality indicates that this bundle with Higgs field cannot be considered as being logarithmic over the tacnodes, since its pullback to $\widehat{X}$ will violate the Bogomolov-Gieseker inequality. However, it is the bundle $f_*(\mathcal{L})$ for which we can obtain a calculation of the Chern classes using the Grothendieck-Riemann-Roch formula.

We would therefore like to compare the Chern classes of our parabolic structure on $\widehat{f}_*(\widehat{\mathcal{L}})$ with those of $f_*(\mathcal{L})$ or rather the pullback $\alpha^*(f_*(\mathcal{L}))$.

The comparison between these two things is a local question. To see this, let us assume that the parabolic weights are rational, as will be the case for the structure to be used. Then, there is a root stack $\rho : \widetilde{X} \to \widehat{X}$ such that the parabolic bundle may be viewed as a vector bundle on the root stack. Let us call this bundle $\mathcal{V}$. The root stack structure occurs over $A + B$. On the other hand, let

$$V := \rho^* \left(\alpha^*(f_*(\mathcal{L})) \right)$$

be the bundle pulled back from a vector bundle on X. Let $\jmath : X^\circ \hookrightarrow X$ be the complement of the tacnode, and we have $\widetilde{\jmath} : X^\circ \hookrightarrow \widetilde{X}$. Let $V^\circ := \widetilde{\jmath}^*(V)$, so we are given a natural isomorphism $V^\circ \cong \widetilde{\jmath}^*(\mathcal{V})$. Thus $f_*(\mathcal{L}) = \jmath_*(V^\circ)$. In particular there are inclusions of quasicoherent sheaves on $\widetilde{X}$

$$V \hookrightarrow \widetilde{\jmath}_*(V^\circ) \hookleftarrow \mathcal{V}.$$

These are isomorphisms away from (the inverse images in $\widetilde{X}$ of) the divisors A and B. The images are both contained in a coherent sheaf of the form, say, $V(nA+nB)$ so we may use these to compare the Chern characters, namely we have coherent sheaves $V(nA+nB)/V$ and $V(nA+nB)/\mathcal{V}$ supported on $A+B$ and

$$\mathrm{ch}(\mathcal{V}) - \mathrm{ch}(V) = \mathrm{ch}(V(nA+nB)/V) - \mathrm{ch}(V(nA+nB)/\mathcal{V}).$$

The Chern characters of $V(nA+nB)/V$ and $V(nA+nB)/\mathcal{V}$ only depend on the local picture at the tacnode.

Because of this locality, we may do the calculation assuming that $\mathcal{L} = \mathcal{O}_Y$ is the trivial line bundle, then multiply by 32 since there are 32 tacnodes, to get the global difference of Chern characters, and add this to the Chern character of $f_*(\mathcal{L})$ for the chosen global line bundle $\mathcal{L}$ on Y, to get the Chern character of the parabolic logarithmic extension.

To further simplify the exposition, we are now going to just state what are the good parabolic weights to use. These were found by doing some computations and then solving the optimization problem in a crude way using a computer, and in fact that was done in a couple of stages: the first time, we noticed that the parabolic weights would involve multiples of 1/4. When we picked up the question some time later, instead of re-doing this optimization we just did a grid search over all the possible weights that were multiples of 1/4. At the present time, instead of exposing this (the details of the computation would be difficult to explain, and the technique was not optimal) we will just state how to get the resulting parabolic structure and then check that it satisfies the parabolic Chern class vanishing conditions. This is made easier by the description of the covering in two stages.

First calculate the vector bundle $\mathcal{U} := \widehat{f}_*(\mathcal{O}_{\widehat{Y}})$. The map $\mu : \widehat{Y} \to \widehat{Z}$ is a double covering in particular it is Galois, so

$$\mu_*(\mathcal{O}_{\widehat{Y}}) = \mathcal{W}^+ \oplus \mathcal{W}^-$$

where $\mathcal{W}^+$ and $\mathcal{W}^-$ are two line bundles on $\widehat{Z}$ where the involution of the covering μ acts by $+1$ and -1. Notice that $\mathcal{W}^+ = \mathcal{O}_{\widehat{Z}}$. Taking the direct image by ν of this decomposition gives

$$\mathcal{U} = \mathcal{U}^+ \oplus \mathcal{U}^-$$

where $\mathcal{U}^+ = \nu_*(\mathcal{W}^+)$ and $\mathcal{U}^- = \nu_*(\mathcal{W}^-)$. These are two bundles of rank 2.

The variety $\widehat{Z}$ is obtained from Z (the double cover of X ramified along D^{Kum}) by blowing up twice, with the first time generating an exceptional divisor whose strict transform is B_Z and the second time generating an exceptional divisor A_Z; the other strict transforms are K_Z and T_Z. Note that $B_Z = \nu^*(B)$ whereas $\nu^*(A) = 2A_Z$ and $\nu^*(K) = 2K_Z$ while $\nu^*(T)$ is T_Z plus another piece. The self intersections are $A_Z^2 = -1$ and $B_Z^2 = -2$.

The decomposition of $\hat{\nu}^*(T)$ into two pieces comes from a decomposition of the inverse image of T in the double cover $Z \to X$, with one of the two pieces being the image in Z of our chosen $T_Z \subset \widehat{Z}$. In particular, this divisor is principal in the neighborhood in Z. Its inverse image in the blow-up $\widehat{Z}$ is $A_Z + B_Z + T_Z$, which is therefore also principal.

In taking the double cover ramified over a principal divisor, it means in our situation to use the trivial bundle as a square root. Thus, the covering $\widehat{Y}$ is a double cover of $\widehat{Z}$ defined by using the trivial bundle as the square root of the ramification divisor $A_Z + B_Z + T_Z$. This divisor gives a line bundle whose restriction to B_Z is trivial as is its restriction to A_Z. It follows that

$$\mathcal{W}^- \cong \mathcal{O}_{\widehat{Z}}$$

too. This gives to $\mathcal{W}^-$ an equivariant structure for the double covering below.

Now $\widehat{Z}$ is a double cover of $\widehat{X}$ branched over the divisor $A + T$. It came from normalizing the inverse image of the double cover branched over D^{Tro} and the pullback of that divisor in $\widehat{X}$ is $A + 2B + T$. This is principal and its square-root used for the covering is the trivial bundle. Taking the normalization has the effect of declaring that the square-root of the $2B$ term is B. We have

$$\mathcal{O}_{\widehat{X}} = \sqrt{\mathcal{O}_{\widehat{X}}(A + 2B + T)} = \mathcal{O}_{\widehat{X}}(B) \otimes \sqrt{\mathcal{O}_{\widehat{X}}(A + T)}$$

so $\sqrt{\mathcal{O}_{\widehat{X}}(A + T)} = \mathcal{O}_{\widehat{X}}(-B)$. That is to say, the covering $\widehat{Z}$ is defined by using $\mathcal{O}_{\widetilde{X}}(-B)$ as square-root of $\mathcal{O}_{\widetilde{X}}(A + T)$. Checking, A has self-intersection -2 so $(-B).A = (A + T).A/2$. Thus, the formula for the structure sheaf of the double cover says

$$\nu_*(\mathcal{O}_{\widehat{Z}}) \cong \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(B).$$

This gives for both pieces

$$\nu_*(\mathcal{W}^+) = \nu_*(\mathcal{O}_{\widehat{Z}}) \cong \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(B)$$

and

$$\nu_*(\mathcal{W}^-) \cong \nu_*(\mathcal{O}_{\widehat{Z}}) \cong \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(B).$$

The subsheaves generated by global sections are given by the subsheaf $\mathcal{O}_{\widehat{X}} \subset \mathcal{O}_{\widehat{X}}(B)$ in the second factor in each case.

We can therefore write

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

as a direct sum of trivial line bundles, with

$$\mathcal{U}^+ = V_1 \oplus V_3(B), \qquad \mathcal{U}^- = V_2 \oplus V_4(B),$$

so

$$\nu_*(\mathcal{O}_{\widehat{Y}}) = V_1 \oplus V_2 \oplus V_3(B) \oplus V_4(B).$$

We now consider the constraints on the parabolic structure. Over B, the $\mathcal{U}^-$ piece constitutes the natural subspace of the filtration. The parabolic weights of any piece of $\mathcal{U}^-$ should be $\leq$ the weights of any piece of $\mathcal{U}^+$.

Near A, let b be a local coordinate defining A in X (this is in keeping with the notations we'll have later: A is the a-axis defined by $b = 0$). The local coordinate defining A_Z is $b^{1/2}$, and the local coordinate defining A' is $b^{1/4}$. This is the square-root of the coordinate defining A_Z. The functions $\mathcal{O}_{\widehat{Z}}$ include $1, b^{1/2}, \ldots$. Now $\mathcal{W}^+$ is generated over $\mathcal{O}_{\widehat{Z}}$ by 1 so it includes the functions 1 and $b^{1/2}$. On the other hand, $\mathcal{W}^-$ is generated over $\mathcal{O}_{\widehat{Z}}$ by $b^{1/4}$ so it includes the functions $b^{1/4}$ and $b^{3/4}$.

The subspaces $\mathcal{O}_{\widehat{X}}(B)$ correspond to the functions that are multiples of $b^{1/2}$.

If we say that $V_1 \oplus V_3(B)$ corresponds to $\mathcal{U}^+$ and $V_2 \oplus V_4(B)$ corresponds to $\mathcal{U}^-$ then V_1 is generated over $\mathcal{O}_{\widehat{X}}$ by 1, $V_3(B)$ is generated by $b^{1/2}$, V_2 is generated by $b^{1/4}$ and $V_4(B)$ is generated by $b^{3/4}$.

The parabolic filtration should therefore be adapted to our decomposition, in the order $1, 2, 3, 4$ along A.

We will now declare a precise collection of parabolic levels. Our parabolic structure is now a direct sum of four parabolic line bundles. Recall that when we have a bundle $\mathcal{O}$ with filtration level that is placed at parabolic level $-c \in (-1, 0]$ at a divisor div it corresponds to a line bundle that is written as $\mathcal{O}(c \cdot \mathrm{div})$. Let's denote the parabolic line bundles as P_1, P_2, P_3, P_4.

Define

$$
\begin{aligned}
P_1 &= \mathcal{O}_{\widehat{X}}(-\frac{1}{4}A - \frac{1}{2}B) \\
P_2 &= \mathcal{O}_{\widehat{X}} \\
P_3 &= \mathcal{O}_{\widehat{X}} \\
P_4 &= \mathcal{O}_{\widehat{X}}(\frac{1}{4}A + \frac{1}{2}B).
\end{aligned}
$$

Over A the parts that have been added are in the correct order.

Over B, this is obtained from $V_1 \oplus V_3(B)$ and $V_2 \oplus V_4(B)$ by adding $-B/2$ to V_1, adding $-B$ to $V_3(B)$ (which really means to make an elementary transformation), and adding $-B/2$ to $V_4(B)$. Thus on the $\mathcal{U}^+$ pieces we added $-B/2$ and $-B$ whereas on the $\mathcal{U}^-$ pieces we added 0 and $-B/2$. This satisfies the required criterion. The decomposition into a sum of line bundles is compatible with the Higgs field in the tangential directions, because of the blowing up: tangential vector fields along B map to zero in the tangent bundle of Y. This is the short reason why this parabolic structure is allowable. In subsection 5.5 below, we will do a calculation in local coordinates to make sure that the Higgs field is logarithmic with respect to this structure.

Technically the definition of the local contribution to ch_2 is as was described previously, but it may now be expressed as

$$
\begin{aligned}
&\mathrm{ch}_2(P_1 \oplus P_2 \oplus P_3 \oplus P_4) = \\
&\qquad = \frac{\left(-\frac{1}{4}A - \frac{1}{2}B\right)^2}{2} + \frac{\left(\frac{1}{4}A + \frac{1}{2}B\right)^2}{2} \\
&\qquad = \frac{1}{16}(A + 2B)^2 \\
&\qquad = \frac{1}{16}(A^2 + 4AB + 4B^2).
\end{aligned}
$$

We have $A^2 = -2$ and $B^2 = -1$ with $AB = 1$. So our contribution is

$$(-2 + 4 - 4)/16 = -1/8.$$

The local contribution is therefore $-1/8$. When we multiply by the 32 tacnodes in a plane section, we obtain a global adjustment of -4 to ch_2. In view of the calculations in Corollary 5.11, this adjustment leads to a parabolic Higgs bundle with vanishing first and second Chern characters.

We need to check that the local parabolic structure we have been considering in this section is one for which the Higgs field becomes logarithmic.

5.5 Calculations in coordinates

The easiest way to make sure that the parabolic structure we are defining will induce a logarithmic property of the Higgs field, is to write things out in local coordinates.

Recall that X denotes a slice by a plane in $X_0 = \mathbb{P}^3$, and localize at a tacnode point of the wobbly locus. Then, we will look near a point in the covering Y/X that is the center of the piece of degree 4 over the neighborhood in X. Fix coordinates x, y for a small ball around the point in X, such that the Kummer is given by $y = 0$ and the trope is given by $x^2 - y = 0$. Let $\mathbb{C}\{x, y\}$ be the coordinate ring of convergent series in x and y.

Express the neighborhood in Y as a composition of two coverings of degree 2. The first has coordinates x, v with $y = v^2$, so it has coordinate ring $\mathbb{C}\{x, v\}$. The inverse image of the tacnode divisor in here has equation $x^2 - v^2 = 0$ that decomposes as $(x + v)(x - v)$ so the pullback divisor decomposes into two irreducible components. Choose one of these, say $x - v = 0$, as the branch locus for the second covering. Introduce the coordinate u with $u^2 = x - v$. This gives the system of coordinates u, v for the neighborhood in Y, with ring $\mathbb{C}\{u, v\}$ and equations for the degree 4 map f to the neighborhood in X are:

$$x = u^2 + v, \qquad y = v^2.$$

Calculation of the Jacobian matrix showed that the branch locus downstairs is the union of the components $y = 0$ and $x^2 - y = 0$. Set $\Delta := x^2 - y$.

The direct image $f_*\mathcal{O}$ is a rank 4 vector bundle V over the neighborhood in X, which may be viewed as given by the module $\mathbb{C}\{u, v\}$ considered as a module of rank 4 over $\mathbb{C}\{x, y\}$. Write the decomposition into a direct sum of line bundles $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ corresponding to the module decomposition

$$\mathbb{C}\{u, v\} = 1 \cdot \mathbb{C}\{x, y\} \oplus u \cdot \mathbb{C}\{x, y\} \oplus v \cdot \mathbb{C}\{x, y\} \oplus uv \cdot \mathbb{C}\{x, y\}.$$

Suppose our tautological form on Y is written as $\lambda = \varphi(u, v)du + \psi(u, v)dv$. Note that it satisfies a constraint, due to the fact that the trope plane is the exceptional divisor for a blow-up of the Prym, and the form comes from a form on the Prym. This says that ψ is a multiple of u, although that condition does not seem to be needed later since the dv term does not pose a problem.

The equation $v^2 = y$ tells us that

$$dv = vdy/2y.$$

The equation $u^2 = x - v$ tells us that

$$du = u^{-1}(dx - dv)/2$$

where dv may be written in terms of dy.

The actions of multiplication by u and v my be expressed in the form of 4×4 matrices acting on the direct sum decomposition. Use as basis vectors $1, u, v, uv$. We have

$$u = \begin{pmatrix} 0 & x & 0 & -y \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

One calculates:

$$u^{-1} = \frac{1}{\Delta} \begin{pmatrix} 0 & \Delta & 0 & 0 \\ x & 0 & y & 0 \\ 0 & 0 & 0 & \Delta \\ 1 & 0 & x & 0 \end{pmatrix}$$

as may be verified by multiplying together with the matrix for u. Also, $v^{-1} = v/y$ so

$$v^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/y & 0 & 0 & 0 \\ 0 & 1/y & 0 & 0 \end{pmatrix}.$$

The differentials du and dv may now be expressed as matrices with entries in terms of dx and dy:

$$dv = (v/2y)dy = \begin{pmatrix} 0 & 0 & dy/2 & 0 \\ 0 & 0 & 0 & dy/2 \\ dy/2y & 0 & 0 & 0 \\ 0 & dy/2y & 0 & 0 \end{pmatrix},$$

and

$$du = u^{-1}(dx - dv)/2 = \frac{1}{2\Delta} \begin{pmatrix} 0 & \Delta & 0 & 0 \\ x & 0 & y & 0 \\ 0 & 0 & 0 & \Delta \\ 1 & 0 & x & 0 \end{pmatrix} \cdot \begin{pmatrix} dx & 0 & -dy/2 & 0 \\ 0 & dx & 0 & -dy/2 \\ -dy/2y & 0 & dx & 0 \\ 0 & -dy/2y & 0 & dx \end{pmatrix}$$

$$= \frac{1}{4\Delta}\left(\begin{array}{c:c:c:c} 0 & 2\Delta dx & 0 & -\Delta dy \\ \hdashline 2xdx - dy & 0 & -xdy + 2ydx & 0 \\ \hdashline 0 & -\Delta dy/y & 0 & 2\Delta dx \\ \hdashline 2dx - xdy/y & 0 & 2xdx - dy & 0 \end{array}\right).$$

The terms $2xdx - dy$ are equal to $d\Delta$. We also note that $y = x^2 - \Delta$ so $dy = 2xdx - d\Delta$ and

$$-xdy + 2ydx = (2y - 2x^2)dx + xd\Delta = xd\Delta - 2\Delta dx,$$

hence

$$2dx - xdy/y = (x/y)d\Delta - (2/y)\Delta dx.$$

We may therefore write

$$du = \frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline (x/y)\frac{d\Delta}{\Delta} - 2dx/y & 0 & 0 & 0 \end{array}\right)$$

$$+\frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline \frac{d\Delta}{\Delta} & 0 & x\frac{d\Delta}{\Delta} & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \frac{d\Delta}{\Delta} & 0 \end{array}\right) + \frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 2dx & 0 & -dy \\ \hdashline 0 & 0 & -2dx & 0 \\ \hdashline 0 & -dy/y & 0 & 2dx \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right).$$

The second and third matrices have terms that are either holomorphic forms, or multiples of $d\Delta/\Delta$ and dy/y. When we pull back to a blow-up, these will remain as (at worst) logarithmic forms. The first matrix, with a single non-zero coefficient in the lower left, is not logarithmic. This term is however logarithmic on points of the divisors $y = 0$ or $\Delta = 0$ away from the origin, indeed that is clear from the expression here when $y \neq 0$ and it is clear from the original expression $(1/4\Delta)(2dx - xdy/y)$ when $\Delta \neq 0$. This term will lead to a non-logarithmic term when we blow up twice the tacnode, so the elementary transformations and parabolic structures will need to take that into account.

Consider the filtration of V that has three steps, with quotient V_1, subquotient $V_2 \oplus V_3$ in the middle, and subbundle V_4. This filtration is preserved by the operators of multiplication by u or v, modulo the maximal ideal (x, y). Furthermore, it is preserved by the residues, although not strictly because of the terms $xd\Delta/\Delta$ in position $(2, 3)$ and $-dy/y$ in position $(3, 2)$.

We can therefore use this filtration to put parabolic structures after blowing up, as will be done in the next subsection.

5.6 Pulling back to the blow-up and a root cover

Blowing up the origin introduces the coordinate $a = y/x$ so $y = ax$ and the coordinate chart on the blow up has coordinates (x, a).

The main term in the lower left corner of du becomes

$$(x/y)\frac{d\Delta}{\Delta} - 2dx/y = \frac{1}{a} d \log \left(\frac{x-a}{x} \right) = \frac{-1}{x-a} d \log(ax).$$

Then blow up again using $b = x/a$ so $x = ab$ and $y = a^2 b$. Our lower left corner term becomes

$$\frac{1}{a(b-1)} \left(2\frac{da}{a} + \frac{db}{b} \right).$$

This is logarithmic along the divisor A which is $b = 0$ (the a-axis) and has a pole of order 2 along B (the b-axis given by $a = 0$).

We therefore need to make an elementary transformation in order to get a logarithmic pole along B. This corresponds to the normalization of the spectral covering in the previous discussion.

In all, we are going to use the same decomposition into 4 line bundles, but putting parabolic levels on these pieces, to get a decomposition into four parabolic line bundles of the form

$$\mathcal{O}(-A/4 - B/2) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(A/4 + B/2).$$

We would like to calculate that this indeed gives a parabolic logarithmic Higgs bundle when we use a Higgs field that is a combination of du and dv.

To do this calculation, let us pull back to a root covering. Since we want the divisors $A/4$ and $B/2$, with A given by $b = 0$ and B given by $a = 0$, let's introduce the root coordinates

$$\alpha = a^{1/2}, \quad \beta = b^{1/4}.$$

Thus

$$a = \alpha^2, \ \ b = \beta^4, \ \ x = \alpha^2\beta^4, \ \ y = \alpha^4\beta^4,$$

and we have

$$dx = 2\alpha\beta^4 d\alpha + 4\alpha^2\beta^3 d\beta, \quad dy = 4\alpha^3\beta^4 d\alpha + 4\alpha^4\beta^3 d\beta$$

and

$$\Delta = \alpha^4\beta^8 - \alpha^4\beta^4 = \alpha^4\beta^4(\beta^4 - 1).$$

The lower left corner term is

$$(*) = \frac{-1}{x-a} d\log(y) = \frac{-4}{\alpha^2(\beta^4-1)}\left(\frac{d\alpha}{\alpha} + \frac{d\beta}{\beta}\right).$$

The full matrix for du becomes:

$$\frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 2dx & 0 & -dy \\ \hdashline d\Delta/\Delta & 0 & -2dx + xd\Delta/\Delta & 0 \\ \hdashline 0 & -dy/y & 0 & 2dx \\ \hdashline (*) & 0 & d\Delta/\Delta & 0 \end{array}\right)$$

$$= \frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 4\alpha\beta^4 d\alpha + 8\alpha^2\beta^3 d\beta & 0 & -4\alpha^3\beta^4 d\alpha + 4\alpha^4\beta^3 d\beta \\ \hdashline d\Delta/\Delta & 0 & -4\alpha\beta^4 d\alpha - 8\alpha^2\beta^3 d\beta + \alpha^2\beta^4 d\Delta/\Delta & 0 \\ \hdashline 0 & -d\log(\alpha^4\beta^4) & 0 & 4\alpha\beta^4 d\alpha + 8\alpha^2\beta^3 d\beta \\ \hdashline (*) & 0 & d\Delta/\Delta & 0 \end{array}\right).$$

Instead of the basis e_1, e_2, e_3, e_4 (which was originally $1, u, v, uv$), the new frame for the new bundle over the root covering is $f_1 = \alpha^{-1}\beta^{-1}e_1, f_2 = e_2, f_3 = e_3, f_4 = \alpha\beta e_4$. In this frame, the new matrix is obtained by multiplying the top row and last column by $\alpha^{-1}\beta^{-1}$, and multiplying the last row and first column by $\alpha\beta$. This yields the new matrix:

$$du_{\mathbf{f}} = \frac{1}{4}\left(\begin{array}{c:c:c:c} 0 & 4\beta^3 d\alpha + 8\alpha\beta^2 d\beta & 0 & -4\alpha\beta^2 d\alpha + 4\alpha^2\beta d\beta \\ \hdashline \alpha\beta d\Delta/\Delta & 0 & -4\alpha\beta^4 d\alpha - 8\alpha^2\beta^3 d\beta + \alpha^2\beta^4 d\Delta/\Delta & 0 \\ \hdashline 0 & -d\log(\alpha^4\beta^4) & 0 & 4\beta^3 d\alpha + 8\alpha\beta^2 d\beta \\ \hdashline \alpha^2\beta^2\cdot(*) & 0 & \alpha\beta d\Delta/\Delta & 0 \end{array}\right).$$

Now,

$$\alpha^2\beta^2\cdot(*) = \frac{-4\beta^2}{(\beta^4-1)}\left(\frac{d\alpha}{\alpha} + \frac{d\beta}{\beta}\right).$$

This is logarithmic at general points of $\alpha = 0$ and $\beta = 0$. Note that the locus $\beta^4 = 1$ corresponds to a strict transform of one of our original divisors and we have verified the logarithmic property over those. The remaining terms of $du_{\mathbf{f}}$ are also logarithmic or better.

Our Higgs field is obtained as a combination of du and dv times $1, u, v, uv$ times functions of x, y. To complete the verification, we need to express the matrices for u, v and dv in terms of the coordinates α and β and in the new frame. In the frame e_1, e_2, e_3, e_4 we had

$$u = \begin{pmatrix} 0 & x & 0 & -y \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^2\beta^4 & 0 & -\alpha^4\beta^4 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \alpha^2\beta^4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which becomes, in the new frame (multiplying the top and bottom rows and the first and last columns as previously):

$$u_{\mathbf{f}} = \begin{pmatrix} 0 & \alpha\beta^3 & 0 & -\alpha^2\beta^2 \\ \alpha\beta & 0 & 0 & 0 \\ 0 & -1 & 0 & \alpha\beta^3 \\ 0 & 0 & \alpha\beta & 0 \end{pmatrix}.$$

Similarly, in the original frame

$$v = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha^4\beta^4 & 0 \\ 0 & 0 & 0 & \alpha^4\beta^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

so in the new frame

$$v_{\mathbf{f}} = \begin{pmatrix} 0 & 0 & \alpha^3\beta^3 & 0 \\ 0 & 0 & 0 & \alpha^3\beta^3 \\ \alpha\beta & 0 & 0 & 0 \\ 0 & \alpha\beta & 0 & 0 \end{pmatrix}.$$

Also, in the previous frame $dv = (dy/2y) \cdot v$ so in the new frame,

$$dv_{\mathbf{f}} = 2\left(\frac{d\alpha}{\alpha} + \frac{d\beta}{\beta}\right) \begin{pmatrix} 0 & 0 & \alpha^3\beta^3 & 0 \\ 0 & 0 & 0 & \alpha^3\beta^3 \\ \alpha\beta & 0 & 0 & 0 \\ 0 & \alpha\beta & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c:c:c:c} 0 & 0 & \alpha^2\beta^3 d\alpha + \alpha^3\beta^2 d\beta & 0 \\ \hdashline 0 & 0 & 0 & \alpha^2\beta^3 d\alpha + \alpha^3\beta^2 d\beta \\ \hdashline \beta d\alpha + \alpha d\beta & 0 & 0 & 0 \\ \hdashline 0 & \beta d\alpha + \alpha d\beta & 0 & 0 \end{array}\right).$$

We see that any products of $1, u, v, uv$ times du or dv in the new frame are logarithmic, so the Higgs field will induce a logarithmic Higgs field on this bundle as desired. This completes the proof of theorem to be stated in the concluding subsection below.

5.7 Degree zero case—conclusion

Theorem 5.16. *Suppose $\mathcal{L}_0$ is a flat line bundle on the abelian variety $\mathcal{P}$, and define the spectral line bundle $\mathcal{L} = \boldsymbol{\varepsilon}_0^*(\mathcal{L}_0) \otimes \mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F})$ on Y. Put*

$$\mathcal{E} := f_*(\mathcal{L})$$

as a meromorphic Higgs bundle on $X = \mathbb{P}^3$, then blow up twice at the tacnodes in a general planar section, and put the parabolic structure we have defined above so that the Higgs field becomes logarithmic. For this parabolic structure, $\mathrm{ch}_i^{\mathrm{par}} = 0$ for $i = 1, 2$. Therefore, this parabolic Higgs bundle admits an extension to a purely imaginary twistor $\mathscr{D}$-module as in Theorem 3.13. The associated local system on $X_0 - \mathrm{Wob}_0$ has rank 8. Its monodromy around the trope planes in Wob_0 is unipotent with a single Jordan block of size 2. Its monodromy around the Kummer surface in Wob_0 is unipotent consisting of a direct sum of 4 Jordan blocks of size 2.

Proof. In keeping with the result of Corollary 5.11, choose the line bundle $\mathcal{L}$ to be anything numerically equivalent to $\mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F})$, that is to say anything of the form $\mathcal{L} = \boldsymbol{\varepsilon}_0^*(\mathcal{L}_0) \otimes \mathcal{O}_Y(\boldsymbol{E} + 2\boldsymbol{F})$ for $\mathcal{L}_0$ a flat line bundle on the Hitchin fiber $\mathcal{P}$.

The tautological 1-form on $\mathcal{P}$ pulls back to a 1-form on Y, which we may view as a meromorphic section of $f^*(\Omega^1_X)$ on Y, yielding a meromorphic Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ satisfying the commutativity condition $\Phi \wedge \Phi = 0$. A local calculation at the ramification points of $f : Y \to X$ shows that Φ has logarithmic singularities along smooth points of the branch divisor of f. However, the fact that $\mathcal{P}$ may be viewed as a subvariety of the cotangent bundle of X over the very stable points, shows that Φ has no poles along smooth points of the branch divisor not on the wobbly locus. Thus, we obtain a logarithmic Higgs bundle $(\mathcal{E}^\circ, \Phi^\circ)$ over X° defined to be the complement of the singular locus of Wob.

Let $\zeta : \widehat{X}^+ \to X^+$ be obtained by blowing up twice the trope conics, on the open subset $X^+ \subset X$ complement of the 16 singular points of Kum.

Our discussion in the transverse plane sections to the trope conics shows how to create a parabolic bundle $\mathcal{E}_\cdot^+$ on $(\widehat{X}^+, D^+)$ where D^+ is the reduced inverse image of the wobbly divisor in $\widehat{X}^+$. The nontrivial parabolic structure is concentrated on the parts of D^+ lying over the trope conics. The parabolic structure extends the given bundle $\mathcal{E}^\circ$ from the open subset $X^\circ \subset \widehat{X}^+$. We have seen, in the transverse plane sections, that the Higgs field Φ^+ (unique extension of Φ°) becomes logarithmic for this parabolic structure. This verification was completed in the previous subsection.

Let $\mathcal{E}^+_{\mathrm{raw}} := \zeta^*\mathcal{E}|_{X^+}$. If $X_H \subset X$ is a plane (general so that it misses the singular points of Kum) it intersects the trope conics in a total of 32 points (two points for each of the 16 conics). Its inverse image $\widehat{X}_H$ is obtained by combining 32 times the local picture we have seen above. We took care to make sure that

$$\mathrm{ch}_1^{\mathrm{par}}\left(\mathcal{E}^+_{\cdot}|_{\widehat{X}_H}\right) = 0.$$

As pointed out in Subsection 5.4, the difference in second Chern characters between $\mathcal{E}^+_{\mathrm{raw}}|_{\widehat{X}_H}$ and $\mathcal{E}^+_{\cdot}|_{\widehat{X}_H}$ is local, so it is 32 times the quantity $(-1/8)$ calculated in Subsection 5.4. In other words,

$$\mathrm{ch}_2^{\mathrm{par}}\left(\mathcal{E}^+_{\cdot}|_{\widehat{X}_H}\right) = \mathrm{ch}_2\left(\mathcal{E}^+_{\mathrm{raw}}|_{\widehat{X}_H}\right) + 32 \cdot (-1/8).$$

On the other hand, from Corollary 5.11, for our choice of line bundle $\mathcal{L}$ we have

$$\mathrm{ch}_2\left(\mathcal{E}^+_{\mathrm{raw}}|_{\widehat{X}_H}\right) = 4.$$

We conclude that $\mathrm{ch}_2^{\mathrm{par}}\left(\mathcal{E}^+_{\cdot}|_{\widehat{X}_H}\right) = 0$.

We now have a parabolic logarithmic Higgs bundle $(\mathcal{E}^+_{\cdot}, \Phi^+)$ on $(\widehat{X}^+, D^+)$ such that the first and second parabolic Chern characters vanish on a plane section $\widehat{X}_H$. The divisor D^+ has normal crossings. Mochizuki's theory [Moc06, Moc09] implies that there exists a tame purely imaginary harmonic bundle on $\widehat{X}^+ - D^+ \cong X - \mathrm{Wob}$ whose corresponding parabolic Higgs bundle is this one. The rank is 8. The monodromy along smooth points of Wob is given by the residue of the Higgs field, which in turn corresponds to the ramification of Y/X over the different components of the wobbly locus, giving the stated properties. $\square$

Remark 5.17. Our technique of construction yields more precise information about the monodromy of the local system along the exceptional divisors in $\widehat{X}$ lying over the trope conics. For example, the eigenvalues around the A divisors are 4-th roots of unity and the eigenvalues around the B-divisors are ± 1. There is also a unipotent piece around the A divisor. It is left to the reader to make a more explicit statement.

6 Hecke operators

6.1 Hecke transformations in terms of bundles

In Chapter 2 we encountered the geometric picture of the Hecke correspondences in the context of pencils of quadrics in $\mathbb{P}^5$. Let us review the Hecke transformations on bundles.

If E is a rank 2 vector bundle and $t \in C$ is a point, the ***Hecke line*** of E at t is the projectivization $\mathbb{P}E_t$ parametrizing rank 1 quotients $E_t \twoheadrightarrow \mathbb{C}$. A vector space quotient corresponds to a surjection of coherent sheaves $E \twoheadrightarrow \mathbb{C}_t$ where $\mathbb{C}_t$ denotes the skyscraper sheaf at t. Let E' denote the kernel of this map, usually known as the ***down-Hecke transform*** of E centered at $E_t \twoheadrightarrow \mathbb{C}$. It is a torsion-free coherent sheaf, hence locally free and therefore it is itself a bundle. We have

$$\det(E') = \det(E) \otimes \mathcal{O}_C(-t).$$

In order to get a map between points of our moduli spaces we need to correct the determinant by tensoring with a line bundle. Thus, consider a point $(A, t) \in \overline{C}$, meaning that A is a line bundle with $A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$. Now, if E is a vector bundle with determinant $\det(E) \cong \mathcal{O}_C(\mathbf{p})$, and $E_t \twoheadrightarrow \mathbb{C}$, we can use the associated down-Hecke $E' = \ker(E \to \mathbb{C}_t)$ with determinant $\mathcal{O}_C(\mathbf{p} - t)$ to form the transformed bundle

$$E' \otimes A.$$

Note that by definition

$$\det(E' \otimes A) = \det(E') \otimes A^{\otimes 2} = \det(E) \otimes \mathcal{O}_C(-t) \otimes \mathcal{O}_C(t - \mathbf{p}) = \mathcal{O}_C.$$

Thus, if E was stable, i.e. representing a point of the moduli space X_1, then the Hecke transform $E' \otimes A$ is a bundle (that one may verify is semistable) with trivial determinant so it corresponds to a point of X_0. We obtain the Hecke $\mathbb{P}^1$ parametrizing the trivial determinant down-Hecke transforms of E at t. It is always a line in $X_0 \cong \mathbb{P}^3$ as we will see in Theorem 6.3.

In the other direction, suppose E is a stable bundle with trivial determinant. We need to transform the "down-Hecke" into an "up-Hecke" to get a point of X_1 with determinant $\mathcal{O}_C(\mathbf{p})$. Again, this will depend on the choice of a point $(A, t) \in \overline{C}$ and a choice of a quotient $E_t \twoheadrightarrow \mathbb{C}$. Once these choices are made we define the transform to be

$$E' \otimes A(\mathbf{p}),$$

where again $E' = \ker(E \to \mathbb{C}_t)$. By definition $\det(E') = \mathcal{O}_C(-t)$ and so

$$\det(E' \otimes A(\mathbf{p})) = \det(E') \otimes A^{\otimes 2}(2\mathbf{p}) = \det(E) \otimes \mathcal{O}_C(-t) \otimes \mathcal{O}_C(t - \mathbf{p}) \otimes \mathcal{O}_C(2\mathbf{p}) = \mathcal{O}_C(\mathbf{p}).$$

It is straightforward again to verify that $E' \otimes A(\mathbf{p})$ is also stable so it is a point of X_1. The image of the Hecke line is a smooth rational curve in X_1, which in fact is a smooth plane conic as we will see in Theorem 6.4. These are the Hecke curves over points of $X_0 - \mathsf{Kum}$.

If we start with a point of the Kummer surface, there are several choices of semistable bundle corresponding to that S-equivalence class. The above formulas, applied to these different bundles, yield the Hecke fiber. For smooth points of Kum it is the union of two lines corresponding to the two semistable bundles; for nodes of Kum it is a single line counted twice. See Theorem 6.4.

In the next section we will put the Hecke correspondences together in a family over the base $\overline{C}$. The fiber of $\overline{\mathcal{H}}$ over $a = (A, t) \in \overline{C}$ will be

$$\overline{\mathcal{H}}(a) := \left\{ (E, \mathsf{k}) \;\middle|\; \begin{array}{l} E \in X_1 \text{ and } \mathsf{k} : E_t \twoheadrightarrow \mathsf{K} \text{ is a one dimensional} \\ \text{quotient of the fiber of } E \text{ at } t. \end{array} \right\}.$$

We have maps

$$X_1 \xleftarrow{p} \overline{\mathcal{H}}(a) \xrightarrow{q} X_0$$

where p is just the projection. It is a $\mathbb{P}^1$-bundle whose fiber over $E \in X_1$ is the projective line of rank 1 quotients of E_t. Existence of the moduli space and the fibration property of the projection are due to the fact that all points of X_1 are stable.

For a point $(E, \mathsf{k}) \in \overline{\mathcal{H}}(a)$ the corresponding point $q(E') \in X_0$ is the S-equivalence class of the down-Hecke of E at the point t, defined by the quotient k and normalized using A as described above.

6.2 The big Hecke correspondences

Putting together these fibers yields the big Hecke correspondences

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0 \times \overline{C} \end{tikzcd} \qquad \begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1 \times \overline{C} \end{tikzcd} \tag{27}$$

whose notation was introduced in Subsection 3.4.

These Hecke correspondences are the ones that most conveniently encode the eigensheaf property for the de Rham data ($\mathscr{D}$-modules) or the Dolbeault data (parabolic Higgs complexes) on the moduli of $\mathbb{P}SL_2(\mathbb{C})$-bundles on C. These are the de Rham and Dolbeault objects that under the Langlands correspondence should correspond to flat $SL_2(\mathbb{C})$-bundles on C or to a semistable Higgs $SL_2(\mathbb{C})$-bundle on C respectively.

To spell this out, note that the moduli of $\mathbb{P}SL_2(\mathbb{C})$-bundles on C is a disconnected Deligne-Mumford stack

$$\mathsf{M}_0 \sqcup \mathsf{M}_1, \quad \mathsf{M}_i = [X_i/\mathsf{J}[2]]\,, \text{ for } i = 0, 1.$$

Here $\mathsf{J}[2]$ is the group of 2-torsion points in the abelian surface $\mathsf{J} := \mathrm{Jac}^0(C)$, and $\mathfrak{a} \in \mathsf{J}[2]$ acts on the moduli space $X_0 \sqcup X_1$ of bundles with fixed determinant by $E \mapsto E \otimes \mathfrak{a}$.

The big M_1 to M_0 Hecke correspondence is a correspondence

$$\begin{tikzcd}
 & \textbf{Hecke} \arrow[dl] \arrow[dr] & \\
\mathsf{M}_1 & & \mathsf{M}_0 \times C
\end{tikzcd}$$

It is described explicitly in terms of a correspondence

$$\begin{tikzcd}
 & \mathcal{H} \arrow[dl] \arrow[dr] & \\
X_1 & & \mathscr{X}_0
\end{tikzcd},$$

Here

- $\mathscr{X}_0$ is the moduli space of semistable rank two bundles with determinant of the form $\mathcal{O}_C(\mathbf{p} - t)$ for some $t \in C$. It fibers $\mathscr{X}_0 \to C$ over C, the fiber over a given $t \in C$ being the moduli $\mathscr{X}_0(t)$ is the moduli of bundles with determinant equal to the specific line bundle $\mathcal{O}_C(\mathbf{p} - t)$.

- $\mathcal{H}$ is the moduli space of triples

$$\mathcal{H} = \left\{ (E, E', \beta) \;\middle|\; \begin{array}{c} E \in X_1,\ E' \in \mathscr{X}_0(t) \\ \beta : E' \hookrightarrow E,\ \mathrm{supp}(\mathrm{coker}(\beta)) = t \end{array} \right\}.$$

- the South-West map is given by by $(E, E', \beta) \mapsto E$ and the South-East map is given by $(E, E', \beta) \mapsto E'$.

In fact $\mathscr{X}_0 \to C$ is an algebraic $\mathbb{P}^3$ bundle, equipped with a flat (Heisenberg-like) connection with monodromy $\mathsf{J}[2]$. The group $\mathsf{J}[2]$ also acts on $\mathscr{X}_0$ by tensorization, and the action preserves the flat connection. The quotient

$$[\mathscr{X}_0/\mathsf{J}[2]] \to C$$

inherits a flat structure with trivial monoromy and taking $\mathbf{p} \in C$ as a base point we get an algebraic isomorphism

$$\begin{tikzcd} {[\mathscr{X}_0/\mathsf{J}[2]]} \arrow[r] \arrow[d, "\cong"'] & C \arrow[d, equal] \\ \mathsf{M}_0 \times C \arrow[r, "\mathrm{pr}_C"'] & C. \end{tikzcd}$$

The group $\mathsf{J}[2]$ also acts on $\mathcal{H}$ by tensoring with 2-torsion line bundles:

$$\mathsf{J}[2] \times \mathcal{H} \to \mathcal{H}, \quad (\mathfrak{a}, (E, E', \beta)) \mapsto (E \otimes \mathfrak{a}, E' \otimes \mathfrak{a}, \beta \otimes \mathrm{id}_{\mathfrak{a}}),$$

and passing to quotients we get

$$\left[\begin{tikzcd} & {[\mathcal{H}/\mathsf{J}[2]]} \arrow[dl] \arrow[dr] & \\ {[X_1/\mathsf{J}[2]]} & & {[\mathscr{X}_0/\mathsf{J}[2]]} \end{tikzcd}\right] \cong \left[\begin{tikzcd} & \textbf{Hecke} \arrow[dl] \arrow[dr] & \\ \mathsf{M}_1 & & \mathsf{M}_0 \times C \end{tikzcd}\right].$$

Similar comments apply to the action of the big Hecke correspondence in the other direction. Thus we can recast the problem of finding a Hecke eigensheaf on $\mathsf{M}_0 \sqcup \mathsf{M}_1$ as the equivalent problem of finding a $\mathsf{J}[2]$-equivariant Hecke eigensheaf on $X_1 \sqcup \mathscr{X}_0$.

We can refine this further by observing that $\mathscr{X}_0$ trivializes on a finite cover of C. Indeed, recall the curve $\mathsf{sq} : \overline{C} \to C$ which is an étale $\mathsf{J}[2]$-Galois cover of C defined as the fiber product

$$\begin{tikzcd} \overline{C} \arrow[r, "\imath_{\overline{C}}"] \arrow[d, "\mathsf{sq}"'] & \mathrm{Jac}^0(C) \arrow[d, "\mathsf{mult}_2"] \\ C \arrow[r, "\mathsf{AJ}_{\mathbf{p}}"'] & \mathrm{Jac}^0(C) \end{tikzcd}$$

where the bottom horizontal arrow is the $\mathbf{p}$-based Abel-Jacobi map $\mathsf{AJ}_{\mathbf{p}} : C \to \mathrm{Jac}^0(C)$, $t \mapsto \mathcal{O}_C(t - \mathbf{p})$. In other words we have $\overline{C} = \{(A, t) \in \mathrm{Jac}^0(C) \times C \,|\, A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(t)\}$.

Because the monodromy of $\mathscr{X}_0$ over C is $\mathsf{J}[2]$, the pullback of $\mathscr{X}_0$ trivializes canonically if we use $(\mathcal{O}_C, \mathbf{p})$ as the base point on $\overline{C}$. That is, we have a fiber square

$$\begin{tikzcd} X_0 \times \overline{C} \arrow[r] \arrow[d, "\mathrm{pr}_{\overline{C}}"'] & \mathscr{X}_0 \arrow[d] \\ \overline{C} \arrow[r, "\mathsf{sq}"'] & C \end{tikzcd}$$

where the top horizontal map is given by $(E, (A, t)) \mapsto E \otimes A^{-1}$. Thus we get a base changed Hecke diagram

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0 \times \overline{C} \end{tikzcd}$$

where the base changed big Hecke correspondence is the moduli

$$\overline{\mathcal{H}} = \left\{ ((E, E', \beta), (A, t)) \;\middle|\; \begin{array}{l} E \in X_1,\ E' \in X_0,\ (A, t) \in \overline{C} \\ \beta : E' \otimes A^{-1} \hookrightarrow E,\ \mathrm{supp}(\mathrm{coker}(\beta)) = t \end{array} \right\}$$

and the maps p and q are defined by

$$p((E, E', \beta), (A, t)) := E, \quad \text{and} \quad q((E, E', \beta), (A, t)) := (E', (A, t)).$$

For future reference, note that $\overline{\mathcal{H}}$ can also be viewed as a correspondence

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1 \times \overline{C} \end{tikzcd}$$

where $d = \mathrm{pr}_{X_0} \circ q$ and $b = p \times (\mathrm{pr}_{\overline{C}} \circ q)$.

Thus the Hecke eigensheaf problem on $\mathsf{M}_0 \sqcup \mathsf{M}_1$ can be reformulated as the problem of finding a $\mathsf{J}[2] \times \mathsf{J}[2]$-equivariant $\overline{\mathcal{H}}$-eigensheaf on $X_0 \sqcup X_1$. Here an element $(\mathfrak{a}_1, \mathfrak{a}_2) \in \mathsf{J}[2] \times \mathsf{J}[2]$ acts by

$$\begin{aligned} &(\mathfrak{a}_1, \mathfrak{a}_2) \cdot E = E \otimes \mathfrak{a}_1 \otimes \mathfrak{a}_2^{-1},\ \text{for} E \in X_1, \\ &(\mathfrak{a}_1, \mathfrak{a}_2) \cdot (E', (A, t)) = (E' \otimes \mathfrak{a}_1, (A \otimes \mathfrak{a}_2, t)),\ \text{for } (E', (A, t)) \in X_0 \times \overline{C}, \\ &(\mathfrak{a}_1, \mathfrak{a}_2) \cdot \beta = \beta \otimes \mathrm{id}_{\mathfrak{a}_1 \otimes \mathfrak{a}_2^{-1}}. \end{aligned}$$

With this in place we can formulate the Dolbeault version of the $\overline{\mathcal{H}}$-Hecke eigensheaf problem as follows.

Problem 6.1 (**Dolbeault $\overline{\mathcal{H}}$-Hecke eigensheaf problem**)**.** Fix a general semistable $SL_2(\mathbb{C})$-Higgs bundle (E, θ) on C. Construct $\mathsf{J}[2] \times \mathsf{J}[2]$-equivariant tame parabolic Higgs bundles $(\mathscr{F}_{0,\bullet}, \Phi_0)$ on X_0 and $(\mathscr{F}_{1,\bullet}, \Phi_1)$ on X_1 so that

- $\mathscr{F}_{0,\bullet}$ and $\mathscr{F}_{1,\bullet}$ have rank 8.
- The parabolic structure on $\mathscr{F}_{i,\bullet}$ and the poles of Φ_i are along the wobbly divisors in X_i for $i = 0, 1$.
- the first and second parabolic Chern classes of each $\mathscr{F}_{i,\bullet}$ are trivial and $(\mathscr{F}_{i,\bullet}, \Phi_i)$ are stable for $i = 0, 1$.
- $\mathscr{F}_{i,\bullet}$ satisfy the $\overline{\mathcal{H}}$-eigensheaf property with eigenvalue (E, θ). In other words we have

  $$(X_1 \textbf{ to } X_0) \qquad q_* p^* (\mathscr{F}_{1,\bullet}, \Phi_1) = (\mathscr{F}_{0,\bullet}, \Phi_0) \boxtimes \mathsf{sq}^*(E, \theta),$$
  $$(X_0 \textbf{ to } X_1) \qquad b_* d^* (\mathscr{F}_{0,\bullet}, \Phi_0) = (\mathscr{F}_{1,\bullet}, \Phi_1) \boxtimes \mathsf{sq}^*(E, \theta).$$

  where all pullbacks, pushforwards, and tensoring are induced from the corresponding operations on polarized twistor $\mathscr{D}$-modules.

6.3 Comparison with the synthetic approach

In the modular direction used for most of the present paper, we start with the curve C having a chosen Weierstrass point $\mathbf{p} \in C$, define the moduli spaces X_0 and X_1, and observe that X_0 is isomorphic to $\mathbb{P}^3$ and X_1 is the base locus of a pencil of quadrics in $\mathbb{P}^5$ [NR69]. In the following discussion, the Narasimhan-Ramanan projective space will be denoted by $\mathbb{P}^5_{NR}$.

Choose $a = (A, t)$ in the covering curve $\overline{C}$. The Hecke correspondence

$$\overline{\mathcal{H}}(a) \subset X_0 \times X_1$$

has Hecke fibers that are lines in X_0, over every point of X_1. This yields a map $X_1 \to \mathsf{Grass}(2, 4)$ to the Grassmanian of lines in X_0. If we write $X_0 = \mathbb{P}(V)$ for a four-dimensional vector space V then the Grassmanian embedds by the Plücker embedding $\mathsf{Grass}(2,4) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ with image a quadric hypersurface. We get the composed map

$$X_1 \to \mathsf{Grass}(2,4) \hookrightarrow \mathbb{P}(\bigwedge^2 V).$$

Proposition 6.2 (cf Theorem 4, [NR69])**.** *The restriction $\mathcal{O}_{\mathbb{P}(\bigwedge^2 V)}(1)|_{X_1}$ is isomorphic to the line bundle $\mathcal{O}_{X_1}(1)$ corresponding to the embedding of X_1 as an intersection of two quadrics in $\mathbb{P}^5_{NR}$. There is an isomorphism $\mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5_{NR}$ such that $\mathsf{Grass}(2,4)$ becomes one of the quadrics in the pencil containing X_1, and the above composed map identifies with the embedding $X_1 \hookrightarrow \mathbb{P}^5_{NR}$. Under these identifications, the modular Hecke correspondence $\overline{\mathcal{H}}(a)$ is equal to the synthetic incidence Hecke correspondence of Section 2.4.*

Proof. Consider a line $\ell \subset X_1$ corresponding to a family of bundles fitting into an exact sequence

$$0 \to L \to E \to L^\vee(\mathbf{p}) \to 0.$$

The family of Hecke lines over bundles E in this family consists of the lines in X_0 contained in a plane and passing through a point. The point is the S-equivalence class of $(L \otimes A) \oplus (L^\vee \otimes A(\mathbf{p} - t))$, while the plane is the subset of X_0 consisting of bundles that have a nontrivlal map from $L \otimes A(-t)$ (as pointed out in the proof of Proposition 5.2, the discussion of [NR69, Proposition 6.1, Theorem 2] implies that this is a plane).

The Schubert cycle σ_1 of codimension 1 in $\mathsf{Grass}(2,4)$, the divisor of $\mathcal{O}_{\mathsf{Grass}(2,4)}(1)$, is the set of lines passing through a given general line $V \subset X_0$. Then V intersects the plane in a general point, and there is exactly one line in the family that passes through that point. Thus, the pullback of the Schubert cycle to the line ℓ via the embedding $\ell \subset X_1 \to \mathsf{Grass}(2,4)$ is a single point. This shows that the pullback of $\mathcal{O}_{\mathsf{Grass}(2,4)}(1)$ is $\mathcal{O}_{X_1}(1)$ since the Picard group of X_1 is $\mathbb{Z}$.

The 6-dimensional space $\bigwedge^2 V$ of sections that embedd the Grassmanian pulls back to a 6-dimensional space of sections of $\mathcal{O}_{X_1}(1)$, and this is also the 6-dimensional space of sections of $\mathcal{O}_{\mathbb{P}^5_{NR}}(1)$. We get the required identification between $\mathbb{P}^5_{NR}$ and $\mathbb{P}(\bigwedge^2 V)$. The Grassmanian is a quadric containing X_1 so it is one of the members of the pencil.

Now $X_0 = \mathbb{P}(V)$ is recovered as one of the rulings of the quadric Grassmanian, namely the ruling of Schubert cycles σ_2: a point $x \in X_0$ corresponds to the Schubert cycle $\sigma_2(x)$ of lines through x, which is a plane in $\mathsf{Grass}(2,4)$. The dual projective space parametrizes the Schubert cycles $\sigma_{1,1}(h)$ of lines in a given plane $h \subset X_0$.

The synthetic Hecke correspondence of Section 2.4 is the incidence correspondence saying when a point of X_1 is in an element of the ruling. Now, the ruling is isomorphic to X_0 and a point E of X_1 is in the element $\sigma_2(x)$ of the ruling corresponding to $x \in X_0$, if and only if the Hecke line corresponding to E contains x. This is the same as the Hecke correspondence $\overline{\mathcal{H}}(a)$. This equality is indeed tautological, because we already used $\overline{\mathcal{H}}(a)$ to get the map $X_1 \to \mathsf{Grass}(2,4)$ that led to the identification $\mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5_{NR}$. $\square$

6.4 Description of Hecke curves

The previous proposition shows that the Hecke correspondence $\overline{\mathcal{H}}(a)$ may be viewed, in the synthetic picture of Section 2.4, as the incidence correspondence between points x of X_1, corresponding to lines $\ell_x \subset \mathbb{P}^3$ and hence also to points of the Grassmanian quadric G, thought of as points in $\mathbb{P}^5$ (that happen to be on the other quadric G' defining the pencil too), and points y of X_0, corresponding to planes $\Pi_y \subset \mathbb{P}^5$ (that happen to be contained in the Grassmanian quadric G and to belong to ruling R):

$$\begin{aligned}\overline{\mathcal{H}}(a) &= \left\{(x,y) \in X_1 \times X_0 \;\middle|\; x \in \Pi_y \subset \mathbb{P}^5\right\}\\ &= \left\{(x,y) \in X_1 \times X_0 \;\middle|\; y \in \ell_x \subset \mathbb{P}^3\right\}.\end{aligned}$$

More precisely, if $a = (A, t)$ we have that $t \in C$ corresponds to the ruling R of the quadric G which is identified as the Grassmannian.

We now get a clear description of the Hecke curves:

- Given $x \in X_1$, the Hecke curve $p^{-1}(x)$ is the line ℓ_x itself.
- Given $y \in X_0$, the Hecke curve $q^{-1}(y)$ is the conic $\Pi_y \cap X_1 = \Pi_y \cap G'$.

As explained in section 2.4 we can interpret this globally as describing a subvariety $\overline{\mathcal{H}}$ of $\overline{C} \times X_0 \times X_1$. Recall that we have three copies of $(\mathbb{Z}/2)^4$, acting respectively on $\overline{C}$, X_0, and X_1 with quotients C, M_0, and M_1. The quotient $\mathscr{R}$ of $\overline{C} \times X_0$ by the diagonal action of $(\mathbb{Z}/2)^4$ fibers $\mathbf{rul} : \mathscr{R} \to C$ over C with fibers non-canonically isomorphic to X_0. In section 2 we described $\mathscr{R}$ as the universal ruling for the pencil of quadrics: it parametrizes the family of planes Π contained in ***some*** member of our pencil. If we choose a Weierstrass point $\mathbf{p} \in C$, we can also identify $\mathscr{R}$ with the family $\mathscr{X}_0$ of pairs (t, V) with $t \in C$ and V a rank 2 bundle on C with determinant $\mathcal{O}_C(\mathbf{p} - t)$. A point $a \in \overline{C}$ above $t \in C$ determines a square root A of the line bundle $\mathcal{O}_C(t - \mathbf{p})$, and tensoring with A converts V to an $SL(2)$ bundle, i.e. kills its determinant. So the pullback of $\mathscr{X}_0$ to $\overline{C}$ is identified with $\overline{C} \times X_0$, proving the identification of $\mathscr{R}$ with $\mathscr{X}_0$.

Most naturally, the incidence description above gives a Hecke subvariety $X_1 \times \mathscr{R}$:

$$\left\{(x,y) \in X_1 \times \mathscr{R} \,\middle|\, x \subset \Pi_y \subset \mathbb{P}^5.\right\}, \tag{28}$$

which is identified with $\mathcal{H} \subset X_1 \subset \mathscr{X}_0$ via the isomorphism $\mathscr{R} \cong \mathscr{X}_0$. As noted above, the Hecke actual $\mathbb{P}GL(2)$ Hecke correspondence that we need is a substack $\mathbf{Hecke} \subset C \times \mathsf{M}_0 \times \mathsf{M}_1$. The relationship is that the subvariety (28) is the inverse image of $\mathbf{Hecke}$ under the $(\mathbb{Z}/2)^8$-quotient map $\mathscr{R} \times X_1 \to C \times \mathsf{M}_0 \times \mathsf{M}_1$. Alternatively, what we are doing mainly in this paper, is to pull back further to $\overline{\mathcal{H}} \subset \overline{C} \times X_0 \times X_1$ where the Hecke eigensheaf calculations may be viewed as most straightforward.

The set of points $p(q^{-1}(y))$ is the set of points in X_1 that admit a Hecke transform equal to y. In the correspondence with the quadric line complex, it is the set of points whose associated line ℓ_x passes through y. That was called X_p in [GH94]. The Hecke curves $p^{-1}(x)$ are projective lines, so they never degenerate. The following theorem uses the results of section 2 to restate the results of [NR69] in the synthetic language.

Theorem 6.3. *The fiber $p^{-1}(x)$ over any point $x \in X_1$ is mapped by q isomorphically to a line in $X_0 \cong \mathbb{P}^3$. This provides a map $X_1 \to G = \mathsf{Grass}(2,4)$ to the Grassmanian of lines in $\mathbb{P}^3$. Furthermore, the embedding $X_1 \subset \mathbb{P}^5$ extends uniquely to an embedding of $\mathsf{Grass}(2,4)$ in $\mathbb{P}^5$ identifying it with one of the quadrics in the pencil that cuts out X_1. The family of projective planes in $\mathbb{P}^3$ consisting of all the lines through a given point, is a ruling of the quadric G, and this ruling is identified with the point $t \in C$ (image of $a \in \overline{C}$) via the identification between C and the set of pairs of a quadric in the pencil and a ruling of that quadric.*

On the other hand, the Hecke curves $q^{-1}(y)$ are conics, embedded in planes in $G \subset \mathbb{P}^5$ so they can degenerate into pairs of lines. This happens when two distinct lines $\ell, \mathsf{m} \subset X_1$ intersect, and Π_y is their span in $\mathbb{P}^5$. Our choice of a Weierstrass point $\mathbf{p} \in C$ allows us to identify the variety of lines in X_1 with the degree zero Jacobian $\mathrm{Jac}^0(C)$ of C. In particular a line $\ell \subset X_1$ corresponds to a line bundle $L \in \mathrm{Jac}^0(C)$ in such a way that distinct lines ℓ, m intersect if and only if for the corresponding points $L, M \in \mathrm{Jac}^0(C)$ the degree 1 line bundle $L \otimes M \otimes \mathcal{O}_C(\mathbf{p})$ is effective on C, i.e. it is $\mathcal{O}_C(t)$ for some $t \in C$. The plane Π_y spanned by such a pair of lines ℓ, m is then in the ruling $\mathscr{R}_t$ indexed by this t. The locus of such Π_y's in $R = \mathscr{R}_t$, for a fixed $t \in C$, is isomorphic to the Kummer surface $\mathrm{Jac}^0(C)/(\pm 1) \subset R$.

It is also possible for the Hecke curve $q^{-1}(y)$ to degenerate into a double line. This happens when $\ell = \mathsf{m}$, which occurs for 16 lines ℓ if we fix t. If we vary t, these lines ℓ are parametrized by the cover $\overline{C} \to C$, and their union in X_1 is the wobbly locus of X_1. However, in $X_0 = R$, their image is a curve (=quotient of $\overline{C}$ by the action of the hyperelliptic involution of C) which is contained in the Kummer.

This is all analyzed and proved synthetically in section 2. For ease of reference we summarize the above conclusions as follows:

Theorem 6.4. *The Hecke fibers over points of the trivial determinant moduli space X_0 fall into three categories:*

- *Over $X_0 - \mathsf{Kum}$, the map q is smooth with fibers that are identified with their images in X_1. These images are conics inside planes in $\mathbb{P}^5$. If $y \in X_0 - \mathsf{Kum}$, the corresponding conic $q^{-1}(y)$, isomorphic to $\mathbb{P}^1$, is the intersection of the plane $\Pi_y \subset \mathbb{P}^5$ with X_1. The conic $q^{-1}(y)$ is identified with the space of rank 1 quotients of the fiber at $t \in C$ of the stable rank 2 bundle with trivial determinant corresponding to y. This gives the viewpoint of a Hecke correspondence going back from X_0 to X_1.*

- *Over smooth points of Kum the fibers of q degenerate into reducible conics composed of two distinct lines meeting at a single point.*

- *Over the 16 singular points of Kum, the fiber degenerates further into a double line.*

Proof. The trichotomy of possibilities was discussed above in Corollary 2.10, see [GH94, pp 762-763]. The synthetic Hecke correspondence used there is identified with the modular Hecke correspondence in question here, by Proposition 6.2. □

Let $\mathsf{K3}(a) \subset \overline{\mathcal{H}}(a)$ denote the closure of the subset of points lying over Kum that are the intersection points of the two lines in the fibers of q over general points of Kum.

Remark 6.5. The subvariety $\mathsf{K3}(a)$ is the K3 surface obtained by blowing up the 16 nodes of Kum. The double lines over the nodes are the exceptional divisors. Its image $p(\mathsf{K3}(a)) \subset X_1$ is the K3 surface denoted by Σ in [GH94]. This embedding depends on $a \in \overline{C}$.

Remark 6.6. In fact the surface $\mathsf{K3}(a)$ only depends on the choice of point $t \in C$ rather than $a \in \overline{C}$, due to the fact that the Kummer surface is invariant by the action of $(\mathbb{Z}/2\mathbb{Z})^4$ and in another viewpoint, the Hecke correspondence between X_1 and a moduli space $\mathscr{X}_0(t)$ depends only on the choice of t; the image of the Kummer K3 surface back into X_1 should therefore only depend on t. We do not need this fact so a full proof is not given.

6.5 The Heisenberg group

For our given curve C the group $\mathsf{J}[2]$ of points of order 2 on $\mathrm{Jac}^0(C)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and we will adopt informally the latter notation for this group. It may be viewed as the group of bundles on C with structure group $\mathbb{Z}/2\mathbb{Z}$, a group that we can view in turn as the center of $SL_2(\mathbb{C})$.

There is a natural ***Heisenberg central extension***

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathsf{Heisen} \to (\mathbb{Z}/2\mathbb{Z})^4 \to 1 \tag{29}$$

whose extension class is given by the natural $\mathbb{Z}/2\mathbb{Z}$-valued symplectic bilinear form on $(\mathbb{Z}/2\mathbb{Z})^4 = H^1(C, \mathbb{Z}/2\mathbb{Z})$ given by the intersection pairing. Indeed, this $\mathbb{Z}/2\mathbb{Z}$-valued symplectic bilinear form on $(\mathbb{Z}/2\mathbb{Z})^4 = H^1(C, \mathbb{Z}/2\mathbb{Z})$ defines a finite Heisenberg central extension

$$1 \to \mathbb{Z}/2 \to \mathsf{Heisen}^{\mathrm{fin}} \to (\mathbb{Z}/2\mathbb{Z})^4 \to 1$$

which under the natural inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{G}_{\mathrm{m}}$ induces the extension (29). In particular, by construction, the finite group $\mathsf{Heisen}^{\mathrm{fin}}$ is a normal subgroup in Heisen.

Consider the theta divisor

$$\Theta = \left\{ L \in \mathrm{Jac}^0(C) \,\middle|\, h^0(C, L(\mathbf{p})) \geq 1 \right\}$$

corresponding to the theta characteristic $\mathcal{O}_C(\mathbf{p})$ on C. From the work of Mumford [Mum66, Mum07a, Mum08, Mum07b, BL04a] it is known that the group Heisen can be identified with the ***theta group*** of authomorphisms of the total space of the line bundle $\mathcal{O}(2\Theta) \in \mathrm{Pic}(\mathrm{Jac}^0(C))$ that lift the translation action of $\mathsf{J}[2]$ on $\mathrm{Jac}^0(C)$.

In particular Heisen acts on the vector space $H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta)) \cong \mathbb{C}^4$ and this action can be identified [Mum66, Mum08] with the Schrödinger representation, i.e. with the unique irreducible representation of Heisen with a tautological central character. The projectivisation of $H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta))$ is the Narasimhan-Ramanan model of the moduli space of rank two bundles with trivial determinant giving an identification $X_0 \cong \mathbb{P}^3$ [NR69]. The center of Heisen acts by scalars, so the action descends to an action of $(\mathbb{Z}/2\mathbb{Z})^4$ on X_0.

The resulting action of Heisen on $\mathbb{C}^6 = \bigwedge^2(\mathbb{C}^4)$ is also an action for which the center of Heisen acts by scalars and thus again provides an action of $(\mathbb{Z}/2\mathbb{Z})^4$ on $\mathbb{P}^5$, preserving and hence acting on X_1. If we view $(\mathbb{Z}/2\mathbb{Z})^4$ as being the group of bundles on C with structure group the center of $SL_2(\mathbb{C})$ then these actions are the actions obtained by tensoring semistable bundles with finite order line bundles that we described in section 6.2. It is worth noting that the action of $(\mathbb{Z}/2\mathbb{Z})^4$ on X_1 can be naturally linearized on the hyperplane bundle $\mathcal{O}_{\mathbb{P}^5}(1)_{|X_1}$ while the action of $(\mathbb{Z}/2\mathbb{Z})^4$ on X_0 is only projective, i.e. it does not linearize on the hyperplane bundle $\mathcal{O}_{X_0}(1) = \mathcal{O}_{\mathbb{P}^3}(1)$. Indeed, the group Heisen and hence its subgroup $\mathsf{Heisen}^{\mathrm{fin}} \subset \mathsf{Heisen}$ both act linearly on $\mathbb{C}^4$ and on $\mathbb{C}^6 = \bigwedge^2 \mathbb{C}^4$. Since the center of $\mathsf{Heisen}^{\mathrm{fin}}$ acts by multiplication by ± 1 on $\mathbb{C}^4$ it follows that this center acts trivially on $\mathbb{C}^6 = \bigwedge^2 \mathbb{C}^4$. In particular the action of $\mathsf{Heisen}^{\mathrm{fin}}$ factrors through a linear action of $(\mathbb{Z}/2\mathbb{Z})^4$ on $\mathbb{C}^6$. In contrast, since $\mathbb{C}^4$ is the Schrödinger representation of $\mathsf{Heisen}^{\mathrm{fin}}$, and this representation determines the non-trivial extension class defining $\mathsf{Heisen}^{\mathrm{fin}}$, it follows that the projective action of $(\mathbb{Z}/2\mathbb{Z})^4$ on $\mathbb{P}^3$ can not be lifted to a linear action on $\mathbb{C}^4$.

The natural family of moduli spaces $\mathscr{X}_1 \to C$ whose fiber over $t \in C$ is the moduli space $\mathscr{X}_1(t)$ of stable bundles of determinant $\mathcal{O}_C(t)$, is étale locally trivial but not globally trivial. It is obtained by dividing $X_1 \times \overline{C}$ by the action of $(\mathbb{Z}/2\mathbb{Z})^4$, where a 2-torsion line bundle $\mathfrak{a} \in (\mathbb{Z}/2\mathbb{Z})^4 = \mathsf{J}[2]$ acts by sending a piar $(E, (A, t)) \in X_1 \times \overline{C}$ to the pair $(E \otimes \mathfrak{a}, (A \otimes \mathfrak{a}, t))$.

This is similar to the family $\mathscr{X}_0 \to C$ of moduli spaces, whose fiber over $t \in C$ we described before, where the fiber over $t \in C$ is the space $\mathscr{X}_0(t)$ of bundles determinant $\mathcal{O}_C(t - \mathbf{p})$. Note that while $\mathscr{X}_1$ depends only on the curve C, the space $\mathscr{X}_0$ depends on having fixed the Weierstrass point $\mathbf{p}$.

As mentioned above, pulling back to the covering $\overline{C} \to C$ gives a trivialization

$$\mathscr{X}_1 \times_C \overline{C} \cong X_1 \times \overline{C}.$$

This yields the isomorphism between $\mathscr{X}_1(t)$ and X_1, through which we pass to obtain the Hecke correspondence between X_0 and X_1 depending on the point $a = (A, t) \in \overline{C}$. We similarly have a trivialization

$$\mathscr{X}_0 \times_C \overline{C} \cong X_0 \times \overline{C}.$$

If one wants to work with structure group $G = \mathbb{P}GL(2)$, the moduli of G-bundles on C is a disjoint union $\mathsf{M} = \mathsf{M}_0 \sqcup \mathsf{M}_1$. Each M_i is a quotient of X_i by the action of $(\mathbb{Z}/2\mathbb{Z})^4$. The action on X_1 flips the sign of an even subset of the 6 coordinates. As the point t varies, M_1 is constant, while X_1 is fixed only up to this subgroup of its finite group of symmetries. The action on X_0 is the Heisenberg action on $\mathbb{P}^3 = \mathbb{P}H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta))$. Neither X_0 nor M_0 depend on $t \in C$. In this viewpoint, the Hecke correspondence is a 5-dimensional subvariety of $C \times \mathsf{M}_0 \times \mathsf{M}_1$. Fixing $t \in C$ and lifting from M_i to X_i gives the Hecke correspondence between X_0 and $\mathscr{X}_1(t)$, and then fixing a lifting of t to $a \in \overline{C}$ yields the Hecke correspondence between X_0 and X_1.

6.6 Hecke fibers over the nodes

The Hecke fibers over nodes of Kum are lines in X_1 counted with multiplicity two. In this subsection we indicate their locations, in particular they will be lines in the wobbly locus. For comparison, we look also at the special lines on the wobbly locus that correspond to the trope planes.

For the latter question, suppose given a line in the wobbly locus parametrizing all non-split extensions

$$0 \to L \to E \to L^\vee(\mathbf{p}) \to 0,$$

for some line bundle L for which $L^{\otimes 2} = \mathcal{O}(q - \mathbf{p})$. If $a = (A, t) \in \overline{C}$ is given, then the corresponding Hecke transforms of such an E are of the form $E' \otimes A$ where E' is the kernel of some map $E \twoheadrightarrow \mathbb{C}_t$. Now, for all Hecke transforms except a special one that we will ignore here, the bundle E' will be an extension

$$0 \to L(-t) \to E' \to L^\vee(\mathbf{p}) \to 0.$$

Thus the Hecke transform contains as subbundle $U = L \otimes A(-t)$. The equation saying that this collection is equal to a trope plane is the equation $U^{\otimes 2} = \mathcal{O}(-2\mathbf{p})$. For $U = L \otimes A(-t)$ we have

$$U^{\otimes 2} = L^{\otimes 2} \otimes A^{\otimes 2}(-2t) = \mathcal{O}(q - \mathbf{p} + t - \mathbf{p} - 2t) = \mathcal{O}(q - t - 2\mathbf{p}).$$

The condition that we are on a trope plane is that this bundle is $\mathcal{O}(-2\mathbf{p})$, i.e. the equation $\mathcal{O}(q - t) = \mathcal{O}$ in other words, $q = t$. Thus, we conclude that the 16 lines on the wobbly locus that correspond to bundles whose Hecke transforms are trope planes, are the 16 lines corresponding to solutions of $L^{\otimes 2} = \mathcal{O}(q - \mathbf{p})$ with $q = t$.

Now to the main question: suppose given a node of $\mathsf{Kum} \subset X_0$, and let us do the Hecke transformation. The Hecke transformation is unstable if we take the polystable representative to start with, so let us look at a bundle E that is an extension of V by itself with $V^{\otimes 2} = \mathcal{O}_C$. Then the subsheaf E' has as line subbundle $V(-t)$, so the Hecke transform $E' \otimes A(\mathbf{p})$ has as line subbundle $L = V \otimes A(\mathbf{p} - t)$. Let us verify that this is on the wobbly locus: we have

$$L^{\otimes 2} = V^{\otimes 2} \otimes A^{\otimes 2}(2\mathbf{p} - 2t) = \mathcal{O}(t - \mathbf{p}) \otimes \mathcal{O}(2\mathbf{p} - 2t) = \mathcal{O}(\mathbf{p} - t).$$

We may also set $q = t'$ to be the conjugate point of t, and note that $\mathcal{O}(t + q) = \mathcal{O}(2\mathbf{p})$ so we can write

$$L^{\otimes 2} = \mathcal{O}(t - \mathbf{p}) \otimes \mathcal{O}(2\mathbf{p} - 2t) = \mathcal{O}(t + q - \mathbf{p} - t) = \mathcal{O}(q - \mathbf{p}).$$

Thus the bundle $E' \otimes A(\mathbf{p})$ being an extension containing L as subbundle, is on the wobbly locus. It corresponds to the line over the point (L, q) of $\overline{C}$ with $L = V \otimes A(\mathbf{p} - t)$ and $q = t'$ is the conjugate point of t.

We now have a description of the 16 lines, that are on the wobbly locus, that are Hecke transforms of the nodes. They are the 16 lines corresponding to solutions of $L^{\otimes 2} = \mathcal{O}(q - \mathbf{p})$ with $q = t'$ being the conjugate point of t.

Putting together these two collections of 16 lines, we get 32 lines on the wobbly locus that correspond to the 32 points in $\overline{C}$ whose images in C are either t or t'.

It looks like these should be the 32 lines under discussion on [GH94, pp 775-77]. In particular, the 32 lines of [GH94] should have the property that they are 'special' in the sense of the definition of page 792 (i.e. being lines of the wobbly locus). This is certainly known in the classical theory but does not seem to have been mentioned in [GH94].

One may also ask to describe the planes in X_0 that correspond to the second collection of 16 lines i.e. the Hecke fibers over nodes. We think that the plane corresponding to the Hecke fiber over a node will contain that node, and will have tangent cone that is the tangent line to the conic (tangent cone of Kum) at the point of the hyperelliptic line given by the image of t. We do not have a proof of that; it should follow from a closer look at the theory of the relationship between the Kummer surface Kum and its dual [Keu97, GH94], but that goes beyond our present scope.

6.7 Pullbacks of wobbly divisors and ramification

Let

$$\overline{W}_0 := d^{-1}(\mathrm{Wob}_0) \subset \overline{\mathcal{H}}, \quad \overline{W}_1 := p^{-1}(\mathrm{Wob}_1) \subset \overline{\mathcal{H}}$$

be the inverse images of the wobbly divisors in the big Hecke correspondence.

Fix a point $a = (A, t) \in \overline{C}$. Let $\overline{W}_0(a)$ and $\overline{W}_1(a)$ denote the fibers of $\overline{W}_0 \to \overline{C}$ and $\overline{W}_1 \to \overline{C}$ over $a \in \overline{C}$. We have a diagram

$$\begin{tikzcd} \overline{W}_1(a) \arrow[r, hook] \arrow[dr] & \overline{\mathcal{H}}(a) \arrow[d] \\ & X_0 \end{tikzcd}$$

Recall that the map $p\overline{\mathcal{H}}(a) \to X_1$ is a $\mathbb{P}^1$-bundle. Therefore, the map $\overline{W}_1(a) \to \mathrm{Wob}_1$ is smooth. It follows that the singularities of $\overline{W}_1(a)$ are the same as those of Wob_1 pulled back. Namely, we have a locus of cusps and a locus of nodes in codimension 1.

Let $\mathrm{Wob}_1^{\mathsf{n}} \to \mathrm{Wob}_1$ denote the normalization, so $\mathrm{Wob}_1^{\mathsf{n}} = \overline{C} \times \mathbb{P}^1$. Similarly denote by $\overline{W}_1(a)^{\mathsf{n}}$ the normalization of $\overline{W}_1(a)$, which maps by a smooth $\mathbb{P}^1$-fibration to $\mathrm{Wob}_1^{\mathsf{n}}$.

We give here some statements about the wobbly from X_1 pulled back and ramifying over X_0.

6.7.1. The map $\overline{W}_1(a) \to X_0$ is a proper morphism which is a finite 16-sheeted cover away from a codimension 2 subset of X_0.

Proof: We have computed that the class of the divisor $\mathrm{Wob}_1 \in X_1$ is $8H$, where H is the hyperplane class in $\mathbb{P}^5$. This means that Wob_1 intersects each line in X_1 at 8 points and each conic in X_1 at 16 points. Since the fibers of $\overline{\mathcal{H}}(a) \to X_0$ are the conics in X_1 this shows that $\overline{W}_1(a) \to X_0$ is a map of degree 16. $\square$

6.7.2. Let $J = \mathrm{Jac}^0(C))$ denote the Jacobian of C viewed as the moduli of lines $\ell \subset X_1$. Let

$$\Gamma = \{(x, \ell) \in X_1 \times J \mid x \in \ell\}$$

be the incidence correspondence. Γ is a $\mathbb{P}^1$-bundle over J. In the interpretation of X_1 as the moduli of vector bundles of determinant $\mathcal{O}_C(\mathbf{p})$ a point of the Jacobian $A \in J$ corresponds to the line $\ell_A \subset X_1$ parametrizing all non-trivial extensions

$$0 \to A \to E \to A^\vee(\mathbf{p}) \to 0.$$

Thus the fiber ℓ_A of $\Gamma \to J$ over A is identified canonically with the line $\mathbb{P}(H^1(C, A^{\otimes 2}(-\mathbf{p})))$. In particular if $A \in \overline{C} \subset J$, then $A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$ for some point $t \in C$ and so we have $H^1(C, A^{\otimes 2}(-\mathbf{p})) = H^1(C, \mathcal{O}_C(-t'))$ where t' is the hyperelliptic conjugate of t. This space in turn is equal to $H^1(C, \mathcal{O}_C)$ under the natural embedding $\mathcal{O}_C(-t') \subset \mathcal{O}_C$. In other words the restriction of the $\mathbb{P}^1$-bundle $\Gamma \to J$ to the curve $\overline{C} \subset J$ is trivial, i.e. $\Gamma_{|\overline{C}} \cong \mathbb{P}^1 \times \overline{C}$ - the product of the hyperelliptic $\mathbb{P}^1$ and $\overline{C}$. The projection map $\Gamma \to X_1$ is known [New68] to be a finite morphism of degree 4.

Claim 6.7.3. (a) For every line $\ell \subset X_1$ there is natural line bundle $\gamma(\ell)$ of degree three giving a rational map $C \dashrightarrow \ell$. The map has a base point if and only if $\ell \subset \mathrm{Wob}_1$ (ℓ "wobbles").

(b) The branch divisor of the map $\Gamma \to X_1$ contains Wob_1, and the ramification divisor above Wob_1 is equal to

$$\Gamma_{|\overline{C}} \cong \mathbb{P}^1 \times \overline{C} \to \mathbb{P}^1 \times (\overline{C} \times \overline{C}) = (\mathbb{P}^1 \times \overline{C}) \times \overline{C} \to \mathrm{Wob}_1 \times J \subset X_1 \times J,$$

where the maps between products are the natural maps on the components. Also, $\Gamma \to X_1$ is simply ramified at the general point of $\mathbb{P}^1 \times \overline{C} \subset \Gamma$.

(c) If $\ell \subset \mathrm{Wob}_1$ does not wobble, then the intersection points $\ell \cap \mathrm{Wob}_1$ are the 8 branch points for the cover $C \to \mathbb{P}^1$.

Proof. We identify the curve C with the family of rulings of the pencil of quadrics. A point $t \in C$ Corresponds to a ruling R_t , which is a family (whose parameter space is isomorphic to $\mathbb{P}^3$) of planes $\mathbb{P}^2$ contained in quadric $Q_{\mathsf{h}(t)}$, where $\mathsf{h} : C \to \mathbb{P}^1$ is the hyperelliptic map. We get a morphism $i : C \times J \to J$: if ℓ is a line in X_1, it is contained in a unique plane of ruling R_t. The intersection of this plane with X_1 equals its intersection with the quadric Q_λ for any $\lambda \neq \mathsf{h}(t)$, so it consists of ℓ plus another line $i(t, \ell)$. If we use the Weierstrass point $\mathbf{p}$ to embed C in $J = \mathrm{Jac}^0(C)$, then this morphism becomes $i(t, \ell_A) = \ell_{A^\vee(t-\mathbf{p})}$. For each $t \in C$, the restriction $i_t : J \to J$ is an involution, so it has 16 fixed points, namely $\{\mathsf{a} \mid 2\mathsf{a} = \mathcal{O}(t - \mathbf{p})\}$. We see that $\overline{C}$ is the union of these fixed loci as the point t varies over

C. On the other hand, for each $\ell \in X_1$, the restriction i_ℓ identifies C with the family of lines in X_1 intersecting ℓ. So we get a morphism $j : \Gamma \times_{X_1} \Gamma \to \Gamma \times C$.

The fiber product $\Gamma \times_{X_1} \Gamma$ is reducible, consisting of the diagonal plus a 3-sheeted cover $\Gamma' \to \Gamma$. The image $j(\Gamma') \subset \Gamma \times C$ gives a morphism $\Gamma \to \mathrm{Sym}^3 C$ which induces the desired $\gamma : J \to \mathrm{Jac}^3 C$. For a given ℓ, the sections of $\gamma(\ell)$ map C to ℓ itself. A point $x \in C$ is a base point of $\gamma(\ell)$ iff ℓ is a fixed point of the involution i_x. So as we noted above, $\gamma(\ell)$ has a base point iff ℓ is a line contained in Wob_1. More generally, for a line $\ell \subset X_1$, the points where ℓ meets Wob_1 are the points above which two of the three points of the corresponding divisor in $|\gamma(\ell)|$ come together . $\square$

Corollary 6.7.4. The general line $\ell \subset X_1$ intersects Wob_1 at 8 distinct points and therefore $\overline{W}_1(a) \to X_0$ is unramified over the general point of the Kummer surface Kum.

The non-smooth divisor of the map $\overline{W}_1(a) \to X_0$ decomposes into three pieces that we will denote as

$$\overline{W}_1(a)^{\mathrm{ramif}}, \quad \overline{W}_1(a)^{\mathrm{node}}, \quad \overline{W}_1(a)^{\mathrm{cusp}}$$

where $\overline{W}_1(a)^{\mathrm{node}}$ is the nodal locus of $\overline{W}_1(a)$, $\overline{W}_1(a)^{\mathrm{cusp}}$ is the cuspidal locus, and $\overline{W}_1(a)^{\mathrm{ramif}}$ is the remainder of the ramification locus.

We note that the ramification locus of the map from the normalization $\overline{W}_1(a)^{\mathsf{n}} \to X_0$ will consist of the pieces mapping to $\overline{W}_1(a)^{\mathrm{ramif}}$ and $\overline{W}_1(a)^{\mathrm{cusp}}$, the latter because a cusp gives ramification. Furthermore, the three pieces are not disjoint since they intersect in codimension 2 of $\overline{W}_1(a)$ but we ignore this aspect.

As $\overline{W}_1(a)$ is a 3-dimensional variety these pieces are two-dimensional, so they map to divisors in X_0. The previous Corollary 6.7.4 tells us that none of these images meet the Kummer surface.

Claim 6.7.5. The images of $\overline{W}_1(a)^{\mathrm{node}}$ and $\overline{W}_1(a)^{\mathrm{cusp}}$ do not contain any trope planes.

Proof. The locus $\overline{W}_1(a)^{\mathrm{cusp}}$ is irreducible and the locus $\overline{W}_1(a)^{\mathrm{node}}$ has 6 pieces. On the other hand, when all the data moves around, the trope planes are permuted in an orbit of size 16, so a monodromy argument in terms of our general parameters implies that these divisors can not contain trope planes. $\square$

Proposition 6.7. *Fix a point $a = (A, t) \in \overline{C}$. Away from the inverse image of a codimension 2 subset of X_0,*

$$\overline{W}_1(a)^{\text{ramif}} = \bigcup_{\mathfrak{a} \in \mathsf{J}[2]} \overline{W}_1(a)^{\text{ramif}}_{\mathfrak{a}}$$

is a union of 16 divisor pieces $\overline{W}_1(a)^{\text{ramif}}_{\mathfrak{a}} \subset \overline{W}_1(a)$ each of which is a ruled surface $\mathbb{F}_1$ mapping to the corresponding trope plane $\mathsf{Trope}_{\mathfrak{a}(\mathbf{p})} \cong \mathbb{P}^2$.

The map back to Wob_1 decomposes as a sum of maps (ruled surfaces)

$$\overline{W}_1(a)^{\text{ramif}}_{\mathfrak{a}} \to (\text{line})_{\mathfrak{a}}(a) \subset \mathrm{Wob}_1$$

where $(\text{line})_{\mathfrak{a}}(a)$ is one of the $\mathbb{P}^1$'s in the expression $\mathrm{Wob}_1^{\mathsf{n}} = \overline{C} \times \mathbb{P}^1 \to \mathrm{Wob}_1$ and it corresponds to the point $b = (A \otimes \xi, t) \in \overline{C}$.

Proof. If $b = (B, u) \in \overline{C}$ then the line $\{b\} \times \mathbb{P}^1 \subset \mathrm{Wob}_1$ pulls back to an $\mathbb{F}_1$-surface in $\overline{\mathcal{H}}(a)$. Such a surface then contracts to a plane that we will denote by $\mathbb{P}^2(b) \subset X_0$. The contraction has an exceptional $\mathbb{P}^1$ that maps to a point we will denote by $x(b) \subset X_0$.

The line corresponds to bundles fitting into an extension

$$0 \to B \to E \to B^{-1}(\mathbf{p}) \to 0$$

and the Hecke transformed bundle will be of the form $E' \otimes A$ where E' is the kernel of a length 1 quotient of E supported on $t \in C$. In particular, the Hecke transformed bundle contains the line bundle $B \otimes A(-t)$, and this determines the plane $\mathbb{P}^2(b)$ as being the plane of semistable bundles in X_0 that contain the degree -1 line bundle $B \otimes A(-t)$.

Let $\overline{W}_1(a)^{\mathsf{p}}$ denote the resulting family of $\mathbb{P}^2$'s. It is a $\mathbb{P}^2$-bundle over $\overline{C}$ with a dominant map to $X_0 = \mathbb{P}^3$.

The map $b \mapsto x(b)$ has image a curve being the subset of codimension 2 appearing in the statement of the proposition. Outside of this curve, $\overline{W}_1(a)^{\mathsf{p}}$ and $\overline{W}_1(a)^{\mathsf{n}}$ coincide. Therefore it is enough to locate the ramification of $\overline{W}_1(a)^{\mathsf{p}} \to X_0$.

The map $b \mapsto \mathbb{P}^2(b)$ may be viewed as a map to the dual projective space that we will denote by $X_0^{\vee}$. It factors as

$$\overline{C} \to \mathrm{Jac}^{-1}(C) \to \mathsf{Kum}^{\vee} \to X_0^{\vee}$$

where $\mathsf{Kum}^{\vee}$ is the ***dual Kummer surface*** appearing in the classical synthetic theory [Keu97, GH94]. It is the dual surface of Kum. The birational map sending a point to its tangent plane blows up the nodes of Kum and blows down the trope conics to get the dual

Kummer surface $\mathsf{Kum}^{\vee}$. We remark that an important element of the classical theory says that $\mathsf{Kum}^{\vee}$ is projectively isomorphic to Kum although not by this birational map.

The map $\mathrm{Jac}^{-1}(C) \to \mathsf{Kum}^{\vee} \subset X_0^{\vee}$ is not an immersion, exactly over the nodes of the dual surface, and there the tangent map vanishes. Any translation of the map $\overline{C} \to \mathrm{Jac}^{-1}(C)$ is an immersion. Thus, the map $\overline{C} \to X_0^{\vee}$ is non-immersive exactly at points that go to nodes of $\mathsf{Kum}^{\vee}$.

This implies that the family of $\mathbb{P}^2(b)$'s becomes stationary exactly whenever the plane in question is a trope plane. In particular, the map $\overline{W}_1(a)^{\mathsf{p}} \to X_0$ is ramified along the planes in $\overline{W}_1(a)^{\mathsf{p}}$ that map to trope planes.

We claim that there are 16 of these, and then we will see by a characteristic class calculation that this accounts for all of the ramification.

The condition that the plane $\mathbb{P}^2(b)$ of vector bundles containing the line bundle $B \otimes A(-t)$ should be a trope plane, is equivalent to saying that $B \otimes A(-t)$ is the dual of a square root of the canonical bundle. We calculate

$$[B \otimes A(-t)]^{\otimes 2} = B^{\otimes 2} \otimes A^{\otimes 2} \otimes \mathcal{O}_C(-2t) = \mathcal{O}_C(u - \mathbf{p} + t - \mathbf{p} - 2t) = \mathcal{O}_C(u - t - 2\mathbf{p}).$$

This is isomorphic to the dual of the canonical bundle exactly when $u = t$. Thus the set of choices of b making $\mathbb{P}^2(b)$ into a trope plane is the set of choices of square-root B such that $B^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$. There are 16 of these.

To complete the proof, we need to see that we have accounted for all of the ramification. Notice that there is another reason for ramification, namely the fact that the map $\overline{W}_1(a)^{\mathsf{n}} \to X_0$ factors through $\overline{W}_1(a) \to X_0$ that has a cuspidal locus. This provides a divisor of ramification that is the inverse image of the cuspidal locus in Wob_1 and consists of a line in each of the planes $\mathbb{P}^2(b)$.

Let ω denote here the canonical class of $\overline{W}_1(a)^{\mathsf{p}}$. Its restriction to each fiber $\mathbb{P}^2(b)$ is $\mathcal{O}_{\mathbb{P}^2}(-3)$. Therefore, the relative canonical class of $\overline{W}_1(a)^{\mathsf{p}}$ over X_0, which is $\mathcal{O}_{X_0}(4) \otimes \omega$, restricts to $\mathcal{O}_{\mathbb{P}_2}(1)$ on each plane $\mathbb{P}^2(b)$.

We already know a divisor, the ramification coming from the cuspidal locus, that is a line in each $\mathbb{P}^2(b)$. This therefore accounts for any ramification that restricts to a divisor in the $\mathbb{P}^2(b)$. The leftover possibility is of ramification consisting of a union of fibers. But, if the map were ramified along a fiber then the family of planes $\mathbb{P}^2(b) \subset X_0$ will be stationary at that fiber. Above we saw that the cases where the planes are stationary correspond exactly to the cases where they are trope planes. We conclude that the ramification locus of the map $\overline{W}_1(a)^{\mathsf{p}} \to X_0$ consists of the 16 planes corresponding to lines in Wob_1 as are described in the statement of the proposition, plus the divisor that maps to $\overline{W}_1(a)^{\mathrm{cusp}}$. $\square$

Similar analysis proves the following

Corollary 6.7.6. The ramification along the planes given in the proposition is simple.

7 Abelianized Hecke

7.1 Setup for abelianization of the Hecke property

Let as before C be our smooth genus 2 curve, and let $\mathsf{p} \in C$ be our fixed Weierstrass point. Let (E, θ) be an eigenvalue $SL_2(\mathbb{C})$-Higgs bundle for our Hecke eigensheaf problem. That is

- E is a rank two algebraic vector bundle on C with trivial determinant.
- $\theta : E \to E \otimes \omega_C$ is a traceless Higgs field satisfyig the genericity condition that the spectral curve
$$\widetilde{C} \, : \, \det(\lambda \cdot \mathrm{id} - \boldsymbol{\pi}^*\theta) = 0 \ \subset \ T^\vee C$$
is smooth.

Note that this genericity condition automatically implies that the Higgs bundle (E, θ) is stable, i.e. corresponds to a flat bundle and a polarized twistor $\mathscr{D}$-module on C by the non-abelian Hodge correspondence [Sim92].

Let $\boldsymbol{N} \in \mathrm{Pic}(\widetilde{C})$ be the spectral line bundle corresponding to (E, θ). That is $\boldsymbol{N}$ is a line bundle on $\widetilde{C}$ such that
$$E = \pi_* \boldsymbol{N}, \quad \theta = \pi_*(\lambda \otimes (-)).$$
The spectral data $(\widetilde{C} \subset T^\vee C, \boldsymbol{N})$ is the ***abelianization*** of the Higgs bundle (E, θ). We used this spectral data to construct modular spectral data defininig a tame parabolic rank 8 Higgs bundle on the moduli of semistable $\mathbb{P}SL_2(\mathbb{C})$-bundles on C. We already checked that this rank 8 Higgs bundle is stabe and has vanishing first and second parabolic Chern class, hence by Mochizuki's theorem [Moc07a, Moc07b] it gives rise to a polarized twistor $\mathscr{D}$-module and a tame parabolic flat bundle. Now we will use the modular spectral data to rewrite the Hecke eigensheaf property on this rank 8 Higgs bundle in terms of abelianized

information and the abelianized Hecke correspondence. We will then explain how to use the abelianized picture to prove that the rank 8 Higgs bundle is indeed a Hecke eigensheaf satisfying the conditions in Problem 6.1.

Our strategy in constructing $(\mathscr{F}_{i,\bullet}, \Phi_i)$ and checking the eigensheaf property is to use abelianization. To abelianize the problem we look at the $\widetilde{C}$-Hitchin fibers inside the moduli of Higgs bundles Higgs_0 and Higgs_1 respectively. These are the degree 2 and degree 3 Prym varieties $\mathcal{P}_2$ and $\mathcal{P}_3$ of the cover $\pi : \widetilde{C} \to C$:

$$\mathcal{P}_2 = \left\{ M \in \mathrm{Jac}^2(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(M) = \omega_C \right\},$$
$$\mathcal{P}_3 = \left\{ M \in \mathrm{Jac}^3(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(M) = \omega_C(\mathbf{p}) \right\}.$$

We have natural rational maps

$$\begin{array}{ccccccc} \mathcal{P}_2 & \dashrightarrow & X_0, & \quad \text{and} \quad & \mathcal{P}_3 & \dashrightarrow & X_1, \\ M & \longmapsto & \pi_* M & & M & \longmapsto & \pi_* M \end{array} \tag{30}$$

which are surjective, quasifinite, and finite of degree 8 over the very stable loci in X_0 and X_1. These maps fit in the following commutative diagrams of correspondences

$$\begin{tikzcd} & \mathcal{P}_2 \times \widehat{C} \arrow[dl, "\mathsf{sum}"'] \arrow[d, dashed] \arrow[dr, "\mathsf{id}"] & \\ \mathcal{P}_3 \arrow[dd, dashed, "\pi_*"'] & & \mathcal{P}_2 \times \widehat{C} \arrow[dd, dashed, "\pi_* \times \widehat{\pi}"] \\ & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0 \times \overline{C} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{P}_3 \times \widehat{C} \arrow[dl, "\mathsf{diff}"'] \arrow[d, dashed] \arrow[dr, "\mathsf{id}"] & \\ \mathcal{P}_2 \arrow[dd, dashed, "\pi_*"'] & & \mathcal{P}_3 \times \widehat{C} \arrow[dd, dashed, "\pi_* \times \widehat{\pi}"] \\ & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1 \times \overline{C} \end{tikzcd} \tag{31}$$

where $\widehat{C}$ is the curve defined by the fiber product

$$\begin{tikzcd} \widehat{C} \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{\mathsf{sq}}"'] & \overline{C} \arrow[d, "\mathsf{sq}"] \\ \widetilde{C} \arrow[r, "\pi"'] & C \end{tikzcd},$$

the maps $\mathsf{sum}/\mathsf{diff}$ are the sum/difference maps

$$\begin{array}{lrcl} \mathsf{sum}: & \mathcal{P}_2 \times \widehat{C} & \longrightarrow & \mathcal{P}_3, \\ & (L,(A,\tilde{t})) & \longmapsto & L\otimes \pi^* A^{-1}(\tilde{t}) \end{array} \qquad \begin{array}{lrcl} \mathsf{diff}: & \mathcal{P}_3 \times \widehat{C} & \longrightarrow & \mathcal{P}_2, \\ & (L,(A,\tilde{t})) & \longmapsto & L\otimes \pi^* A(-\tilde{t}) \end{array}$$

while the rational maps $\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}$ and $\mathcal{P}_3 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}$ are defined by

$$\begin{aligned} &\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}, \qquad (L,(A,\tilde{t})) \mapsto ((E,E',\beta),(A,t)), \\ &\mathcal{P}_3 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}, \qquad (M,(A,\tilde{t})) \mapsto ((V,V',\gamma),(A,t)) \end{aligned}$$

where

$$\left| \begin{array}{l} E = \pi_*\left((L\otimes \pi^*A^{-1})(\tilde{t})\right), \\ E' = \pi_* L, \\ \beta = \pi_*\left[L\otimes\pi^*A^{-1} \hookrightarrow (L\otimes \pi^*A^{-1})(\tilde{t})\right], \\ t = \pi(\tilde{t}). \end{array}\right. \quad \text{and} \quad \left| \begin{array}{l} V = \pi_* M, \\ V' = \pi_*\left((M\otimes \pi^*A(-\tilde{t})\right), \\ \gamma = \pi_*\left[M(-\tilde{t}) \hookrightarrow M\right], \\ t = \pi(\tilde{t}). \end{array}\right. \tag{32}$$

The modular spectral covers Y_0 and Y_1 corresponding to $\widetilde{C}$ are minimal resolutions

$$\begin{tikzcd} Y_0 \arrow[r, "f_0"] \arrow[d, "\varepsilon_o"'] & X_0 \\ \mathcal{P}_2 \arrow[ur, dashed] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_1 \arrow[r, "f_1"] \arrow[d, "\varepsilon_1"'] & X_1 \\ \mathcal{P}_3 \arrow[ur, dashed] & \end{tikzcd}$$

of the rational maps in (30). We will recall the explicit construction of these resolutions below. The resulting covering maps $f_i : Y_i \to X_i$ are finite of degree 8 and we will use them to construct the Hecke eigensheaf.

Let $(\widetilde{C} \subset T^\vee C, \boldsymbol{N})$ be the spectral data abelianizing the Higgs bundle (E,θ) on C. Our approach to solving Problem 6.1 is to use the Fourier-Mukai transform on the Jacobian of $\widetilde{C}$ to convert the skyscraper sheaf $\mathcal{O}_{\boldsymbol{N}}$ to line bundles on $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. After that we pull back these line bundles to Y_0, Y_1, modify them appropriately along the exceptional divisors of the blow-up maps $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ and $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$ and push them forward via f_0 and f_1 to get the eigensheaf: a pair of parabolic Higgs bundles $(\mathscr{F}_{0,\bullet},\Phi_0)$ $(\mathscr{F}_{1,\bullet},\Phi_1)$ on $X_)$ and X_1 respectively, which are stable and satisfy the vanishing Chern class conditions from Problem 6.1.

Once the parabolic Higgs bundles $(\mathscr{F}_{0,\bullet}, \Phi_0)$ and $(\mathscr{F}_{1,\bullet}, \Phi_1)$ are constructed via this procedure, we can use the diagram (31) to rewrite the eigensheaf equation from Problem 6.1 as an equation of line bundles. Specifically, the (X_1 to X_0) part of the eigensheaf condition becomes an equation of line bundles on $Y_0 \times \widehat{C}$ while the (X_0 to X_1) part of eigensheaf condition becomes an equation of line bundles on $Y_1 \times \widehat{C}$. To carry this out and to show that our Fourier-Mukai constructions satisfy the equations on $Y_0 \times \widehat{C}$ and $Y_1 \times \widehat{C}$ we will need the abelianized Hecke correspondence $\widehat{\mathcal{H}}^{ab}$ which can be described either as the minimal blow-up of $\mathcal{P}_2 \times \widehat{C}$ resolving the rational map $\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}$ or as the minimal blow-up of $\mathcal{P}_3 \times \widehat{C}$ resolving the rational map $\mathcal{P}_3 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}$. In the next section we will construct and analyze this minimal resolution in detail.

7.2 The abelianized Hecke correspondence in context

Recall that the (big) abelianized Hecke space $\widehat{\mathcal{H}}^{\mathrm{ab}}$ depends on the choice of a fixed spectral curve $\pi : \widetilde{C} \to C$ and can be viewed either as a correspondence between Y_1 and $Y_0 \times \widehat{C}$ or as a correspondence between Y_0 and $Y_1 \times \widehat{C}$:

$$
\begin{tikzcd}
 & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "p^{\mathrm{ab}}"'] \arrow[dr, "q^{\mathrm{ab}}"] & \\
Y_1 & & Y_0 \times \widehat{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "d^{\mathrm{ab}}"'] \arrow[dr, "b^{\mathrm{ab}}"] & \\
Y_0 & & Y_1 \times \widehat{C}
\end{tikzcd}
\tag{33}
$$

Here

- Y_0 is the blow-up of the Prym variety

$$\mathcal{P}_2 = \left\{ M \in \mathrm{Jac}^2(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(M) = \omega_C \right\},$$

  at the 16 points $\{\pi^*\kappa\}_{\kappa \in \mathsf{Spin}(C)} \subset \mathcal{P}_2$, where $\mathsf{Spin}(C) = \{\kappa \in \mathrm{Jac}^1(C) \,|\, \kappa^{\otimes 2} = \omega_C\}$ denotes the set of theta characteristics on C. We will write $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ for the blow up morphism and $\boldsymbol{E}_0 = \sqcup_{\kappa \in \mathsf{Spin}(C)} \boldsymbol{E}_{0,\kappa}$ for the corresponding exceptional divisor.

- Y_1 is the blow-up of the Prym variety

$$\mathcal{P}_3 = \left\{ M \in \mathrm{Jac}^3(\widetilde{C}) \;\middle|\; \mathsf{Nm}_\pi(M) = \omega_C(\mathbf{p}) \right\},$$

  at the image of the map (4) which is explicitly given by

$$\boldsymbol{\imath}_{\widehat{C}} : \widehat{C} \to \mathcal{P}_3, \quad (A, \tilde{t}) \mapsto \pi^*(A^{-1}(\mathbf{p})) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t}).$$

We will write $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$ for the blow up morphism and $\boldsymbol{E}_1$ for the corresponding exceptional divisor.

Note that $\boldsymbol{E}_0$ is disconnected with 16 connected components $\boldsymbol{E}_{0,\kappa}$ each isomorphic to $\mathbb{P}^2$. In contrast the divisor $\boldsymbol{E}_1$ is irreducible. In fact, as we saw in Lemma 3.2 and Lemma 3.9**(b)** it is smooth and isomorphic to $\widehat{C} \times \mathbb{P}^1$.

We are now ready to describe the space $\widehat{\mathcal{H}}^{\mathrm{ab}}$ as a common blow-up of $Y_0 \times \widehat{C}$ and $Y_1 \times \widehat{C}$. Namely $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is the blow up of $Y_0 \times \widehat{C}$ centered at the strict transform of the surface $\widehat{C} \times \widehat{C} \subset \mathcal{P}_2 \times \widehat{C}$, where the inclusion $\widehat{C} \times \widehat{C} \subset \mathcal{P}_2 \times \widehat{C}$ is given by

$$\begin{array}{rccc} \boldsymbol{\imath}_{\widehat{C}\times\widehat{C}} : & \widehat{C} \times \widehat{C} & \hookrightarrow & \mathcal{P}_2 \times \widehat{C} \\ & ((A_1, \tilde{t}_1), (A_2, \tilde{t}_2)) & \longrightarrow & \left(\pi^* \left(A_2 \otimes A_1^{\vee}(\mathbf{p})\right)(\tilde{t}_1 - \tilde{t}_2), (A_2, \tilde{t}_2)\right). \end{array} \tag{34}$$

The fact that (34) is a closed embedding follows from Lemma 3.2 since the restriction of the map (34) to $\widehat{C} \times \{(A_2, \tilde{t}_2)\}$ is a translate of the map (4).

Note next that for each theta characteristic $\kappa \in \mathsf{Spin}(C)$, the curve $\{\pi^*\kappa\} \times \widehat{C} \subset \mathcal{P}_2 \times \widehat{C}$ is contained in the image of (34). Indeed, if $(A, \tilde{t}) \in \widehat{C}$, then $(A \otimes \kappa^{\vee}(\mathbf{p}), \tilde{t})$ is also in $\widehat{C}$, and the image of the pair $((A \otimes \kappa^{\vee}(\mathbf{p}), \tilde{t}), (A, \tilde{t}))$ under the map (34) is precisely $(\pi^*\kappa, (A, \tilde{t})) \in \mathcal{P}_2 \times \widehat{C}$. This implies that the strict transform of $\widehat{C} \times \widehat{C}$ in

$$Y_0 \times \widehat{C} = \mathsf{Bl}_{\bigsqcup_{\kappa \in \mathsf{Spin}(C)} \{\pi^*\kappa\} \times \widehat{C}} \left(\mathcal{P}_2 \times \widehat{C}\right)$$

is still equal to $\widehat{C} \times \widehat{C}$. Hence we can describe the abelianized Hecke correspondence $\widehat{\mathcal{H}}^{\mathrm{ab}}$ as the iterated blow up

$$\widehat{\mathcal{H}}^{\mathrm{ab}} = \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \left(Y_0 \times \widehat{C}\right) = \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \left(\mathsf{Bl}_{\bigsqcup_{\kappa \in \mathsf{Spin}(C)} \{\pi^*\kappa\} \times \widehat{C}} \left(\mathcal{P}_2 \times \widehat{C}\right)\right) \xrightarrow{\boldsymbol{\varepsilon}} \mathcal{P}_2 \times \widehat{C}.$$

Furthermore, since the centers of these succesive blowups are smooth and nested in each other

$$\bigsqcup_{\kappa \in \mathsf{Spin}(C)} \{\pi^*\kappa\} \times \widehat{C} \subset \widehat{C} \times \widehat{C}$$

we can also describe $\widehat{\mathcal{H}}^{\mathrm{ab}}$ by performing the blow ups in the opposite order, i.e. first blow up $\mathcal{P}_2 \times \widehat{C}$ along $\widehat{C} \times \widehat{C}$, and then blow up along the strict transforms of the curves $\pi^*\kappa \times \widehat{C}$.

In other words, we also have

$$\widehat{\mathcal{H}}^{\mathrm{ab}} = \mathsf{Bl}_{\bigsqcup_{\kappa\in\mathsf{Spin}(C)} \{\pi^*\kappa\}\times\widehat{C}} \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C}.$$

For ease of reference we introduce notation for the exceptional divisors of these blow ups. We let $\boldsymbol{Exc}_1 \subset \widehat{\mathcal{H}}^{\mathrm{ab}}$ denote the exceptional divisor of the blow up map $\widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_0 \times \widehat{C}$, and let $\boldsymbol{Exc}_0 = \sqcup_\kappa \boldsymbol{Exc}_{0,\kappa} \subset \widehat{\mathcal{H}}^{\mathrm{ab}}$ denote the strict transform of the divisor $\boldsymbol{E}_0 \times \widehat{C} \subset Y_0 \times \widehat{C}$. Equivalently $\boldsymbol{Exc}_0$ is the exceptional divisor of the blow up map $\widehat{\mathcal{H}}^{\mathrm{ab}} \to \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C}$, and $\boldsymbol{Exc}_1$ is the strict transform of the exceptional divisor of the map $\mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C} \to \mathcal{P}_2 \times \widehat{C}$.

The map $q^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_0 \times \widehat{C}$ is just the blow up morphism, while the map $p^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_1$ lifts the Abel-Jacobi sum map

$$\mathsf{sum} : \mathcal{P}_2 \times \widehat{C} \to \mathcal{P}_3, \quad \mathsf{sum}(L, (A, \tilde{t})) = L \otimes \pi^* A^{-1}(\tilde{t})$$

to the blow ups of the source and target. Recall that $Y_1 = \mathsf{Bl}_{\widehat{C}} \mathcal{P}_3$ is the blow up of $\mathcal{P}_3$ centered at the curve $\widehat{C} \hookrightarrow \mathcal{P}_3$, $(A, \tilde{t}) \to \pi^*(A^{-1}(\mathbf{p}))(\tilde{t})$.

The preimage $\mathsf{sum}^{-1}(\widehat{C})$ of this curve under sum is exactly the surface $\widehat{C} \times \widehat{C} \subset \mathcal{P}_2 \times \widehat{C}$ given by the image of (34). So by the universal property of blow ups we see that the map sum lifts to a morphism

$$\mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C} \to \mathsf{Bl}_{\widehat{C}} \mathcal{P}_3 = Y_1$$

which we can further precompose with the second blow up map

$$\widehat{\mathcal{H}}^{\mathrm{ab}} = \mathsf{Bl}_{\bigsqcup_{\kappa\in\mathsf{Spin}(C)} \{\pi^*\kappa\}\times\widehat{C}} \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C} \longrightarrow \mathsf{Bl}_{\widehat{C}\times\widehat{C}} \mathcal{P}_2 \times \widehat{C}$$

to obtain the morphism $p^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_1$. This exhibits $\widehat{\mathcal{H}}^{\mathrm{ab}}$ as a blow-up of $Y_0 \times \widehat{C}$ and gives the first correspondence diagram in (33).

To realize $\widehat{\mathcal{H}}^{\mathrm{ab}}$ as a blow-up of $Y_1 \times \widehat{C}$ and describe the second correspondence diagram in (33) note that the products $\mathcal{P}_2 \times \widehat{C}$ and $\mathcal{P}_3 \times \widehat{C}$ are naturally isomorphic via the pair of mutually inverse maps

$$\begin{aligned} (\mathsf{diff}, \mathrm{id}) : \ \mathcal{P}_3 \times \widehat{C} & \longrightarrow \mathcal{P}_2 \times \widehat{C}, \\ (M, (A, \tilde{t})) & \longmapsto (M \otimes \pi^* A(-\tilde{t}), (A, \tilde{t})) \end{aligned}$$

and

$$\begin{aligned} (\mathsf{sum}, \mathrm{id}) : \ \mathcal{P}_2 \times \widehat{C} & \longrightarrow \mathcal{P}_3 \times \widehat{C}. \\ (L, (A, \tilde{t})) & \longmapsto (L \otimes \pi^* A^{-1}(\tilde{t}), (A, \tilde{t})) \end{aligned}$$

But postcomposing the inclusion (34) with the map $\mathsf{sum} \times \mathrm{id}$ gives the map

$$\begin{aligned} \boldsymbol{\imath}_{\widehat{C}} \times \mathrm{id} : \qquad \widehat{C} \times \widehat{C} & \longrightarrow \mathcal{P}_3 \times \widehat{C}, \\ ((A_1, \tilde{t}_1), (A_2, \tilde{t}_2)) & \longmapsto (\pi^*(A_1^{-1}(\mathbf{p}))(\tilde{t}), (A_2, \tilde{t}_2)) \end{aligned}$$

gives the product of $\widehat{C} \subset \mathcal{P}_3$ with $\widehat{C}$. Therefore the composition

$$\mathrm{Bl}_{\widehat{C}\times\widehat{C}}(\mathcal{P}_2 \times \widehat{C}) \longrightarrow \mathcal{P}_2 \times \widehat{C} \xrightarrow{(\mathsf{sum},\mathrm{id})} \mathcal{P}_3 \times \widehat{C}$$

is simply the blow up of $\mathcal{P}_3 \times \widehat{C}$ along the product surface $\boldsymbol{\imath}_{\widehat{C}}(\widehat{C}) \times \widehat{C}$. Thus

$$\mathrm{Bl}_{\widehat{C}\times\widehat{C}}(\mathcal{P}_2 \times \widehat{C}) = Y_1 \times \widehat{C},$$

which in turn identifies $\widehat{\mathcal{H}}^{\mathrm{ab}}$ with a blow up of $Y_1 \times \widehat{C}$.

The morphism $b^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_1 \times \widehat{C}$ is again the blow-up morphism, while the morphism $d^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_0$ now lifts the difference map $\mathsf{diff} : \mathcal{P}_3 \times \widehat{C} \to \mathcal{P}_2$ to the blow-ups of the source and target. This describes the second correspondence in diagram (33).

Observe also that the two maps $(\mathsf{diff}, \mathrm{id})$ and $(\mathsf{sum}, \mathrm{id})$ not only identify $\mathcal{P}_3 \times \widehat{C}$ and $\mathcal{P}_2 \times \widehat{C}$ but also identify the two diagrams of correspondences in (31) when the maps between all of the other corresponding nodes in the two diagrams are taken to be identities.

Recall next that the blown up Prym varieties Y_0 and Y_1 are equipped with finite degree 8 morphisms $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ which are the minimal resolutions of the rational maps $\mathcal{P}_2 \dashrightarrow X_0$ and $\mathcal{P}_3 \dashrightarrow X_1$ both given by $L \mapsto \pi_* L$, for L in either $\mathcal{P}_2$ or $\mathcal{P}_3$. These maps realize Y_0 and Y_1 as the modular spectral covers of X_9 and X_1 which are used to define the putative Hecke eigen Higgs bundle on $X_0 \sqcup X_1$.

Following our general strategy, we will check the Hecke eigensheaf property for this prarbolic Higgs bundle by reducing the question to checking an abelianized Hecke eigensheaf property for the corresponding modular spectral data. This is facilitated by the observation that the Pryms $\mathcal{P}_2$, $\mathcal{P}_3$, the moduli X_0, X_1 of bundles, their modular spectral covers Y_0, Y_1, the classical Hecke correspondence $\overline{\mathcal{H}}$, and the abelianized Hecke correspondence $\widehat{\mathcal{H}}^{\mathrm{ab}}$ can all be organized in a single geometric context, which, for the (X_1 to X_0) direction of the Hecke property, is most compactly recorded in the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{P}_2 \times \widehat{C} \arrow[dl, "\mathsf{sum}"'] \arrow[dr, "\mathsf{id}"] \arrow[dd, dashed] & & & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[lll, "\boldsymbol{\varepsilon}"'] \arrow[dl, "p^{\mathrm{ab}}"'] \arrow[dr, "q^{\mathrm{ab}}"] \arrow[dd, "g"] & \\
\mathcal{P}_3 \arrow[dd, dashed, "\pi_*"'] & & \mathcal{P}_2 \times \widehat{C} \arrow[dd, dashed, "\pi_* \times \widehat{\pi}"] & Y_1 \arrow[lll, bend right, "\boldsymbol{\varepsilon}_1"'] \arrow[dd, "f_1"] & & Y_0 \times \widehat{C} \arrow[lll, bend right, "\boldsymbol{\varepsilon}_0 \times \mathrm{id}"'] \arrow[dd, "f_0 \times \widehat{\pi}"] \\
 & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rrr, equal] & & & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_1 \arrow[rrr, equal, bend right] & & X_0 \times \overline{C} \arrow[rrr, equal, bend right] & X_1 & & X_0 \times \overline{C}
\end{tikzcd} \tag{35}$$

We used f_0, f_1, and g to denote the resolutions of the rational maps from the Pryms to the moduli of bundles and the Hecke correspondence to maps from the modular spectral covers, and the abelianized Hecke correspondence. Note that the map $g : \widehat{\mathcal{H}}^{\mathrm{ab}} \to \overline{\mathcal{H}}$ is indeed a morphism because the the maps $f_1 : Y_1 \to X_1$ and $f_0 \times \widehat{\pi} : Y_0 \times \widehat{C} \to X_0 \times \overline{C}$ are morphisms, and also the map $p \times q : \overline{\mathcal{H}} \to X_1 \times (X_0 \times \overline{C})$ is a closed embedding.

Finally note that we also have a companion diagram which compactly records all spaces and maps needed to abelianize and check the (X_0 to X_1) direction of the Hecke property, namely

$$\begin{tikzcd}
 & \mathcal{P}_3 \times \widehat{C} \arrow[dl, "\mathsf{diff}"'] \arrow[dr, "\mathsf{id}"] \arrow[dd, dashed] & & & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[lll, "(\mathsf{sum},\mathsf{id}) \circ \boldsymbol{\varepsilon}"'] \arrow[dl, "d^{\mathrm{ab}}"'] \arrow[dr, "b^{\mathrm{ab}}"] \arrow[dd, "g"] & \\
\mathcal{P}_2 \arrow[dd, dashed, "\pi_*"'] & & \mathcal{P}_3 \times \widehat{C} \arrow[dd, dashed, "\pi_* \times \widehat{\pi}"] & Y_0 \arrow[lll, bend right, "\boldsymbol{\varepsilon}_0"'] \arrow[dd, "f_0"] & & Y_1 \times \widehat{C} \arrow[lll, bend right, "\boldsymbol{\varepsilon}_1 \times \mathrm{id}"'] \arrow[dd, "f_1 \times \widehat{\pi}"] \\
 & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] \arrow[rrr, equal] & & & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\
X_0 \arrow[rrr, equal, bend right] & & X_1 \times \overline{C} \arrow[rrr, equal, bend right] & X_0 & & X_1 \times \overline{C}
\end{tikzcd} \tag{36}$$

7.3 The pullback of the relative dualizing sheaf

For the verification of the Hecke property we need to understand the line bundles $g^*\omega_q$ and $g^*\omega_b$ on $\widehat{\mathcal{H}}^{\mathrm{ab}}$, where $\omega_q = \omega_{\overline{\mathcal{H}}} \otimes q^*\omega^{-1}_{X_0 \times \overline{C}}$ and $\omega_b = \omega_{\overline{\mathcal{H}}} \otimes b^*\omega^{-1}_{X_1 \times \overline{C}}$ are the relative dualizing sheaf of the maps q and b respectively. The key thing is to understand the pullbacks of $\omega_{\overline{\mathcal{H}}}$, ω_q, and ω_b under the rational map $\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}$. Note that the base locus of this map has codimension at least two, and so the pullback of line bundles is well defined.

First we have the following

Proposition 7.1. *The Picard group of $\overline{\mathcal{H}}$ is*

$$\operatorname{Pic}(\overline{\mathcal{H}}) = \operatorname{Pic}(X_1) \times \operatorname{Pic}(X_0) \times \operatorname{Pic}(\overline{C}) \cong \mathbb{Z} \times \mathbb{Z} \times \operatorname{Pic}(\overline{C}).$$

In particular there is a unique line bundle $M \in \operatorname{Pic}(\overline{C})$ so that

$$\omega_{\overline{\mathcal{H}}} = p^*\mathcal{O}_{X_1}(-1) \otimes q^*(\mathcal{O}_{X_0}(-2) \boxtimes M).$$

Proof. Recall that the map $p \times (p_{\overline{C}} \circ q) : \overline{\mathcal{H}} \to X_1 \times \overline{C}$ is a $\mathbb{P}^1$ bundle. In moduli terms this $\mathbb{P}^1$-bundle is described as follows.

The moduli space X_1 has a universal bundle $\mathscr{E} \to X_1 \times C$, satisfying $\mathscr{E}_{|\{E\}\times C} \cong E$ for all $E \in X_1$ [New68, NR69]. The universal bundle $\mathscr{E}$ is not quite unique[6] but is well defined up to tensoring with a line bundle of the form $p_{X_1}^*\mathcal{O}_{X_1}(k)$ for some k. We also have the normalized Poincaré line bundle $\mathsf{Poinc} \to \operatorname{Jac}^0(C) \times C$, i.e. the unique line bundle satisfying $\mathsf{Poinc}_{|\{A\}\times C} \cong A$ for all A, and $\mathsf{Poinc}_{|\operatorname{Jac}^0(C)\times\{p\}} \cong \mathcal{O}_{|\operatorname{Jac}^0(C)}$. Since the curve

$$\overline{C} = \{(A,t) \in \operatorname{Jac}^0(C) \times C \mid A^{\otimes 2}(\mathbf{p}) \cong \mathcal{O}_C(t)\}$$

is embedded in $\operatorname{Jac}^0(C) \times C$, we can consider the pulled back line bundle $\mathscr{A} := \mathsf{Poinc}_{|\overline{C}}$. Then we have

$$\overline{\mathcal{H}} = \mathbb{P}\left((\operatorname{id} \times \mathsf{sq})^*\mathscr{E} \otimes p_{\overline{C}}^*\mathscr{A}\right) \longrightarrow X_1 \times \overline{C}. \tag{37}$$

Also, recall that the fibers of the $\mathbb{P}^1$-bundle $\overline{\mathcal{H}} \to X_1 \times \overline{C}$ map to straight lines in X_0 under the projection $\operatorname{pr}_{X_0} = p_{X_0} \circ q : \overline{\mathcal{H}} \to X_0$, and so the line bundle $\operatorname{pr}_{X_0}^* \mathcal{O}_{X_0}(1)$ has degree one on the fibers of $\overline{\mathcal{H}} \to X_1 \times \overline{C}$.

Suppose ξ is a line bundle $\overline{\mathcal{H}}$. If ξ has degree k on the fibers of $\overline{\mathcal{H}} \to X_1 \times \overline{C}$, then $\xi \otimes \operatorname{pr}_{X_0}^* \mathcal{O}_{X_0}(-k)$ is trivial on the fibers of $\overline{\mathcal{H}} \to X_1 \times \overline{C}$, and so by cohomology and base change

$$(\overline{\mathcal{H}} \to X_1 \times \overline{C})_*(\xi \otimes \operatorname{pr}_{X_0}^* \mathcal{O}_{X_0}(-k)) \tag{38}$$

[6] In fact in our setting there the universal sheaf can be normalized by further requiring that $\det(\mathscr{E})_{|X_1\times\{\mathbf{p}\}} \cong \mathcal{O}_{X_1}(1)$. One can show that such a normalized universal bundle is unique but we will not need this fact.

is a line bundle on $X_1 \times \overline{C}$. Since $\mathrm{Pic}(X_1) = \mathbb{Z}$ and is generated by $\mathcal{O}_{X_1}(1)$ it follows that the restriction of the line bundle (38) to any slice $X_1 \times \{\bar{t}\}$, $\bar{t} \in \overline{C}$ is isomorphic to $\mathcal{O}_{X_1}(r)$ for some fixed integer $r \in \mathbb{Z}$, independent of $\bar{t}$. Again by cohomology and base change we conclude that the further pushforward of $(\overline{\mathcal{H}} \to X_1 \times \overline{C})_*(\xi \otimes \mathrm{pr}_{X_0}^* \mathcal{O}_{X_0}(-k)) \otimes p^*\mathcal{O}_{X_1}(-r)$ under the map $p_{\overline{C}} : X_1 \times \overline{C} \to \overline{C}$ is some line bundle α on $\overline{C}$. But now the projection formula implies that

$$\xi = \mathrm{pr}_{X_1}^* \mathcal{O}_{X_1}(r) \otimes \mathrm{pr}_{X_0}^* \mathcal{O}_{X_0}(k) \otimes \mathrm{pr}_{\overline{C}}^* \alpha,$$

where $\mathrm{pr}_{X_1} = p$, $\mathrm{pr}_{X_0} = p_{X_0} \circ q$, and $\mathrm{pr}_{\overline{C}} = p_{\overline{C}} \circ q$.

We can now apply this reasoning to $\omega_{\overline{\mathcal{H}}}$. We will have

$$\omega_{\overline{\mathcal{H}}} = \mathrm{pr}_{X_1}^* \mathcal{O}_{X_1}(r) \otimes \mathrm{pr}_{X_0}^* \mathcal{O}_{X_0}(k) \otimes \mathrm{pr}_{\overline{C}}^* M.$$

Since the restriction of the dualizing sheaf to a smooth fiber of a map is the dualizing sheaf of that fiber, and since the fibers of $\overline{\mathcal{H}} \to X_1 \times \overline{C}$ map to lines in X_0 we conclude that $k = -2$. Also, a smooth fiber of the map $\overline{\mathcal{H}} \to X_0 \times \overline{C}$ maps to a conic inside $X_1 \subset \mathbb{P}^5$, and so we conclude that $r = -1$. Therefore

$$\omega_{\overline{\mathcal{H}}} = \mathrm{pr}_{X_1}^* \mathcal{O}_{X_1}(-1) \otimes \mathrm{pr}_{X_0}^* \mathcal{O}_{X_0}(-2) \otimes \mathrm{pr}_{\overline{C}}^* M,$$

for some line bundle $M \in \mathrm{Pic}(\overline{C})$. This completes the proof of the proposition $\square$

Let now $\omega_q = \omega_{\overline{\mathcal{H}}} \otimes q^*\omega_{X_0 \times \overline{C}}^{-1}$ denote the relative dualizing sheaf for the map q. Then the previous proposition gives

$$\omega_q = p^*\mathcal{O}_{X_1}(-1) \otimes q^* \left(\mathcal{O}_{X_0}(2) \boxtimes \left(M \otimes \omega_{\overline{C}}^{-1}\right)\right),$$

and therefore

$$\begin{aligned} &\left(\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}\right)^* \omega_q = \\ &\quad (\mathsf{sum}^* (\mathcal{P}_3 \dashrightarrow X_1)^* \mathcal{O}_{X_1}(-1)) \otimes \left((\mathcal{P}_2 \dashrightarrow X_0)^* \mathcal{O}_{X_0}(2) \boxtimes \widehat{\pi}^* \left(M \otimes \omega_{\overline{C}}^{-1}\right)\right). \end{aligned} \tag{39}$$

Similarly, if $\omega_b = \omega_{\overline{\mathcal{H}}} \otimes b^*\omega_{X_1 \times \overline{C}}^{-1}$ denotes the relative dualizing sheaf for the map b, then the formula for $\omega_{\overline{\mathcal{H}}}$ implies

$$\omega_b = d^*\mathcal{O}_{X_0}(-2) \otimes b^* \left(\mathcal{O}_{X_1}(1) \boxtimes \left(M \otimes \omega_{\overline{C}}^{-1}\right)\right),$$

and therefore

$$\left(\mathcal{P}_3 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}\right)^* \omega_b = \\ (\mathsf{diff}^* (\mathcal{P}_2 \dashrightarrow X_0)^* \mathcal{O}_{X_0}(-2)) \otimes \left((\mathcal{P}_3 \dashrightarrow X_1)^* \mathcal{O}_{X_1}(1) \boxtimes \widehat{\pi}^* \left(M \otimes \omega_{\overline{C}}^{-1}\right)\right). \tag{40}$$

Next we will compute the line bundle $M \in \mathrm{Pic}(\overline{C})$ appearing in Proposition 7.1 and in the formulas (39) and (40).

Lemma 7.2. $M = \mathsf{sq}^*\omega_C(\mathbf{p})$.

Proof. Let $E \in X_1$ be a general point. Consider the surface $P_E := p^{-1}(E) \subset \overline{\mathcal{H}}$. Then P_E is a smooth geometrically ruled surface over $\overline{C}$ and in fact from the $\mathbb{P}^1$-bundle description (37) of $\overline{\mathcal{H}}$ we get

$$P_E = \mathbb{P}\left((\mathrm{id} \times \mathsf{sq})^* \mathscr{E}_{|\{E\}\times \overline{C}} \otimes \mathscr{A}\right) = \mathbb{P}(\mathsf{sq}^* E \otimes \mathscr{A}) \cong \mathbb{P}(\mathsf{sq}^* E).$$

The map

$$\begin{tikzcd} P_E \arrow[rr, "q_{|P_E}"] \arrow[dr] & & X_0 \times \overline{C} \arrow[dl] \\ & \overline{C} & \end{tikzcd}$$

embeds P_E into $X_0 \times \overline{C}$ by embedding each fiber of $P_E \to \overline{C}$ as a straight line in $X_0 \cong \mathbb{P}^3$. Also, since $P_E = p^{-1}(E)$ is a fiber of a map, we have $\omega_{\overline{\mathcal{H}}|P_E} \cong \omega_{P_E}$ and so by Proposition 7.1 we have

$$\omega_{P_E} \cong \omega_{\overline{\mathcal{H}}|P_E} = \left(q_{|P_E}\right)^* (\mathcal{O}_{X_0}(-2) \boxtimes M).$$

So we can compute M by computing the restrictions of $\mathrm{pr}_{X_0}^* \mathcal{O}_{X_0}(-2)$ and ω_{P_E} to some section $\zeta : \overline{C} \to P_E$ of the ruled surface $P_E \to \overline{C}$.

Any line subundle $L \subset E$ of degree zero will give rise to such a section. There are four such sububundles for a generic E and we can choose any one of them. Let $L \subset E$ be a line subbundle of degree zero. Then we have a short exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^\vee(\mathbf{p}) \longrightarrow 0,$$

and L gives a section of $\mathbb{P}(E) \to C$ which pulls back to a section $\zeta : \overline{C} \to P_E$ of $P_E \to \overline{C}$. Note that for any point $(A, t) \in \overline{C}$ the fiber of $P_E \to \overline{C}$ over (A, t) is canonically $\mathbb{P}(E_t \otimes A_t)$ and the value of the section ζ at (A, t) is given by the line $L_t \otimes A_t$, i.e.

$$\zeta(A, t) = [L_t \otimes A_t] \in \mathbb{P}(E_t \otimes A_t).$$

The bundle $E \otimes A$ fits in a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \otimes A \arrow[r] & E \otimes A \arrow[r] & L^{\vee} \otimes A(\mathbf{p}) \arrow[r] \arrow[d, equal] & 0. \\ & & & (L \otimes A)^{\vee}(t) & \end{tikzcd}$$

By definition the map $P_E \subset \overline{\mathcal{H}} \stackrel{\mathrm{pr}_{X_0}}{\longrightarrow} X_0$ sends the point $\zeta(A,t)$ to the down Hecke transform of $E \otimes A$ centered at the line $(L \otimes A)_t \subset (E \otimes A)_t$, i.e. to the locally free sheaf $\widetilde{E}$ defined by the commutative diagram

$$\begin{tikzcd} & & 0 & 0 & \\ & & ((L \otimes A)^{\vee}(t))_t \arrow[u] \arrow[r, equal] & ((L \otimes A)^{\vee}(t))_t \arrow[u] & \\ 0 \arrow[r] & L \otimes A \arrow[r] & E \otimes A \arrow[r] \arrow[u] & (L \otimes A)^{\vee}(t) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & L \otimes A \arrow[r] \arrow[u, equal] & \widetilde{E} \arrow[r] \arrow[u] & (L \otimes A)^{\vee} \arrow[r] \arrow[u] & 0 \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

The third row of this diagram exhibits $\widetilde{E}$ as an extension of the degree zero line bundle $(L \otimes A)^{\vee}$ by $L \otimes A$ and so the bundle $\widetilde{E}$ goes to the point corresponding to the S-equivalence class $[(L \otimes A) \oplus (L \otimes A)^{\vee}] \in \mathsf{Kum} \subset X_0$.

Hence the map $\mathrm{pr}_{X_0} \circ \zeta : \overline{C} \to X_0$ factors as

$$\begin{tikzcd} \overline{C} \arrow[r, "\mathrm{pr}_{X_0} \circ \zeta"] \arrow[d, hook] & X_0 \\ \mathrm{Jac}^0(C) \arrow[r] & \mathsf{Kum} \arrow[u, hook] \end{tikzcd} \tag{41}$$

Here the left vertical map $\overline{C} \hookrightarrow \mathrm{Jac}^0(C)$ is given by $(A,t) \mapsto A \otimes L$, i.e. is the translation by L of the defining embedding of $\overline{C}$ in $\mathrm{Jac}^0(C)$. The bottom horizontal map $\mathrm{Jac}^0(C) \to \mathsf{Kum}$ is the quotient of $\mathrm{Jac}^0(C)$ by (-1), and the composition $\mathrm{Jac}^0(C) \to \mathsf{Kum} \to X_0 \cong \mathbb{P}^3$ is given by the linear system $|2\Theta_{\mathbf{p}}|$, with $\Theta_{\mathbf{p}}$ being the theta divisor

$$\Theta_{\mathbf{p}} = \left\{ \alpha \in \mathrm{Jac}^0(C) \;\middle|\; h^0(C, \alpha(\mathbf{p})) \geq 1 \right\}.$$

Remark 7.3. Strictly speaking the moduli space X_0 is isomorphic to $\mathbb{P}(H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta_{\mathbf{p}})))$, rather than $\mathbb{P}(H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta_{\mathbf{p}}))^\vee)$. Indeed, by [NR69] the identification

$$X_0 = \mathbb{P}(H^0(\mathrm{Jac}^0(C), \mathcal{O}(2\Theta_{\mathbf{p}})))$$

is given by the map $V \mapsto D_V \in |2\Theta_{\mathbf{p}}|$, where $D_V = \{\alpha \in \mathrm{Jac}^0(C) \mid h^0(C, V \otimes \alpha(\mathbf{p}))\}$. To conclude that the map $\mathrm{Jac}^0(C) \to X_0$ is indeed given by the linear system $|2\Theta_{\mathbf{p}}|$ we use the classical fact [Hud05, Dol20, Keu97] that the embedding of the Kummer surface in $\mathbb{P}^3$ is projectively self-dual.

Suppose $x \in C$ is a fixed point, and $\mathsf{AJ}_x : C \to \mathrm{Jac}^0(C)$, $t \mapsto \mathcal{O}(t-x)$ is the x-based Abel-Jacobi map. Suppose $\kappa \in \mathsf{Spin}(C)$ is a theta characteristic and let $\Theta_\kappa = \{\alpha \in \mathrm{Jac}^0(C) \,|\, h^0(C, \alpha \otimes \kappa) \geq 1\}$ be the associated theta divisor on $\mathrm{Jac}^0(C)$. By Rieman's theorem [GH94] we have that $\mathsf{AJ}_x^* \mathcal{O}(\Theta_\kappa) = \kappa(x)$. In particular, since $\mathcal{O}_C(\mathbf{p})$ is a theta characteristic we get that

$$\mathsf{AJ}_{\mathbf{p}}^* \mathcal{O}(\Theta_{\mathbf{p}}) \cong \mathcal{O}_C(2\mathbf{p}) = \omega_C.$$

If we denote the natural embedding of $\overline{C}$ in $\mathrm{Jac}^0(C)$ by $\boldsymbol{\imath}_{\overline{C}} : \overline{C} \hookrightarrow \mathrm{Jac}^0(C)$, $(A, t) \mapsto A$, then we can write the left vertical map in the diagram (41) as the composition $\boldsymbol{t}_L \circ \boldsymbol{\imath}_{\overline{C}} : \overline{C} \hookrightarrow \mathrm{Jac}^0(C)$, where for a line bundle L on C we use

$$\boldsymbol{t}_L := L \otimes (-) : \mathrm{Jac}^k(C) \to \mathrm{Jac}^{k+\deg L}(C)$$

to denote the map of tensoring by L. With this notation we now have a commutative diagram

$$\begin{tikzcd} \overline{C} \arrow[r, "\mathsf{sq}"] \arrow[d, hook, "\boldsymbol{t}_L \circ \boldsymbol{\imath}_{\overline{C}}"'] & C \arrow[d, hook, "\boldsymbol{t}_{L^{\otimes 2}} \circ \mathsf{AJ}_{\mathbf{p}}"] \\ \mathrm{Jac}^0(C) \arrow[r, "\mathsf{mult}_2"'] & \mathrm{Jac}^0(C) \end{tikzcd} \qquad (42)$$

But $\mathsf{mult}_2^* \mathcal{O}(\Theta_{\mathbf{p}}) = \mathcal{O}(4\Theta_p)$, and so we get

$$\begin{aligned} (\boldsymbol{t}_L \circ \boldsymbol{\imath}_{\overline{C}})^* \mathcal{O}(4\Theta_p) &= \mathsf{sq}^* \circ \mathsf{AJ}_{\mathbf{p}}^* \circ \boldsymbol{t}_{L^{\otimes 2}}^* \mathcal{O}(\Theta_{\mathbf{p}}) \\ &= \mathsf{sq}^* \left(L^{\otimes -2} \otimes \mathsf{AJ}_{\mathbf{p}}^* \mathcal{O}(\Theta_{\mathbf{p}}) \right) \\ &= \mathsf{sq}^* \left(L^{\otimes -2}(2\mathbf{p}) \right). \end{aligned}$$

But by (41) we have

$$(\mathrm{pr}_{X_0} \circ \zeta)^* \mathcal{O}_{X_0}(-2) = (\boldsymbol{t}_L \circ \boldsymbol{\imath}_{\overline{C}})^* \mathcal{O}(-4\Theta_p),$$

and hence

$$(\mathrm{pr}_{X_0} \circ \zeta)^* \mathcal{O}_{X_0}(-2) = \mathsf{sq}^* \left(L^{\otimes 2}(-2\mathbf{p}) \right). \tag{43}$$

Next we need to compute $\zeta^* \omega_{P_E}$. By adjunction we have

$$\zeta^* \omega_{P_E} = \omega_{\overline{C}} \otimes \zeta^* \mathcal{O}_{P_E}(-\zeta).$$

Using the fact that $\zeta : \overline{C} \to P_E = \mathbb{P}(\mathsf{sq}^* E)$ corresponds to the subbundle $\mathsf{sq}^* L$ we compute

$$\begin{aligned}
\zeta^* \mathcal{O}_{P_E}(\zeta) &= N_{\zeta/P_E} \\
&= \underline{Hom}_{\mathcal{O}_{\overline{C}}} (\mathsf{sq}^* L, \mathsf{sq}^*(E/L)) \\
&= \mathsf{sq}^*(L^\vee \otimes L^\vee(\mathbf{p})) \\
&= \mathsf{sq}^* \left(L^{\otimes -2}(\mathbf{p}) \right).
\end{aligned}$$

Hence

$$\zeta^* \omega_{P_E} = \omega_{\overline{C}} \otimes \mathsf{sq}^* \left(L^{\otimes 2}(-\mathbf{p}) \right) = (\mathsf{sq}^* \mathcal{O}_C(2\mathbf{p})) \otimes \mathsf{sq}^* \left(L^{\otimes 2}(-\mathbf{p}) \right) = \mathsf{sq}^* \left(L^{\otimes 2}(\mathbf{p}) \right). \tag{44}$$

Substituting (43) and (44) in the the identity

$$\zeta^* \omega_{P_E} = (\mathrm{pr}_{X_0} \circ \zeta)^* \mathcal{O}_{X_0}(-2) \otimes M$$

we get

$$\mathsf{sq}^* \left(L^{\otimes 2}(\mathbf{p}) \right) = \mathsf{sq}^* \left(L^{\otimes 2}(-\mathbf{p}) \right) \otimes M,$$

that is $M = \mathsf{sq}^* \mathcal{O}_C(3\mathbf{p}) = \mathsf{sq}^* \omega_C(\mathbf{p})$. $\square$

Since $\omega_{\overline{C}} = \mathsf{sq}^* \omega_C$ we therefore get that

$$\omega_q = p^* \mathcal{O}_{X_1}(-1) \otimes q^* \left(\mathcal{O}_{X_0}(2) \boxtimes \widehat{\pi}^* \mathsf{sq}^* \mathcal{O}_C(\mathbf{p}) \right)$$

and so we get a slightly simpler version of (39), namely:

$$\begin{aligned}
&\left(\mathcal{P}_2 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}} \right)^* \omega_q = \\
&\qquad \left(\mathsf{sum}^* \left(\mathcal{P}_3 \dashrightarrow X_1 \right)^* \mathcal{O}_{X_1}(-1) \right) \otimes \left(\left(\mathcal{P}_2 \dashrightarrow X_0 \right)^* \mathcal{O}_{X_0}(2) \boxtimes \widehat{\pi}^* \mathsf{sq}^* \mathcal{O}_C(\mathbf{p}) \right).
\end{aligned} \tag{45}$$

Finally, pulling (45) by $\boldsymbol{\varepsilon} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to \mathcal{P}_2 \times \widehat{C}$ and using the commutativity of the main diagram (35) we get

$$\boxed{g^*\omega_q = \left(p^{\mathrm{ab}}\right)^* f_1^*\mathcal{O}_{X_1}(-1) \otimes \left(q^{\mathrm{ab}}\right)^* \left(f_0^*\mathcal{O}_{X_0}(2) \boxtimes \widehat{\pi}^*\mathsf{sq}^*\mathcal{O}_C(\mathbf{p})\right)} \tag{46}$$

This is exactly what we will need to abelianize and check the (X_1 to X_0) direction of the Hecke property. For understanding the abelianization of the (X_0 to X_1) direction we also need to rewrite (46) in terms of the maps in the second correspondence diagram in (33). This is straighforward. Substituting $M = \mathsf{sq}^*\omega_C(\mathbf{p})$ in (40) gives the simplified identity

$$\begin{aligned}\left(\mathcal{P}_3 \times \widehat{C} \dashrightarrow \overline{\mathcal{H}}\right)^* \omega_b = \\ \left(\mathsf{diff}^* \left(\mathcal{P}_2 \dashrightarrow X_0\right)^* \mathcal{O}_{X_0}(-2)\right) \otimes \left(\left(\mathcal{P}_3 \dashrightarrow X_1\right)^* \mathcal{O}_{X_1}(1) \boxtimes \widehat{\pi}^*\mathsf{sq}^*\mathcal{O}_C(\mathbf{p})\right).\end{aligned} \tag{47}$$

Finally, pulling back (47) via the map $(\mathsf{sum}, \mathrm{id})\circ\boldsymbol{\varepsilon} : \widehat{\mathcal{H}} \to \mathcal{P}_3 \times \widehat{C}$ and using the commutativity of the companion diagram (36) we get

$$\boxed{g^*\omega_b = \left(d^{\mathrm{ab}}\right)^* f_0^*\mathcal{O}_{X_0}(-2) \otimes \left(b^{\mathrm{ab}}\right)^* \left(f_1^*\mathcal{O}_{X_1}(1) \boxtimes \widehat{\pi}^*\mathsf{sq}^*\mathcal{O}_C(\mathbf{p})\right)} \tag{48}$$

7.4 The Hecke property via abelianization

Throughout this section we will fix a base point $\widetilde{\mathbf{p}} \in \widetilde{C}$ such that $\pi(\widetilde{\mathbf{p}}) = \mathbf{p}$. Now suppose $(\widetilde{C} \subset T^\vee C, \boldsymbol{N})$ is the spectral data for (E, θ), i.e. $(E, \theta) = (\pi_*\boldsymbol{N}, \pi_*(\lambda \otimes (-)))$. Then $\boldsymbol{N} \in \mathcal{P}_2$ and it determines two natural line bundles

$$\mathfrak{L}_0 \in \mathrm{Pic}^0(\mathcal{P}_2), \quad \text{and} \quad \mathfrak{L}_1 \in \mathrm{Pic}^0(\mathcal{P}_3),$$

where $\mathfrak{L}_1 = \boldsymbol{t}^*_{-\widetilde{\mathbf{p}}}\mathfrak{L}_0$, and $\mathfrak{L}_0$ is the appropriately defined Fourier-Mukai transform of the skyscraper sheaf $\mathcal{O}_{\boldsymbol{N}}$ on $\mathcal{P}_2$. Explicitly, consider the abelian subvariety $\mathcal{P} \subset \mathrm{Jac}^0(\widetilde{C})$ defined by

$$\mathcal{P} = \left\{ L \in \mathrm{Jac}^0(\widetilde{C}) \mid \mathsf{Nm}_\pi(L) = \mathcal{O}_C \right\}.$$

This abelian subvariety comes with a natural polarization $\boldsymbol{\xi} \to \mathcal{P}$ defined by pulling back the canonical theta line bundle on $\mathrm{Jac}^4(\widetilde{C})$ via the natural map

$$\mathcal{P} \subset \mathrm{Jac}^0(\widetilde{C}) \to \mathrm{Jac}^4(\widetilde{C}), \quad L \mapsto L \otimes \mathcal{O}_{\widetilde{C}}(2\pi^*\mathbf{p}).$$

Explicitly $\boldsymbol{\xi}$ is defined as $\boldsymbol{\xi} = \mathcal{O}_{\mathcal{P}}(\Xi_{2\pi^*\mathsf{p}})$, where $\Xi_{2\pi^*\mathsf{p}}$ is the Prym theta divisor $\Xi_{2\pi^*\mathsf{p}} := \left\{ L \in \mathcal{P} \;\middle|\; h^0(\widetilde{C}, L(2\pi^*\mathsf{p})) \geq 1 \right\}$. The line bundle $\boldsymbol{\xi}$ has polarization type $(1,2,2)$ and the kernel of the associated polarization homomorphism is isomorphic to $\mathsf{J}[2]$ [BNR89]. Concretely, the polarization homomorphism $\phi_{\boldsymbol{\xi}}$ to the dual abelain variety $\mathcal{P}^\vee = \mathrm{Pic}^0(\mathcal{P})$ is defined by

$$\phi_{\boldsymbol{\xi}} : \mathcal{P} \to \mathcal{P}^\vee, \quad L \mapsto \boldsymbol{t}_L^* \boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}.$$

It is a surjective homomorphism of abelian varieties and $\ker \phi_{\boldsymbol{\xi}} = \pi^*\mathsf{J}[2]$.

Since $\mathcal{P}$ and $\mathcal{P}^\vee$ are dual abelian varieties we have a canonical normalized Poincaré line bundle

$$^{\mathcal{P}}\mathsf{Poinc} \longrightarrow \mathcal{P} \times \mathcal{P}^\vee$$

characterized by the conditions

$$\begin{aligned} {}^{\mathcal{P}}\mathsf{Poinc}_{|P\times\{\mathscr{L}\}} &\cong \mathscr{L}, \text{ for all } \mathscr{L} \in \mathcal{P}^\vee, \text{ and} \\ {}^{\mathcal{P}}\mathsf{Poinc}_{|\{\mathcal{O}_{\widetilde{C}}\}\times\mathcal{P}^\vee} &\cong \mathcal{O}_{\mathcal{P}^\vee}. \end{aligned}$$

Let $\mathsf{FM} : D^b(\mathcal{P}) \xrightarrow{\cong} D^b(\mathcal{P}^\vee)$, $F \mapsto p_{2*}\left(p_1^*F \otimes {}^{\mathcal{P}}\mathsf{Poinc}\right)$ denote the Fourier-Mukai transform with kernel ${}^{\mathcal{P}}\mathsf{Poinc}$. For any $L \in \mathcal{P}$ we have

$$\mathsf{FM}(\mathcal{O}_L) = {}^{\mathcal{P}}\mathsf{Poinc}_{|\{L\}\times\mathcal{P}^\vee}$$

which is a line bundle of degree zero on $\mathcal{P}^\vee$.

With this notation we can now define the line bundles on $\mathcal{P}_2$, and $\mathcal{P}_3$ which become part of the modular spectral data for the Hecke eigensheaf.

Definition 7.4. *For any $\boldsymbol{N} \in \mathcal{P}_2$ define degree zero line bundles on $\mathcal{P}$, $\mathcal{P}_2$, and $\mathcal{P}_3$ by setting*

$$\begin{aligned} \mathfrak{L} &:= \phi_{\boldsymbol{\xi}}^* \mathsf{FM}\left(\mathcal{O}_{\boldsymbol{N}(-\pi^*\mathsf{p})}\right) \;\in\; \mathrm{Pic}^0(\mathcal{P}), \\ \mathfrak{L}_0 &:= \boldsymbol{t}^*_{\mathcal{O}_{\widetilde{C}}(-\pi^*\mathsf{p})}\mathfrak{L} \;\in\; \mathrm{Pic}^0(\mathcal{P}_2), \\ \mathfrak{L}_1 &:= \boldsymbol{t}^*_{\mathcal{O}_{\widetilde{C}}(-\widetilde{\mathsf{p}}-\pi^*\mathsf{p})}\mathfrak{L} \;\in\; \mathrm{Pic}^0(\mathcal{P}_3). \end{aligned}$$

Note: By construction The line bundles $\mathfrak{L}$, $\mathfrak{L}_0$, and $\mathfrak{L}_1$ all depend on the Higgs bundle (E,θ) or more precisely on the corresponding spectral data $(\widetilde{C} \subset T^\vee C, \boldsymbol{N})$. We suppress the explicit dependence on $\boldsymbol{N}$ in the labeling of these line bundles to avoid cluttering the notation.

We can describe these line bundles explicitly in terms of $\boldsymbol{\xi}$.

Lemma 7.5. $\mathfrak{L} = \boldsymbol{t}^*_{\boldsymbol{N}(-\pi^*(\mathbf{p}))}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}$.

Proof. The Poincaré line bundle ${}^{\mathcal{P}}\mathsf{Poinc}$ is built explicitly starting from the bi-extension line bundle

$$\mathcal{Q} \to \mathcal{P} \times \mathcal{P}, \qquad \mathcal{Q} = \boldsymbol{m}^*\boldsymbol{\xi} \otimes p_1^*\boldsymbol{\xi}^{-1} \otimes p_2^*\boldsymbol{\xi}^{-1},$$

where $\boldsymbol{m} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is the group operation on the abelian variety $\mathcal{P}$.

Indeed, the translation action of $\mathsf{J}[2]$ on the second copy of $\mathcal{P}$ in $\mathcal{P} \times \mathcal{P}$ lifts to an action on $\mathcal{Q}$. Indeed, since $\boldsymbol{\xi}$ is invariant under translation by elements in $\mathsf{J}[2]$, it follows that $\mathcal{Q}$ is invariant under the translation action by elements of $\mathsf{J}[2] \times \mathsf{J}[2] \subset \mathcal{P} \times \mathcal{P}$. But the extension class defining the theta group of $\mathcal{Q}$ is given by [Mum08, BL04b] a pairing $e^{\mathcal{Q}}$ which is the exponentiation of the first Chern class of $\mathcal{Q}$. By definition the first Chern class of $\mathcal{Q}$ restricts to zero on each of the two summands $H_1(\mathcal{P}, \mathbb{Z}) \oplus H_1(\mathcal{P}, \mathbb{Z})$ in the Künneth decomposition of the first homology of $\mathcal{P} \times \mathcal{P}$. Hence $e^{\mathcal{Q}}$ restricts to the trivial character on each of the two factors in $\mathsf{J}[2] \times \mathsf{J}[2]$. Thus the theta group of $\mathcal{Q}$ splits over $\mathsf{J}[2]$ which implies that $\mathcal{Q}$ is equivariant under the the translation action of $\{\mathcal{O}_{\widetilde{C}}\} \times \mathsf{J}[2]$, as claimed. Note that by definition the restriction of $\mathcal{Q}$ to $\{\mathcal{O}_{\widetilde{C}}\} \times \mathcal{P}$ is canonically trivial and we can use this trivializatio to normalize the $\{\mathcal{O}_{\widetilde{C}}\} \times \mathsf{J}[2]$-action on $\mathcal{Q}$ by requiring that $\mathsf{J}[2]$ acts tautologically on $\mathcal{O}_{\mathcal{P}} = \mathcal{Q}_{|\{\mathcal{O}_{\widetilde{C}}\} \times \mathcal{P}}$.

Using this normalized equivariant structure on $\mathcal{Q}$ we can descend $\mathcal{Q}$ to a biextension line bundle

$$\underline{\mathcal{Q}} \to \mathcal{P} \times \mathcal{P}^\vee$$

on the quotient $\mathcal{P} \times (\mathcal{P}/\mathsf{J}[2]) \cong \mathcal{P} \times \mathcal{P}^\vee$. If $\mathscr{L} \in \mathcal{P}^\vee$ we have that

$$\underline{\mathcal{Q}}_{|\mathcal{P} \times \{\mathscr{L}\}} \cong \mathcal{Q}_{|\mathcal{P} \times \{\alpha\}},$$

where $\alpha \in \mathcal{P}$ is any point, s.t. $\phi_{\boldsymbol{\xi}}(\alpha) = \mathscr{L}$. But then by the definition of $\mathcal{Q}$ we have

$$\mathcal{Q}_{\mathcal{P} \times \{\alpha\}} = \boldsymbol{t}^*_\alpha\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1} = \phi_{\boldsymbol{\xi}}(\alpha) = \mathscr{L}.$$

Hence

$$\underline{\mathcal{Q}}_{|\mathcal{P} \times \{\mathscr{L}\}} \cong \mathscr{L}, \quad \text{for all } \mathscr{L} \in \mathcal{P}^\vee.$$

Also, by our normalization of the equivariant structure, $\underline{\mathcal{Q}}_{|\{\mathcal{O}_{Ctilde}\} \times \mathcal{P}^\vee}$ is the descent of $\mathcal{O}_{\mathcal{P}} = \mathcal{Q}_{|\{\mathcal{O}_{\widetilde{C}}\} \times \mathcal{P}}$ with respect to the tautological $\mathsf{J}[2]$-action. Hence $\underline{\mathcal{Q}}_{|\{\mathcal{O}_{\widetilde{C}}\} \times \mathcal{P}^\vee} \cong \mathcal{O}_{\mathcal{P}^\vee}$. This shows that

$$\underline{\mathcal{Q}} \cong {}^{\mathcal{P}}\mathsf{Poinc},$$

and so

$$\mathcal{Q} \cong (\mathrm{id} \times \phi_{\boldsymbol{\xi}})^* \left({}^{\mathcal{P}}\mathsf{Poinc}\right).$$

This implies that

$$\begin{aligned}
\mathfrak{L} &= \phi_{\boldsymbol{\xi}}^* \left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}^\vee}\right) \\
&= \left((\mathrm{id} \times \phi_{\boldsymbol{\xi}})^* \left({}^{\mathcal{P}}\mathsf{Poinc}\right)\right)_{|\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}} \\
&= \mathcal{Q}_{|\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}} \\
&= \boldsymbol{t}^*_{\boldsymbol{N}(-\pi^*\mathbf{p})}\boldsymbol{\xi} \times \boldsymbol{\xi}^{-1}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Recall next that our candidate Hecke eigensheaf for the eigenvalue (E, θ) was a tame parabolic Higgs bundle $(\mathscr{F}_{0,\bullet}, \Phi_0) \sqcup (\mathscr{F}_{1,\bullet}, \Phi_1)$ on $X_0 \sqcup X_1$ which was constructed from the modular spectral covers

$$f_0 : Y_0 \to X_0, \quad \text{and} \quad f_1 : Y_1 \to X_1$$

by setting

$$\begin{aligned}
(\mathscr{F}_{0,0}, \Phi_0) &= (f_{0*}\mathscr{L}_0, f_{0*}(\boldsymbol{\alpha}_0 \otimes (-))), \\
(\mathscr{F}_{1,0}, \Phi_1) &= (f_{1*}\mathscr{L}_1, f_{1*}(\boldsymbol{\alpha}_1 \otimes (-))).
\end{aligned}$$

Here

- $\boldsymbol{\alpha}_0 : Y_0 \to T^\vee_{X_0}(\log \mathrm{Wob}_0)$ and $\boldsymbol{\alpha}_1 : Y_1 \to T^\vee_{X_1}(\log \mathrm{Wob}_1)$ are the tautological maps (defined away from the preimage of codimension two loci in X_0 and X_1) from the modular spectral covers Y_0 and Y_1 to the logarithmic cotangent bundles of X_0 and X_1 with poles along the (normal crossings part of the) wobbly divisors.

- The modular spectral line bundles $\mathscr{L}_0$ and $\mathscr{L}_1$ are given by

$$\begin{aligned}
\mathscr{L}_0 &= (\boldsymbol{\varepsilon}_0^*\mathfrak{L}_0)(\boldsymbol{E}_0) \otimes f_0^*\mathcal{O}_{X_0}(2), \text{ and} \\
\mathscr{L}_1 &= \boldsymbol{\varepsilon}_1^*\mathfrak{L}_1 \otimes f_1^*\mathcal{O}_{X_1}(1)
\end{aligned}$$

  and $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$ denote the exceptional divisors of the blowup maps $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ and $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$.

- $(\mathscr{F}_{0,0}, \Phi_0)$ and $(\mathscr{F}_{1,0}, \Phi_1)$ are equipped with parabolic structures along Wob_0 and Wob_1 respectively.

In sections 9 and 8 we used these constructions for $(\mathscr{F}_{i,0}, \Phi_i)$ together with the formula for the L^2 Dolbeault pushforward of parabolic Higgs bundles from [DPS16] and its refinements proven in Proposition 12.7 to rewrite the Hecke condition

$$\begin{aligned} q_* p^* \left(\mathscr{F}_{1,\bullet}, \Phi_1\right) &= \left(\mathscr{F}_{0,\bullet}, \Phi_0\right) \boxtimes \mathsf{sq}^*(E, \theta), \\ b_* d^* \left(\mathscr{F}_{0,\bullet}, \Phi_0\right) &= \left(\mathscr{F}_{1,\bullet}, \Phi_1\right) \boxtimes \mathsf{sq}^*(E, \theta) \end{aligned} \tag{49}$$

as a condition on the modular spectral data $(Y_0, \mathscr{L}_0)$ and $(Y_1, \mathscr{L}_1)$. Namely, chasing the maps in the basic diagram (35) we get that the (X_1 to X_0) direction of the Hecke eigensheaf property (49) is equivalent to the (Y_1 to Y_0) abelianized Hecke condition

$$\boxed{\begin{aligned} &(p^{\mathrm{ab}})^* \big(\boldsymbol{\varepsilon}_1^* \mathfrak{L}_1 \otimes f_1^* \mathcal{O}_{X_1}(1)\big) \otimes (g^* \omega_q) \left(\boldsymbol{Exc}_0 + \boldsymbol{Exc}_1\right) \\ &\qquad\qquad = (q^{\mathrm{ab}})^* \left[\left(\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0(\boldsymbol{E}_0) \otimes f_0^* \mathcal{O}_{X_0}(2)\right) \boxtimes \widehat{\mathsf{sq}}^* \boldsymbol{N} \right], \end{aligned}} \tag{50}$$

where (50) is understood as an isomorphism of line bundles on $\widehat{\mathcal{H}}^{\mathrm{ab}}$ which holds away from the q^{ab}-pullback of any codimension two subvariety in $Y_0 \times \widehat{C}$. In particular it suffices to prove that (50) holds modulo multiples of the divisor $\boldsymbol{Exc}_1$.

Similarly, by Proposition 12.7, the analysis of the contribution from the singularities of the horizontal divisor to be carried out in section 10.2, and the companion diagram (36) we get that the (X_0 to X_1) direction of the Hecke eigensheaf property (49) is equivalent to the (Y_0 to Y_1) abelianized Hecke condition

$$\boxed{\begin{aligned} &(d^{\mathrm{ab}})^* \big(\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0(\boldsymbol{E}_0) \otimes f_0^* \mathcal{O}_{X_0}(2)\big) \otimes g^* \omega_b \\ &\qquad\qquad = (b^{\mathrm{ab}})^* \left[\left(\boldsymbol{\varepsilon}_1^* \mathfrak{L}_1 \otimes f_1^* \mathcal{O}_{X_1}(1)\right) \boxtimes \widehat{\mathsf{sq}}^* \boldsymbol{N} \right]. \end{aligned}} \tag{51}$$

Again (51) should be understood as an equality of line bundles on $\widehat{\mathcal{H}}^{\mathrm{ab}}$ away from the b^{ab}-pullback of any codimension two subvariety in $Y_1 \times \widehat{C}$. In particular, it suffices to check that (51) holds modulo integral combinations of the components of the divisor $\boldsymbol{Exc}_0$.

Substituting the formula (46) into (50) and (48) we see that the (Y_1 to Y_0) abelianized Hecke condition (50) reduces to checking that

$$\big((p^{\mathrm{ab}})^* \boldsymbol{\varepsilon}_1^* \mathfrak{L}_1\big) \left(\boldsymbol{Exc}_0 + \boldsymbol{Exc}_1\right) = (q^{\mathrm{ab}})^* \left[\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0(\boldsymbol{E}_0) \boxtimes \widehat{\mathsf{sq}}^* (\boldsymbol{N}(-\pi^* \mathbf{p})) \right], \quad \text{modulo } \mathbb{Z} \cdot \boldsymbol{Exc}_1. \tag{52}$$

Similarly the (Y_0 to Y_1) abelianized Hecke condition (48) reduces to showing that

$$(d^{\mathrm{ab}})^* \left(\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0(\boldsymbol{E}_0)\right) = (b^{\mathrm{ab}})^* \left[(\boldsymbol{\varepsilon}_1^* \mathfrak{L}_1) \boxtimes \widehat{\mathsf{sq}}^* (\boldsymbol{N}(-\pi^* \mathbf{p}))\right], \quad \text{modulo} \sum_{\kappa \in \mathsf{Spin}(C)} \mathbb{Z} \cdot \boldsymbol{Exc}_{0,\kappa}. \tag{53}$$

But in section 7.2 we saw that

$$(q^{\mathrm{ab}})^* \left[\mathcal{O}_{Y_0}(\boldsymbol{E}_0) \boxtimes \mathcal{O}_{\widehat{C}}\right] = (q^{\mathrm{ab}})^* \mathcal{O}_{\mathcal{P}_2 \times \widehat{C}}(\boldsymbol{E}_0 \times \widehat{C}) = \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}}(\boldsymbol{Exc}_0 + \boldsymbol{Exc}_1),$$

and that

$$(d^{\mathrm{ab}})^* \mathcal{O}_{Y_0}(\boldsymbol{E}_0) = \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}}(\boldsymbol{Exc}_0).$$

This shows that full exceptional divisor content cancels in both equations (52) and (53) and so these become equations on line bundles on $\mathcal{P}_2 \times \widehat{C}$ and $\mathcal{P}_3 \times \widehat{C}$ respectively. Concretely we are reduced to checking that

$$\mathsf{sum}^* \mathfrak{L}_1 = \mathfrak{L}_0 \boxtimes \widehat{\mathsf{sq}}^* (\boldsymbol{N}(-\pi^* \mathbf{p})) \text{ in } \operatorname{Pic}(\mathcal{P}_2 \times \widehat{C}). \tag{54a}$$
$$\mathsf{diff}^* \mathfrak{L}_0 = \mathfrak{L}_1 \boxtimes \widehat{\mathsf{sq}}^* (\boldsymbol{N}(-\pi^* \mathbf{p})) \text{ in } \operatorname{Pic}(\mathcal{P}_3 \times \widehat{C}). \tag{54b}$$

Since $(\mathsf{sum}, \mathrm{id})$ and $(\mathsf{diff}, \mathrm{id})$ are inverse isomorphisms, (54a) and (54b) are clearly equaivalent. Therefore, to verify both the (Y_1 to Y_0) and (Y_0 to Y_1) abelianized Hecke conditions we only need the following

Proposition 7.6. *For any $\boldsymbol{N} \in \mathcal{P}_2$ the corresponding modular spectral line bundles $\mathfrak{L}_0$ and $\mathfrak{L}_1$ satisfy the identity $\mathsf{sum}^* \mathfrak{L}_1 = \mathfrak{L}_0 \boxtimes \widehat{\mathsf{sq}}^* (\boldsymbol{N}(-\pi^* \mathbf{p}))$ in $\operatorname{Pic}(\mathcal{P}_2 \times \widehat{C})$.*

Proof. Let us first compute $\mathsf{sum}^* \mathfrak{L}_1$. For this it will be convenient to express $\mathsf{sum}^* \mathfrak{L}_1$ as a pullback of a line bundle on $\mathcal{P} \times \mathcal{P}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{P}_2 \times \widehat{C} \arrow[r, "\mathsf{sum}"] \arrow[d, "(\boldsymbol{t}_{-\pi^* \mathbf{p}}) \times \boldsymbol{\jmath}"'] & \mathcal{P}_3 \arrow[d, "\boldsymbol{t}_{-\widetilde{\mathbf{p}} - \pi^* \mathbf{p}}"] \\
\mathcal{P} \times \mathcal{P} \arrow[r, "\boldsymbol{m}"'] & \mathcal{P}
\end{tikzcd}$$

where $\boldsymbol{\jmath} : \widehat{C} \to \mathcal{P}$ is given by $(A, \tilde{t}) \mapsto \pi^* A^{-1}(\tilde{t} - \widetilde{\mathbf{p}})$, and $\boldsymbol{m} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is the group law on the Prym, i.e. $(L_1, L_2) \mapsto L_1 \otimes L_2$.

Since by definition $\mathfrak{L}_1 = \boldsymbol{t}^*_{-\widetilde{\mathbf{p}} - \pi^* \mathbf{p}} \mathfrak{L}$ we get that

$$\mathsf{sum}^* \mathfrak{L}_1 = \left((\boldsymbol{t}_{-\pi^* \mathbf{p}}) \times \boldsymbol{\jmath}\right)^* \boldsymbol{m}^* \mathfrak{L}.$$

Also

$$\mathfrak{L}_0 = \boldsymbol{t}^*_{-\pi^*\mathbf{p}}\mathfrak{L} = ((\boldsymbol{t}_{-\pi^*\mathbf{p}}) \times \boldsymbol{\jmath})^* \, p_1^*\mathfrak{L},$$

and so it suffices to understand the line bundle

$$\boldsymbol{m}^*\mathfrak{L} \otimes p_1^*\mathfrak{L}^{-1}$$

on the abelian variety $\mathcal{P} \times \mathcal{P}$. By Lemma 7.5 we have $\mathfrak{L} = \boldsymbol{t}^*_{\boldsymbol{N}(-\pi^*(\mathbf{p}))}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}$. To simplify notation write $\boldsymbol{a} = \boldsymbol{N}(-\pi^*\mathbf{p}) \in \mathcal{P}$. Thus we would like to understand the line bundle

$$\boldsymbol{m}^*(\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}) \otimes p_1^*(\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}).$$

To simplify notation consider the $\boldsymbol{a}$-translated biextension line bundle ${}^{\boldsymbol{a}}\mathcal{Q}$ on $\mathcal{P} \times \mathcal{P}$ defined by

$${}^{\boldsymbol{a}}\mathcal{Q} = \boldsymbol{m}^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi} \otimes p_1^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi}^{-1} \otimes p_2^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi}^{-1}.$$

With this notation we now have

$$\begin{aligned} \boldsymbol{m}^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi} \otimes p_1^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi}^{-1} &= {}^{\boldsymbol{a}}\mathcal{Q} \otimes p_2^*\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi}, \text{ and} \\ \boldsymbol{m}^*\boldsymbol{\xi}^{-1} \otimes p_1^*\boldsymbol{\xi} &= \mathcal{Q}^{-1} \otimes p_2^*\boldsymbol{\xi}^{-1}. \end{aligned}$$

In particular we get

$$\boldsymbol{m}^*\mathfrak{L} \otimes p_1^*\mathfrak{L}^{-1} = ({}^{\boldsymbol{a}}\mathcal{Q} \otimes \mathcal{Q}) \otimes p_2^*(\boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi} \otimes \theta^{-1}) = ({}^{\boldsymbol{a}}\mathcal{Q} \otimes \mathcal{Q}) \otimes p_2^*\mathfrak{L}.$$

On the other hand, for any $\boldsymbol{b} \in \mathcal{P}$ we have

$${}^{\boldsymbol{a}}\mathcal{Q}_{|\{\boldsymbol{b}\}\times\mathcal{P}} = \boldsymbol{t}^*_{\boldsymbol{a}+\boldsymbol{b}}\boldsymbol{\xi} \otimes \boldsymbol{t}^*_{\boldsymbol{a}}\boldsymbol{\xi}^{-1} = \boldsymbol{t}^*_{\boldsymbol{a}}\left(\boldsymbol{t}^*_{\boldsymbol{b}}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}\right).$$

But $\boldsymbol{t}^*_{\boldsymbol{b}}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1} \in \mathrm{Pic}^0(\mathcal{P})$, and so is translation invariant. Hence

$${}^{\boldsymbol{a}}\mathcal{Q}_{|\{\boldsymbol{b}\}\times\mathcal{P}} \cong \boldsymbol{t}^*_{\boldsymbol{b}}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1} = \mathcal{Q}_{|\{\boldsymbol{b}\}\times\mathcal{P}}.$$

Similarly we have that ${}^{\boldsymbol{a}}\mathcal{Q}_{|\mathcal{P}\times\{\boldsymbol{b}\}} \cong \mathcal{Q}_{|\mathcal{P}\times\{\boldsymbol{b}\}}$. So by the see-saw principle we have ${}^{\boldsymbol{a}}\mathcal{Q} \cong \mathcal{Q}$ and hence

$$\boldsymbol{m}^*\mathfrak{L} \otimes p_1^*\mathfrak{L}^{-1} = p_2^*\mathfrak{L}.$$

This implies that

$$\mathsf{sum}^*\mathfrak{L}_1 \otimes p_1^*\mathfrak{L}_0^{-1} = ((\boldsymbol{t}_{-\pi^*\mathbf{p}}) \times \boldsymbol{\jmath})^* \, p_2^*\mathfrak{L} = \mathrm{pr}^*_{\widehat{C}}\boldsymbol{\jmath}^*\mathfrak{L} \tag{55}$$

and the proposition reduced to the following

Lemma 7.7. $\boldsymbol{\jmath}^*\mathfrak{L} = \widehat{\mathsf{sq}}^*\left(\boldsymbol{N}(-\pi^*\mathbf{p})\right)$.

Proof. First observe that $\boldsymbol{\jmath}^*\mathfrak{L}$ is a pullback of a line bundle on $\widetilde{C}$ via the map $\widehat{\mathsf{sq}} : \widehat{C} \to \widetilde{C}$. Indeed, by definition

$$\mathfrak{L} = \phi_{\boldsymbol{\xi}}^*\left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\boldsymbol{N}(-\pi^* pw)\}\times\mathcal{P}^\vee}\right).$$

On the other hand, the map

$$\phi_{\boldsymbol{\xi}} \circ \boldsymbol{\jmath} : \widehat{C} \longrightarrow \mathcal{P}^\vee$$

is given by

$$(\phi_{\boldsymbol{\xi}} \circ \boldsymbol{\jmath})(A, \tilde{t}) = \phi_{\boldsymbol{\xi}}(\pi^* A^{-1}(\tilde{t} - \widetilde{\mathbf{p}})).$$

But $\boldsymbol{\jmath} : \widehat{C} \to \mathcal{P}$ is $\mathsf{J}[2]$ equivariant for the Galois action of $\mathsf{J}[2]$ on the source $\widehat{C}$ and the translation action of $\mathsf{J}[2] = \pi^*\mathsf{J}[2] = \ker\phi_{\boldsymbol{\xi}} \subset \mathcal{P}$. Indeed we have

$$\boldsymbol{\jmath}\left(\mathfrak{a}\cdot(A,\tilde{t})\right) = \boldsymbol{\jmath}(A\otimes\mathfrak{a},\tilde{t}) = \pi^*(A\otimes\mathfrak{a})^{-1}(\tilde{t}-\widetilde{\mathbf{p}}) = \pi^*A^{-1}\otimes\pi^*\mathfrak{a}(\tilde{t}-\widetilde{\mathbf{p}}) = \pi^*\mathfrak{a}\otimes\boldsymbol{\jmath}((A,\tilde{t})).$$

Hence

$$(\phi_{\boldsymbol{\xi}}\circ\boldsymbol{\jmath})\left(\mathfrak{a}\cdot(A,\tilde{t})\right) = (\phi_{\boldsymbol{\xi}}\circ\boldsymbol{\jmath})(A,\tilde{t}),$$

and so the map $\phi_{\boldsymbol{\xi}}\circ\boldsymbol{\jmath}$ factors through $\widehat{C}/\mathsf{J}[2] = \widetilde{C}$. In other words we have a comutative triangle

$$\begin{tikzcd} \widehat{C} \arrow[rr, "\phi_{\boldsymbol{\xi}}\circ\boldsymbol{\jmath}"] \arrow[dr, "\widehat{\mathsf{sq}}"'] & & \mathcal{P}^\vee \\ & \widetilde{C} \arrow[ur, "\psi"'] & \end{tikzcd}$$

for a well defiined map $\psi : \widetilde{C} \to \mathcal{P}^\vee$.

In particular

$$\boldsymbol{\jmath}^*\mathfrak{L} = \boldsymbol{\jmath}^*\phi_{\boldsymbol{\xi}}^*\left({}^{\mathcal{P}}\mathsf{Poinc}_{\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}^\vee}\right) = \widehat{\mathsf{sq}}^*\psi^*\left({}^{\mathcal{P}}\mathsf{Poinc}_{\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}^\vee}\right).$$

So the question reduces to computing $\psi^*\left({}^{\mathcal{P}}\mathsf{Poinc}_{\{\boldsymbol{N}(-\pi^*\mathbf{p})\}\times\mathcal{P}^\vee}\right)$. This calls for a better understanding of the map ψ which is a version of the Abel-Prym map.

We have an addition map

$$\mathbf{add} : \mathrm{Jac}^0(C)\times\mathcal{P} \longrightarrow \mathrm{Jac}^0(\widetilde{C}), \quad (A, L) \mapsto \pi^*A\otimes L,$$

which is a surjective homomorphism of abelian varieties with kernel isomorphic to $\mathsf{J}[2]$, embedded in $\mathrm{Jac}^0(C)\times\mathcal{P}$ by the map $\mathfrak{a}\mapsto(\mathfrak{a},\pi^*\mathfrak{a})$. Next note that we have a natural map

$$\widehat{\pi}\times\boldsymbol{\jmath} : \widehat{C} \longrightarrow, \quad (A,\tilde{t}) \mapsto \left(A, \pi^*A^{-1}(\tilde{t}-\widetilde{\mathbf{p}})\right).$$

This map is clearly $\mathsf{J}[2]$-equivariant and so the composition $\widehat{C} \xrightarrow{\widehat{\pi}\times\boldsymbol{\jmath}} \mathrm{Jac}^0(C)\times\mathcal{P} \xrightarrow{\mathbf{add}} \mathrm{Jac}^0(\widetilde{C})$ will factor as

$$\begin{tikzcd} \widehat{C} \arrow[r, "\widehat{\pi}\times\boldsymbol{\jmath}"] \arrow[d, "\widehat{\mathsf{sq}}"'] & \mathrm{Jac}^0(C)\times\mathcal{P} \arrow[r, "\mathbf{add}"] & \mathrm{Jac}^0(\widetilde{C}) \\ \widetilde{C} \arrow[urr, "\mathsf{AJ}_{\widetilde{\mathbf{p}}}"'] & & \end{tikzcd}$$

where $\mathsf{AJ}_{\widetilde{\mathbf{p}}}$ is the $\widetilde{\mathbf{p}}$-based Abel-Jacobi map $\mathsf{AJ}_{\widetilde{\mathbf{p}}} : \widetilde{C} \to \mathrm{Jac}^0(\widetilde{C})$, $\tilde{t} \mapsto \mathcal{O}_{\widetilde{C}}(\tilde{t} - \widetilde{\mathbf{p}})$.

Consider the theta line bundle $\widetilde{\boldsymbol{\theta}}$ on $\mathrm{Jac}^0(\widetilde{C})$ defined by the theta characteristic $\mathcal{O}_{\widetilde{C}}(2\pi^*\mathbf{p}) \in \mathsf{Spin}(\widetilde{C})$. In other words

$$\widetilde{\boldsymbol{\theta}} = \mathcal{O}_{\mathrm{Jac}^0(\widetilde{C})}(\widetilde{\boldsymbol{\Theta}}_{2\pi^*\mathbf{p}}), \quad \text{where} \quad \widetilde{\boldsymbol{\Theta}}_{2\pi^*\mathbf{p}} = \left\{ L \in \mathrm{Jac}^0(\widetilde{C}) \;\middle|\; h^0(\widetilde{C}, L(2\pi^*\mathbf{p})) \geq 1 \right\}.$$

Recall that we used $\widetilde{\boldsymbol{\theta}}$ to define the line bundle $\boldsymbol{\xi}$ on $\mathcal{P}$, i.e. we had $\boldsymbol{\xi} = \widetilde{\boldsymbol{\theta}}_{|\mathcal{P}}$. Also, $\widetilde{\boldsymbol{\theta}}$ is a principal polarization on $\mathrm{Jac}^0(\widetilde{C})$ which defines a surjective group homomorphism

$$\mathbf{q}_{\widetilde{\boldsymbol{\theta}}} : \mathrm{Jac}^0(\widetilde{C}) \to \mathcal{P}^\vee, \quad L \mapsto \left(\boldsymbol{t}_L^* \widetilde{\boldsymbol{\theta}} \otimes \widetilde{\boldsymbol{\theta}}^{-1} \right)_{|\mathcal{P}},$$

which fits in the commutative diagram

$$\begin{tikzcd} \widehat{C} \arrow[r, "\widehat{\pi}\times\boldsymbol{\jmath}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Jac}^0(C)\times\mathcal{P} \arrow[r, "\mathsf{sum}"] \arrow[d, "\mathrm{pr}_{\mathcal{P}}"'] & \mathrm{Jac}^0(\widetilde{C}) \arrow[d, "\mathbf{q}_{\widetilde{\boldsymbol{\theta}}}"] \\ \widehat{C} \arrow[r, "\boldsymbol{\jmath}"'] & \mathcal{P} \arrow[r, "\phi_{\boldsymbol{\xi}}"'] & \mathcal{P}^\vee \end{tikzcd}$$

Therefore

$$\phi_{\boldsymbol{\xi}} \circ \boldsymbol{\jmath} = \mathbf{q}_{\widetilde{\boldsymbol{\theta}}} \circ \mathsf{sum} \circ (\widehat{\pi} \times \boldsymbol{\jmath}) = \mathbf{q}_{\widetilde{\boldsymbol{\theta}}} \circ \mathsf{AJ}_{\widetilde{\mathbf{p}}} \circ \widehat{\mathsf{sq}}.$$

This implies that the map $\psi : \widetilde{C} \to \mathcal{P}^\vee$ factors as

$$\begin{tikzcd} \widetilde{C} \arrow[rr, "\psi"] \arrow[dr, "\mathsf{AJ}_{\widetilde{\mathbf{p}}}"'] & & \mathcal{P}^\vee \\ & \mathrm{Jac}^0(\widetilde{C}) \arrow[ur, "\mathbf{q}_{\widetilde{\boldsymbol{\theta}}}"'] & \end{tikzcd}$$

and so we need to compute pullbacks by $\mathsf{AJ}_{\widetilde{\mathbf{p}}}$ and $\mathbf{q}_{\widetilde{\boldsymbol{\theta}}}$.

But for any $\alpha \in \mathcal{P}$ and any $\mathscr{A} \in \mathcal{P}^\vee$ we have that the fiber of the Poincare line bundle ${}^{\mathcal{P}}\mathsf{Poinc}$ at the point $(\alpha, \mathscr{A}) \in \mathcal{P} \times \mathcal{P}^\vee$ is canonically isomorphic to the fiber of the line bundle

$\mathscr{A} \in \mathcal{P}^\vee = \mathrm{Pic}^0(\mathcal{P})$ at $\alpha \in \mathcal{P}$. Hence for any $L \in \mathrm{Jac}^0(\widetilde{C})$ we have equality of fibers

$$\begin{aligned}\left(\mathsf{q}_{\widetilde{\boldsymbol{\theta}}}^* \left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\alpha\}\times\mathcal{P}^\vee}\right)\right)_L &= {}^{\mathcal{P}}\mathsf{Poinc}_{(\alpha, \mathsf{q}_{\widetilde{\boldsymbol{\theta}}}(L))} \\ &= \left(\mathsf{q}_{\widetilde{\boldsymbol{\theta}}}(L)\right)_\alpha \\ &= \left(\boldsymbol{t}_L^* \widetilde{\boldsymbol{\theta}} \otimes \widetilde{\boldsymbol{\theta}}^{-1}\right)_\alpha .\end{aligned}$$

By the see-saw principle this equality of fibers implies that we have an isomorphism

$$\mathsf{q}_{\widetilde{\boldsymbol{\theta}}}^* \left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\alpha\}\times\mathcal{P}^\vee}\right) \cong \widetilde{\mathcal{Q}}_{|\{\alpha\}\times\mathrm{Jac}^0(\widetilde{C})}$$

of line bundles on $\mathrm{Jac}^0(\widetilde{C})$, where $\widetilde{\mathcal{Q}} \to \mathrm{Jac}^0(\widetilde{C}) \times \mathrm{Jac}^0(\widetilde{C})$ is the biextension line bundle given by

$$\widetilde{\mathcal{Q}} = \boldsymbol{m}^* \widetilde{\boldsymbol{\theta}} \otimes p_1^* \widetilde{\boldsymbol{\theta}}^{-1} \otimes p_2^* \widetilde{\boldsymbol{\theta}}^{-1}.$$

By definition, the pullback of $\widetilde{\mathcal{Q}}_{|\{\alpha\}\times\mathrm{Jac}^0(\widetilde{C})}$ by the Abel-Jacobi map $\mathsf{AJ}_{\widetilde{\mathsf{p}}} : \widetilde{C} \to \mathrm{Jac}^0(\widetilde{C})$ is the line bundle on $\widetilde{C}$ whose fiber at a point $\tilde{t} \in \widetilde{C}$ is equal to the line $\alpha_{\tilde{t}} \otimes \alpha_{\widetilde{\mathsf{p}}}^{-1}$. Thus

$$\mathsf{AJ}_{\widetilde{\mathsf{p}}}^* \left(\widetilde{\mathcal{Q}}_{|\{\alpha\}\times\mathrm{Jac}^0(\widetilde{C})}\right) = \alpha \otimes \alpha_{\widetilde{\mathsf{p}}}^{-1} \cong \alpha.$$

All together this gives an isomorphism

$$\begin{aligned}\psi^* \left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\boldsymbol{N}(-\pi^*\mathsf{p})\}\times\mathcal{P}^\vee}\right) &= \mathsf{AJ}_{\widetilde{\mathsf{p}}}^* \mathsf{q}_{\widetilde{\boldsymbol{\theta}}}^* \left({}^{\mathcal{P}}\mathsf{Poinc}_{|\{\boldsymbol{N}(-\pi^*\mathsf{p})\}\times\mathcal{P}^\vee}\right) \\ &= \mathsf{AJ}_{\widetilde{\mathsf{p}}}^* \left(\widetilde{\mathcal{Q}}_{|\{\boldsymbol{N}(-\pi^*\mathsf{p})\}\times\mathrm{Jac}^0(\widetilde{C})}\right) \cong \boldsymbol{N}(-\pi^*\mathsf{p}),\end{aligned}$$

and hence $\boldsymbol{\jmath}^* \mathfrak{L} = \widehat{\mathsf{sq}}^* \boldsymbol{N}(-\pi^*\mathsf{p})$ as claimed. This proves the lemma. $\square$

The previous lemma together with the identity (55) now implies that $\mathsf{sum}^* \mathfrak{L}_1 = \mathfrak{L}_0 \boxtimes \widehat{\mathsf{sq}}^* \boldsymbol{N}(-\pi^*\mathsf{p})$ which completes the proof of the proposition. $\square$

7.5 Disjointness statements

Lemma 7.8. *The projection map $\overline{\mathcal{H}} \to X_0 \times X_1 \times \overline{C}$ is an isomorphism onto its image, which is smooth.*

Proof. If $a = (A, t) \in \overline{C}$ and a stable bundle $E_1 \in X_1$ are fixed, the fiber of $\overline{\mathcal{H}}$ over (a, E_1) is the Hecke line consisting of all the E_0 obtained by Hecke transformations along 1-dimensional quotients of $(E_1 \otimes A)(t)$. This is a $\mathbb{P}^1$ mapping to a line in $\mathbb{P}^3$. We get a morphism

$X_1 \times \overline{C} \to \mathrm{Grass}(2,4)$ to the Grassmanian of lines in $\mathbb{P}^3$. Since the source is smooth, this map is a regular function. The pullback of the universal fibration over $\mathrm{Grass}(2,4)$ is a smooth $\mathbb{P}^1$-bundle over $X_1 \times \overline{C}$. A point on one of the lines determines uniquely the sheaf E_0 and the map ζ, so $\overline{\mathcal{H}}$ is isomorphic to this $\mathbb{P}^1$-bundle. $\square$

Recall that $\mathcal{P}_m$ denotes the Prym variety of degree m line bundles L on $\widetilde{C}$ such that the norm $\mathsf{Nm}_{\widetilde{C}/C}(L)$ of the associated divisor down to C is linearly equivalent to $m\mathbf{p}$. This condition is equivalent to stating that $\det(\pi_* L) \cong \mathcal{O}_C((m-2)\mathbf{p})$ since $\pi_*(\mathcal{O}_{\widetilde{C}}) = \omega_C^{-1} = \mathcal{O}_C(-2\mathbf{p})$. In particular, $\mathcal{P}_2$ is the Prym variety of degree 2 line bundles L on $\widetilde{C}$ such that $\det(\pi_* L) \cong \mathcal{O}_C$, and $\mathcal{P}_3$ is the Prym variety of degree 3 line bundles L on $\widetilde{C}$ such that $\det(\pi_* L) \cong \mathcal{O}_C(\mathbf{p})$.

We have the maps $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$ and $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$, that are respectively blowing up of 16 points or a curve $\widehat{C} \subset \mathcal{P}_3$. We also have $\mathcal{P}_1$ the Prym variety of degree 1 line bundles on $\widetilde{C}$ whose norm is $\mathcal{O}_C(\mathbf{p})$. Subtraction gives a map

$$\mathbf{m}_{3,2} : \mathcal{P}_3 \times \mathcal{P}_2 \to \mathcal{P}_1.$$

On the other hand, we have a shifted Abel-Jacobi map

$$\mathbf{j} : \widehat{C} \to \mathcal{P}_1$$

defined by $\mathbf{j}(A, \tilde{t}) := \mathcal{O}_{\widetilde{C}}(\tilde{t}) \otimes \pi^*(A^{-1})$. Note that if $L = \mathcal{O}_{\widetilde{C}}(\tilde{t}) \otimes \pi^*(A^{-1})$ and $\pi(\tilde{t}) = t$, then

$$\mathsf{Nm}_{\widetilde{C}/C}(L) = A^{\otimes -2}(t) = \mathcal{O}_C(t - (t - \mathbf{p})) = \mathcal{O}_C(\mathbf{p})$$

as required.

Using the product of the blowup maps and $\mathbf{m}_{3,2}$ gives a composed map

$$Y_1 \times Y_0 \xrightarrow{\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_0} \mathcal{P}_3 \times \mathcal{P}_2 \xrightarrow{\mathbf{m}_{3,2}} \mathcal{P}_1.$$

The abelianized Hecke maps to the fiber product

$$\widehat{\mathcal{H}}^{\mathrm{ab}} \to (Y_1 \times Y_0) \times_{\mathcal{P}_1} \widehat{C}.$$

If we fix a point $\widehat{a} \in \widehat{C}$, and consider the restriction of $\widehat{\mathcal{H}}^{\mathrm{ab}}$ to the subvariety $(Y_1 \times Y_0) \times_{\mathcal{P}_1} \{\widehat{a}\}$ we get a correspondence

$$\begin{tikzcd} & \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \arrow[dl, "p^{\mathrm{ab}}"'] \arrow[dr, "q^{\mathrm{ab}}"] & \\ Y_1 & & Y_0 \end{tikzcd}$$

The map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to Y_1$ is the blow-up of Y_1 along 16 lines contained in the exceptional divisor $\boldsymbol{E}_1 \cong \widehat{C} \times \mathbb{P}^1$ of the map $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$. Similarly the map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to Y_0$ is the blow-up of Y_0 along a curve isomorphic to $\widehat{C}$ which intersects transversally at a single point each of the 16 plane components $\boldsymbol{E}_{0,\kappa} \cong \mathbb{P}^2$ of the exceptional divisor $\boldsymbol{E}_0 = \sqcup_{\kappa \in \mathsf{Spin}(C)} \boldsymbol{E}_{0,\kappa}$.

Consider the maps $r_0 = \mathrm{id} \times f_0 : Y_1 \times Y_0 \to Y_1 \times X_0$ and $r_1 = f_1 \times \mathrm{id} : Y_1 \times Y_0 \to X_1 \times Y_0$. With this notation we have

Theorem 7.9. *Fix a point $\widehat{a} = (A, \tilde{t}) \in \widehat{C}$ and let $\tau\widehat{a} = (A, \tau(\tilde{t}))$ denote its conjugate under the covering involution of the double cover $\widehat{\pi} : \widehat{C} \to \overline{C}$. Assume that $\widehat{a}$ is general in $\widehat{C}$. Then the image in X_0 of the intersection*

$$r_0\left(\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})\right) \cap r_0\left(\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})\right) \subset Y_1 \times X_0$$

has dimension at most 1.

Proof. Suppose we are given two points in $\widehat{\mathcal{H}}^{\mathrm{ab}}$ that map to the points $(y_0, y_1, \widehat{a})$ and $(y_0', y_1', \tau\widehat{a})$ in $(Y_0 \times Y_1) \times_{\mathcal{P}_1} \widehat{C}$. We assume that these points further map to the same point (x_0, x_1, a) of $\overline{\mathcal{H}}$, and that $y_1 = y_1'$. We would like to conclude x_0 lies in a dimension ≤ 1 subset of X_0. Here this dimension is measured for a given fixed $\widehat{a}$.

Suppose first that the point $y_1 = y_1'$ lies in the exceptional divisor $\boldsymbol{E}_1 \subset Y_1$. This part of the exceptional locus in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ projects to a curve in Y_0 and so to a curve in X_0, so we may ignore this case.

We may therefore assume that $y_1 = y_1'$ corresponds to a line bundle L on $\widetilde{C}$ whose direct image is a stable vector bundle $V = \pi_* L$ of determinant $\det V = \mathcal{O}_C(\mathbf{p})$. The Hecke transformations corresponding to the abelianized Hecke at $\widehat{a}$ and $\tau\widehat{a}$ are along the lines $L_{\tilde{t}}$ and $L_{\tau(\tilde{t})}$ in V_t corresponding to the two different points $\tilde{t}$ and $\tau(\tilde{t})$ of the spectral curve $\widetilde{C}$. Thus $L_{\tilde{t}}, L_{\tau(\tilde{t}} \subset V_t$ correspond to distinct points of the Hecke line $\mathbb{P}^1 = \mathbb{P}((V \otimes A)_t)$, so they are distinct points in $\overline{\mathcal{H}}(a)$ and hence distinct points in X_0. This contradicts the hypothesis, which completes the proof starting with $y_1 = y_1'$. □

Corollary 7.10. *Suppose ℓ is a general line in X_0 and let $H_\ell = q^{-1}(\ell) \subset \overline{\mathcal{H}}(a)$ be the restriction of $\overline{\mathcal{H}}(a)$ to ℓ. Then the restrictions $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$ over $\ell \subset X_0$ are disjoint in the spectral surface $\Sigma \to H_\ell$ that is the pullback*

$$\Sigma := H_\ell \times_{X_1} Y_1.$$

Proof. By Theorem 7.9, the image in X_0 of the intersection of the two abelianized Hecke spaces inside $Y_1 \times X_0$, is at most a curve. Thus, a general line does not meet it, and we get the required disjointness property. $\square$

For the Hecke transform in the direction from X_0 to X_1, the following lemma is used.

Lemma 7.11. *For a general point $x_0 \in \mathsf{Kum}$ of the Kummer in X_0, the image $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to \overline{\mathcal{H}}(a)$ intersects the Hecke fiber over x_0 only at the point on the K3 surface where the two lines intersect, corresponding to the Hecke transform of a polystable bundle.*

Proof. Choose one of the four points $y_0 \in Y_0$ over x_0, and let L be the corresponding line bundle on $\widetilde{C}$. Let $E := \pi_*(L)$. We claim that E is polystable. Since $x_0 \in \mathsf{Kum}$ we know that E is strictly semistable, i.e. there is a degree 0 line bundle U with $U \hookrightarrow E$. We would like to show that there is also a nonzero map $U^\vee \to E$.

By adjunction the inclusion $U \subset E$ on C corresponds to a map $\pi^*U \hookrightarrow L$ of rank one locally free sheaves on $\widetilde{C}$. The latter map will have cokernel which is a torsion sheaf of length two since L has degree 2. We can therefore write

$$L = \pi^*U \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t}_1 + \tilde{t}_2).$$

Let $t_1, t_2 \in C$ be the image points of $\tilde{t}_1, \tilde{t}_2$. Let t_1' and t_2' be their conjugates by the hyperelliptic involution $\boldsymbol{\iota}_C$. The norm of L down to C is ω_C, yielding the formula

$$U^{\otimes 2} \cong \omega_C(-t_1 - t_2).$$

In particular,

$$U^\vee \cong U \otimes \mathcal{O}_C(t_1 + t_2 - 2\mathbf{p}).$$

Thus

$$\pi^*(U^\vee) \cong \pi^*(U) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t}_1 + \tilde{t}_2 + \tau\tilde{t}_1 + \tau\tilde{t}_2 - 2\widetilde{\mathbf{p}} - 2\tau\widetilde{\mathbf{p}}).$$

To get a map $\pi^*(U^\vee) \to L$ we therefore need a section of $\mathcal{O}_{\widetilde{C}}(2\widetilde{\mathbf{p}} + 2\tau\widetilde{\mathbf{p}} - \tau\tilde{t}_1 - \tau\tilde{t}_2)$.

Recall that $\widetilde{C}$ is also a hyperelliptic curve, whose hyperelliptic involution σ is a lift of the hyperelliptic involution $\boldsymbol{\iota}_C$ of C. The fixed points of σ are the 12 inverse image points of the Weierstrass points of C. In particular, $2\widetilde{\mathbf{p}}$ and $2\tau\widetilde{\mathbf{p}}$ are linearly equivalent divisors coming from $\mathbb{P}^1$ by pullback along the hyperelliptic map $\widetilde{C} \to \mathbb{P}^1$. Therefore

$$\mathcal{O}_{\widetilde{C}}(2\widetilde{\mathbf{p}} - \tau\tilde{t}_i) \cong \mathcal{O}_{\widetilde{C}}(\sigma\tau\tilde{t}_i) \cong \mathcal{O}_{\widetilde{C}}(2\tau\widetilde{\mathbf{p}} - \tau\tilde{t}_i).$$

Thus

$$\mathcal{O}_{\widetilde{C}}(2\widetilde{\mathbf{p}} + 2\tau\widetilde{\mathbf{p}} - \tau\tilde{t}_1 - \tau\tilde{t}_2) \cong \mathcal{O}_{\widetilde{C}}(\sigma\tau\tilde{t}_1 + \sigma\tau\tilde{t}_2)$$

is effective. This gives the required nonzero section so there is a nonzero map $\pi^*(U^\vee) \to L$ corresponding by adjunction to $U^\vee \to E$. We get a map $U \oplus U^\vee \to E$ that, for general U, has to be an isomorphism. Thus E is polystable. Now, a Hecke transformation of E that becomes a stable bundle has to be by a diagonal line, corresponding to the stated point of $\overline{\mathcal{H}}(a)$. □

Theorem 7.12. *Fix a point $\widehat{a} = (A, \tilde{t}) \in \widehat{C}$ and its conjugate $\tau\widehat{a} = (A, \tau(\tilde{t}))$. Assume that $\widehat{a}$ is general in $\widehat{C}$. Then, up to a subset whose image in X_1 has dimension ≤ 1, the intersection*

$$r_1\left(\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})\right) \cap r_1\left(\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})\right) \subset X_1 \times Y_0$$

consists of the set of points of the form (x_1, y_0) such that y_0 is a point of Y_0 lying over the Kummer variety, and x_1 is a point in X_1 that is in the image of the K3 surface in $\overline{\mathcal{H}}(a)$.

Proof. This will follow the same lines as the proof of Theorem 7.9 and we keep the same notations. However, instead of supposing that $y_1 = y_1'$, we suppose now that $y_0 = y_0'$. If y_0 is a point of one of the 16 exceptional divisors $\boldsymbol{E}_{0,\kappa} \subset Y_0$ lying over the trope planes, these project down to lines in the wobbly locus of X_1 so as before we can ignore this case.

If y_0 corresponds to a line bundle L on $\widetilde{C}$ whose direct image $E = \pi_* L$ is a stable bundle, then by the same argument as in the proof of Theorem 7.9, the two points of the abelianized Hecke correspond to Hecke transformations at different points of the spectral curve, so they are distinct in the Hecke curve $\mathbb{P}^1 = \mathbb{P}((E \otimes A)_t)$ that maps to a conic in X_1. Thus, they map to distinct points in X_1 and so this case can not happen.

We are left with the case that the point $x_0 = f_0(y_0)$ is a point $x_0 \in \mathsf{Kum}$ of the Kummer variety. We may assume that x_0 is general in the Kummer variety, as points on a divisor of the Kummer will lead to subsets of X_1 of dimension ≤ 1. By Lemma 7.11, the point (x_1, x_0) on $\overline{\mathcal{H}}(a)$ lies on the Kummer K3 surface, so x_1 is in the image of the K3 surface in X_1. We get the stated divisor as the image of the intersection of the two abelianized Hecke pieces. □

Corollary 7.13. *Suppose ℓ is a general line in X_1 and let $H_\ell = p^{-1}(\ell) \subset \overline{\mathcal{H}}(a)$ be the restriction of $\overline{\mathcal{H}}(a)$ to ℓ. Let*

$$\Sigma := H_\ell \times_{X_1} Y_1$$

be the spectral surface of the pullback Higgs sheaf over H_ℓ. Then the restrictions $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tilde{a})_\ell$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\tilde{a})_\ell$ over $\ell \subset X_1$ are curves in Σ that meet at each of the double points of Σ lying over one of the two nodes of the curve to be denoted $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$ in Section 8.

Proof. Consider the image $K(a) \subset X_1$ of the Kummer K3 surface inside $\overline{\mathcal{H}}(a)$. The line ℓ intersects $K(a)$ in two points. This may be seen from the discussion of Section 8 where the two points are the images in ℓ of $\mathfrak{p}$ and $\mathfrak{q}$, or alternatively in Remark 11.3 where $K(a)$ is viewed as the intersection of X_1 with an additional quadric that depends on a.

Suppose (x_1, y_0) is an intersection point in Σ of the two curves, that means that $x_1 \in \ell$ and this point is in the intersection of the two abelianized Hecke pieces. We have seen that this means that y_0 lies over a point of the Kummer surface, and indeed that y_0 corresponds to a line bundle $L \in \mathcal{P}_{@}$ such that $\pi_* L \cong U \oplus U^\vee$ for some line bundle $U \in \mathrm{Jac}^0(C)$ with

$$L = \pi^* U \otimes \mathcal{O}_{\widetilde{C}}(\tilde{t}_1 + \tilde{t}_2).$$

But $\widehat{a} = (A, \tilde{t})$ with $\pi(\tilde{t}) = t$ and $A^{\otimes 2} = \mathcal{O}_C(t - p)$, and so the point x_1 corresponds to a bundle V that fits in an exact sequence

$$0 \to U \otimes A^{-1} \to V \to U^\vee \otimes A^{-1} \otimes \mathcal{O}_C(t) \to 0.$$

Assuming that x_0 is a general point of the Kummer surface, then we are given two sub-line bundles of degree 0 in the same Hecke transformation, namely $U \otimes A^{-1}$ and $U^{-1} \otimes A^{-1}$. It means that these two agree as sub-lines in the fiber V_t.

Suppose we now vary the point in a family $x_1(z)$ and thus the bundle $V(z)$. Starting with a line that is not multiple as one of the four lines through $x_1(0)$, and extend it to a family of lines locally at $x_1(z)$; follow this line, making the Hecke transformation at the resulting sub-line of $V(z)_t$. This gives a locally well-defined section of the Hecke correspondence that maps into the Kummer surface in X_0. We see in this way that if two distinct lines through x_1 (neither of which is doubled in the set of four) have the property that they agree in V_t, then the family of intersections of the Hecke lines with the Kummer surface has two distinct branches.

This applies to our previous situation. In Section 8.1 we will see that the family of intersections of the Hecke lines with the Kummer surface is a reducible curve $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$ inside

H_ℓ. We have said that at our intersection point (x_1, y_0) we are given two distinct sub-lines of the bundle V (the point x_1) such that the Hecke transformations at a agree. Our argument then says that (x_1, x_0) is a point in H_ℓ that is a double point of the curve $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$. This shows that it is one of the two points we have identified, over which Σ has four ordinary double points.

We notice that there are four points y_0 over x_0, and this will give intersection points in the four ordinary double points of Σ lying over the given point of H_ℓ. □

Remark 7.14. In the Hecke transformation from X_0 to X_1, the restriction of the resulting rank 16 Higgs bundle on a general line $\ell \subset X_1$ is the direct sum of two rank 8 Higgs bundles isomorphic to the restrictions of our constructed Higgs bundle to ℓ. The discussion of Section 8 will show that the contributions from the two branches of the critical locus became equivalent to direct images of line bundles on the the disjoint unions of the two branches, via the mechanism of the blow-up of the double points of Σ and the correction due to a line bundle on the exceptional locus.

7.6 Pullbacks of the wobbly locus in the abelianized Hecke

We have two composed maps

$$\widehat{\mathcal{H}}^{\mathrm{ab}} \xrightarrow{p^{\mathrm{ab}}} Y_1 \xrightarrow{f_1} X_1$$

and

$$\widehat{\mathcal{H}}^{\mathrm{ab}} \xrightarrow{d^{\mathrm{ab}}} Y_0 \xrightarrow{f_0} X_0.$$

We would like to describe the inverse images of the wobbly loci Wob_1, Wob_0 under these.

For $f_1 : Y_1 \to X_1$ recall that $\boldsymbol{E}_1 \subset Y_1$ is the exceptional divisor of the blowup $\boldsymbol{\varepsilon}_1 : Y_1 \to \mathcal{P}_3$ (see section 3.2). Then

$$(f_1)^{-1}(\mathrm{Wob}_1) = 2\boldsymbol{E}_1 + \mathsf{Residual}_1.$$

Here $\mathsf{Residual}_1$ is a 4-sheeted cover of Wob_1, because f_1 has degree 8, while $\boldsymbol{E}_1$ is a double cover of Wob_1. By definition $\mathsf{Residual}_1$ parametrizes line bundles in the Prym that push forward to wobbly bundles.

Look at the map $p^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_1$. Birationally this is the Abel-Jacobi map: $\mathcal{P}_2 \times \widehat{C} \to \mathcal{P}_3$, sending a pair $(L, \hat{y} = (\tilde{y}, (A, y)))$ to $L \otimes \pi^* A^{-1}(\tilde{y})$. The generic fiber of p^{ab} is $\widehat{C}$. Over the generic point of the exceptional divisor $\boldsymbol{E}_1 = \widehat{C} \times \mathbb{P}^1$, the fiber becomes the union of $\widehat{C}$ with 16 surfaces, each an $\mathbb{F}_1$.

We want to pull back this divisor by the Abel-Jacobi map. Recall from the discussion in Section 7.2 that the map $\widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_1 \times \widehat{C}$ is a blowing-up along 16 disjoint subvarieties $\mathbb{P}^1 \times \widehat{C}$ in $\boldsymbol{E}_1 \times \widehat{C}$. In other words, the variety $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is the blowup of $Y_1 \times \widehat{C}$ along the 16 surfaces, all contained in $\boldsymbol{E}_1 \times \widehat{C} = \mathbb{P}^1 \times \widehat{C} \times \widehat{C}$. The 16 surfaces are

$$\mathbb{P}^1 \times \widehat{C} \times_{\widetilde{C}} \widehat{C} = \sqcup_{\kappa \in \mathsf{Spin}(C)} \mathbb{P}^1 \times \widehat{C} \subset \mathbb{P}^1 \times \widehat{C} \times \widehat{C}.$$

When we pull back our divisor $(f_1)^{-1}(\mathrm{Wob}_1) = 2\boldsymbol{E}_1 + \mathsf{Residual}_1$ by p^{ab}, we may first just take its product with $\widehat{C}$ and then do the blowing-up. Not much happens over $\mathsf{Residual}_1 \times \widehat{C}$ (we get its strict transform $\widehat{\mathsf{Residual}}_1$), while the pullback of $\boldsymbol{E}_1 \times \widehat{C}$ is $\boldsymbol{Exc}_1$ plus 16 new components $\boldsymbol{Exc}_{0,\kappa}$. This proves the following:

Lemma 7.15. *The pullback of* Wob$_1$ *by the composed map* $f_1 \circ p^{\mathrm{ab}}$ *is*

$$\boldsymbol{Exc}_1 \cup \Big(\bigcup_{\kappa \in \mathsf{Spin}(C)} \boldsymbol{Exc}_{0,\kappa} \Big) \cup \widehat{\mathsf{Residual}}_1.$$

Turn now to the pullback of Wob$_0$ in Y_0 and $\widehat{\mathcal{H}}^{\mathrm{ab}}$. For the map $f_0 : Y_0 \to X_0$ and the exceptional divisor $\boldsymbol{E}_0 \subset Y_0$ we get

$$(f_0)^{-1}(\mathrm{Wob}_0) = 2\boldsymbol{E}_0 + \mathsf{Residual}_0 + K$$

Here K is the full inverse image of the Kummer surface $\mathsf{Kum} \subset Y_0$. It is irreducible, in fact it is dominated by the second symmetric product of $\widehat{C}$.

Left over, $\mathsf{Residual}_0$ is a 6-sheeted cover of $\cup_\kappa \mathsf{Trope}_\kappa$, because f_0 has degree 8, while $\boldsymbol{E}_0$ is birational to $\cup_\kappa \mathsf{Trope}_\kappa$. In particular, $\mathsf{Residual}_0$ is a union of 16 pieces corresponding to the 16 trope planes (we are not saying here that these pieces are necessarily irreducible but of course that is strongly suspected).

Consider $d^{\mathrm{ab}} : \widehat{\mathcal{H}}^{\mathrm{ab}} \to Y_0$. Here $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is the blowup of $Y_0 \times \widehat{C}$ along the surface $\widehat{C} \times \widehat{C} \subset Y_0 \times \widehat{C}$. The exceptional divisor for this blowup is $\boldsymbol{Exc}_1$. The pullback of $\mathsf{Residual}_0$ is the strict transform $\widehat{\mathsf{Residual}}_0$ of the divisor $\mathsf{Residual}_0 \times \widehat{C}$. For $\boldsymbol{E}_0 = \sqcup_\kappa \boldsymbol{E}_{0,\kappa}$, the pullback of each component consists of the strict transform of $\boldsymbol{Exc}_{0,\kappa}$.

Lemma 7.16. *The pullback of* Wob_0 *by the composed map* $f_0 \circ p^{\mathrm{ab}}$ *is*

$$\left(\bigcup_{\kappa} \boldsymbol{Exc}_{0,\kappa}\right) \cup \boldsymbol{Exc}_1 \cup \widehat{K} \cup \widehat{\mathsf{Residual}}_0.$$

Proof. We claim that the surface $\widehat{C} \times \widehat{C}$ is contained in the extra component $K \times \widehat{C}$. We have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{H}}^{\mathrm{ab}} & \to & Y_0 \times \widehat{C} \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}} & \to & X_0 \times \overline{C} \end{array}.$$

If we take a general point $(V, \hat{y}) \in \widehat{C} \times \widehat{C} \subset Y_0 \times \widehat{C}$ then the inverse image of $(V, \hat{y})$ in $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is a $\mathbb{P}^1$. We want to say that this $\mathbb{P}^1$ maps to a positive dimensional subvariety in $\overline{\mathcal{H}}$. By construction the image of this $\mathbb{P}^1 \subset \widehat{\mathcal{H}}^{\mathrm{ab}}$ to Y_1 is one of the rulings of the divisor $\widehat{C} \times \mathbb{P}^1 = \boldsymbol{E}_1 \subset Y_1$. In particular if we map the $\mathbb{P}^1 \subset \widehat{\mathcal{H}}^{\mathrm{ab}}$ all the way down to X_1, then the image is one of the lines in Wob_1, i.e. is positive dimensional. But the map $\widehat{\mathcal{H}}^{\mathrm{ab}} \to X_1$ factors through $\overline{\mathcal{H}}$. Hence the image of $\mathbb{P}^1 \subset \widehat{\mathcal{H}}^{\mathrm{ab}}$ in $\overline{\mathcal{H}}$ will be a $\mathbb{P}^1$ as well. Note also that the image $\mathbb{P}^1 \subset \overline{\mathcal{H}}$ is contained in a fiber of the map $q : \overline{\mathcal{H}} \to X_0 \times \overline{C}$. But the only components of fibers of q that map to lines in $X_1 \subset \mathbb{P}^5$ are components of fiber over points of the Kummer (note that by assumption our $\mathbb{P}^1$ maps to a general point in $\overline{C}$ in particular not a preimage of a Weierstrass point of C).

This shows that the locus to be blown up is contained in $K \times \widehat{C}$. Therefore, the pullback of K in $\widehat{\mathcal{H}}^{\mathrm{ab}}$ is the union of the strict transform $\widehat{K}$, and the exceptional divisor $\boldsymbol{Exc}_1$. The pullbacks of the other pieces of $(f_0)^{-1}(\mathrm{Wob}_0)$ yield the other pieces of the stated decomposition. □

Theorem 7.17. *The intersection of the two pullbacks of the wobbly loci up to codimension* 1 *in* $\widehat{\mathcal{H}}^{\mathrm{ab}}$ *consists of just the exceptional pieces*

$$(f_0 \circ p^{\mathrm{ab}})^{-1}(\mathrm{Wob}_0) \cap (f_1 \circ p^{\mathrm{ab}})^{-1}(\mathrm{Wob}_1) = \boldsymbol{Exc}_1 \cup (\bigcup_{\kappa} \boldsymbol{Exc}_{0,\kappa}).$$

Proof. Using the previous lemmas, it means we need to look at the intersection

$$\left[(\bigcup_{\kappa} \boldsymbol{Exc}_{0,\kappa}) \cup \boldsymbol{Exc}_1 \cup \widehat{K} \cup \widehat{\mathsf{Residual}}_0\right] \cap \left[\boldsymbol{Exc}_1 \cup (\bigcup_{\kappa} \boldsymbol{Exc}_{0,\kappa}) \cup \widehat{\mathsf{Residual}}_1\right].$$

We need to show that $\widehat{\mathsf{Residual}}_1$ does not have any irreducible components in common with $\widehat{K} \cup \widehat{\mathsf{Residual}}_0$.

Choose $a = (A, t) \in \overline{C}$ with a lifting $\widehat{a} = (A, \tilde{t})$ and look at the fiber over $\widehat{a}$. Here, the subvarieties have dimension 2 and we want to show (for general a) that they do not share any 2-dimensional components.

In view of the decomposition of the fiber $\widehat{\mathsf{Residual}}_0(a)$ into 16 pieces that move around under the monodromy operation when we move a, whereas on the other hand $\widehat{\mathsf{Residual}}_1(a)$ can't have more than 4 irreducible components since it is a 4-sheeted covering of the irreducible Wob_1, we see that $\widehat{\mathsf{Residual}}_0$ and $\widehat{\mathsf{Residual}}_1$ can not share irreducible components. Thus, we are reduced to the question of showing that $\widehat{\mathsf{Residual}}_1$ and $\widehat{K}$ do not share any components.

Now, by Lemma 7.11, the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tilde{a})$ in $\overline{\mathcal{H}}(a)$ only intersects the Hecke fibers over the Kummer Kum, in the intersection points of the two lines. But this collection of points maps to the Kummer K3 surface inside X_1 that isn't the same as Wob_1. Thus, the intersection $\widehat{\mathsf{Residual}}_1 \cap \widehat{K}$ does not have any 2-dimensional pieces. This completes the proof of the theorem. □

Corollary 7.18. *Inside the big Hecke correspondence $\overline{\mathcal{H}}$, the intersection of the two pullbacks of the wobbly loci with the abelianized Hecke is, as for its 3-dimensional pieces:*

$$p^{-1}(\mathrm{Wob}_1) \cap d^{-1}(\mathrm{Wob}_0) \cap g(\widehat{\mathcal{H}}^{\mathrm{ab}}) = g(\boldsymbol{Exc}_0 \cup \boldsymbol{Exc}_1).$$

Proof. This is the image by $g : \widehat{\mathcal{H}}^{\mathrm{ab}} \to \overline{\mathcal{H}}$ of the statement in the previous theorem. □

Recall that for each $a \in \widehat{C}$, the piece $g(\boldsymbol{Exc}_0)(a)$ in the fiber $\overline{\mathcal{H}}(a)$ contracts to a lower-dimensional subvariety (in this case the union of 16 lines in the wobbly locus) under the projection to X_1. And similarly, the piece $g(\boldsymbol{Exc}_1)(a)$ contracts to a lower-dimensional subvariety (a birational copy of the curve $\overline{C}$ in Kum) under the projection to X_0.

The piece $g(\boldsymbol{Exc}_0)(a)$ mapping to X_0 consists of 16 planes over the trope planes, which are the ramification of the horizontal divisor in the Hecke correspondence that contribute to the singularities of the Hecke transformed system from X_1 to X_0. The singularities over the Kummer surface come from the degenerations of Hecke conics to pairs of lines.

On the other hand, the piece $g(\boldsymbol{Exc}_1)(a)$ mapping to X_1 is the ramification of the inverse image of the Kummer surface in the Hecke correspondence, giving the singularities of the Hecke transformed system from X_0 to X_1.

In the next chapters we will be testing the Hecke transforms by restricting over lines in the target spaces, so the ramification locations contributing to singularities will show up as intersected with the inverse images of the lines in the Hecke correspondence.

8 Hecke transformation from X_0 to X_1

We recall the standard diagrams. The big Hecke correspondence fits into a diagram of the form

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1 \times \overline{C}. \end{tikzcd}$$

Fix a point $a = (A, t) \in \overline{C}$ where $A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$ and consider the Hecke correspondence $\overline{\mathcal{H}}(\overline{a})$ fitting into the diagram

$$\begin{tikzcd} & \overline{\mathcal{H}}(a) \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1. \end{tikzcd}$$

The objective is to pull-back the constructed Higgs bundle from X_0 and take the higher direct image along b to X_1.

In order to simplify the measurement of the result, fix a general line $\ell \subset X_1$. Let $\overline{\mathcal{H}}(a)_\ell$ be the inverse image of ℓ in the Hecke variety $\overline{\mathcal{H}}(a)$. This is a Hirzebruch surface $\mathbb{F}_1$, which is a $\mathbb{P}^1$-bundle over $\ell \cong \mathbb{P}^1$.

Let $P \subset X_0$ be the image of the map from $\overline{\mathcal{H}}(a)_\ell$ to X_0. This is a plane $P \cong \mathbb{P}^2 \subset \mathbb{P}^3$. The map $d : \overline{\mathcal{H}}(a)_\ell \to P$ is blowing up a point $\mathfrak{p} \in P$. This point is a general point contained in the Kummer $\mathsf{Kum} \subset X_0$. There is also a point $\mathfrak{q} \in P \cap \mathsf{Kum}$ where P is tangent to Kum. In particular $\mathfrak{p}$ and $\mathfrak{q}$ are not on the trope planes.

Let $\boldsymbol{\mathcal{T}} \subset \overline{\mathcal{H}}(a)_\ell$ be the strict transform of $\mathsf{Kum} \cap P$, and let $\boldsymbol{\mathcal{N}} \subset \overline{\mathcal{H}}(a)_\ell$ be the exceptional divisor. It is the unique section of the $\mathbb{F}_1$ surface that has self-intersection -1.

We have that $\boldsymbol{\mathcal{T}} \cap \boldsymbol{\mathcal{N}} = \mathfrak{p}'$ is a single point in $\overline{\mathcal{H}}(a)_\ell$. The point $\mathfrak{q}$ corresponds to a unique point $\mathfrak{q}' \in \overline{\mathcal{H}}(a)_\ell$ where the curve $\boldsymbol{\mathcal{T}}$ has a node. The distinction in notation here is that $\mathfrak{p} \in P$ is a point of the plane inside X_0, while $\mathfrak{p}'$ is one of the points in the exceptional divisor above $\mathfrak{p}$, namely the intersection point of $\mathcal{T}$ and $\mathcal{N}$; it also corresponds to the tangent direction of the Hecke line at $\mathfrak{p}$ or equivalently the tangent direction of $P \cap \mathsf{Kum}$ at $\mathfrak{p}$. The points $\mathfrak{q}$ and $\mathfrak{q}'$ are basically identical since there is no blowing-up, we just use the notation $\mathfrak{q}'$ rather than $\mathfrak{q}$ for uniformity.

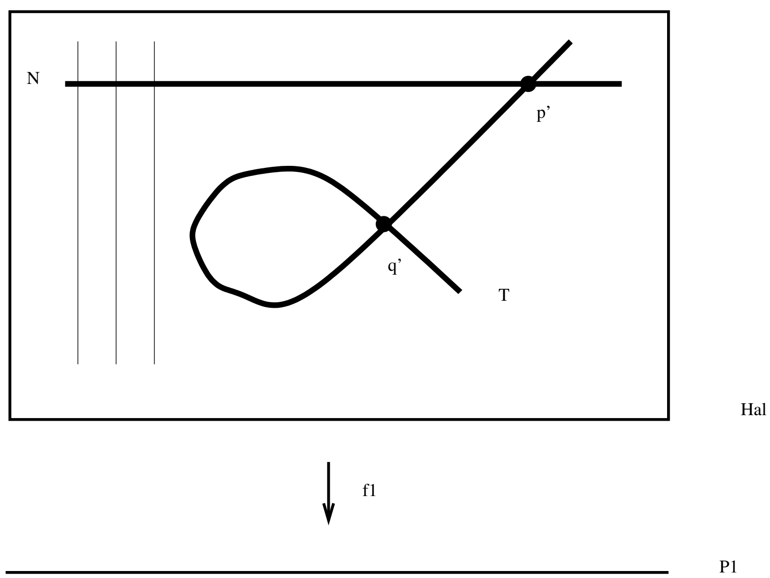

Figure 1: The Hirzebruch surface $\overline{\mathcal{H}}(a)_\ell$

Let $Y_{1,\ell} \subset Y_1$ be the inverse image of ℓ. This is a smooth curve in view of the moving properties of the family of lines ℓ, indeed, locally this family looks like the family of complete intersections of two moving divisors in X_1.

8.1 The abelianized Hecke as a critical locus

The big abelianized Hecke fits into a diagram of the form

$$\begin{tikzcd} & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "d^{\mathrm{ab}}"'] \arrow[dr, "b^{\mathrm{ab}}"] & \\ Y_0 & & Y_1 \times \widehat{C}. \end{tikzcd}$$

The inverse image in $\widehat{C}$ of $a \in \overline{C}$ consists of two points that we will note $\widehat{a}$ and $\tau\widehat{a}$. We obtain two abelianized Hecke varieties $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})$. Denote their disjoint union by

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a}) := \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \sqcup \widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a}).$$

The map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to Y_1$ is the blow-up along 16 lines contained in the wobbly locus, as was discussed in Section 7.2.

A general line $\ell \subset X_1$ does not meet those lines. Thus, if we let $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ denote the inverse image of ℓ in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$, the projection induces an isomorphism

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \xrightarrow{\cong} Y_{1,\ell}.$$

The same holds for the other piece $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$.

The curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has degree 8 over ℓ. The full curve of degree 16 over ℓ is

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell = \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \sqcup \widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell.$$

The 16 exceptional divisors in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ contract to the 16 exceptional planes in Y_0. It follows that the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ in Y_0 does not meet those.

Let $Y_{0,P} := Y_0 \times_{X_0} P$, and set

$$\Sigma := Y_{0,P} \times_P \overline{\mathcal{H}}(a)_\ell.$$

This is the spectral variety for the Higgs bundle $(\mathscr{F}_{0,\bullet}, \Phi_0)_{\overline{\mathcal{H}}(a)_\ell}$ which is $(\mathscr{F}_{0,\bullet}, \Phi_0)$ pulled back to $\overline{\mathcal{H}}(a)_\ell$ via the map $d : \overline{\mathcal{H}}(a)_\ell \to X_0$. In particular Σ has degree 8 over $\overline{\mathcal{H}}(a)_\ell$. We have a lifting

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \to \Sigma.$$

Proposition 8.1. *The map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \to \Sigma$ identifies the abelianized Hecke as the upper critical locus (see Proposition 12.1)*

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \cong \widetilde{\mathrm{Crit}}\left(\overline{\mathcal{H}}(a)_\ell/\ell, (\mathscr{F}_{0,\bullet}, \Phi_0)_{\overline{\mathcal{H}}(a)_\ell}\right)$$

away from the points $\mathfrak{p}'$ and $\mathfrak{q}'$.

Proof. Suppose we are given a point of Σ. It corresponds to a point of Y_0 together with a Hecke operation leading to a Hecke transformed bundle which is a point of $\ell \subset X_1$.

If the point of Y_0 is on an exceptional divisor, it means that the point of Σ is a ramification point of Σ over the horizontal divisor in $\overline{\mathcal{H}}(a)_\ell$. The residues of the Higgs field Φ_0 are nilpotent, so their Jordan types must be constant along the parts of the horizontal divisor that are etale over ℓ, and this implies that ramification points of Σ over this subset of the horizontal divisor can not be in the relative critical locus (we will see this argument again in the next subsection). The points $\mathfrak{p}'$ and $\mathbf{q}'$, as well as the ramification points of the horizontal divisor, are contained in the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ and also in the relative critical locus.

We claim that the upper critical locus is smooth at the ramification points of the horizontal divisor, i.e. at points where $\boldsymbol{\mathcal{T}}$ ramifies over ℓ. In local coordinates x, t where the map to ℓ is given by t, we can write $\boldsymbol{\mathcal{T}}$ as $x^2 - t = 0$, and Σ is $x^2 - t = w^2$. Thus x, w are coordinates on Σ and $t = x^2 - w^2$. Write α as $adx + bdw$. The vertical direction over the ramification point is the w-axis, and the fact that the Jordan form of the residue of the Higgs field is constant along smooth points of Wob_1 (our ramification is such a smooth point) implies that α is nonzero on the vertical direction, as was pointed out in Lemma 8.3. This says that $b(0) \neq 0$. Now, the relative differentials are obtained by setting $dt = 0$ so $xdx = wdw$. Thus α^{rel} is written as

$$\alpha^{\mathrm{rel}} = (a + (x/w)b)dx$$

and the condition $\alpha^{\mathrm{rel}} = 0$ may be written as $wa + xb = 0$. As $b(0) \neq 0$, the linear term $wa(0) + xb(0)$ is nonzero, so this defines a smooth curve which is the local branch of the relative critical locus at this point of Σ.

This shows that the relative critical locus is smooth at ramification points of $\boldsymbol{\mathcal{T}}/\ell$, so a set-theoretical identification between the two subvarieties (neither of which has embedded points) is an isomorphism.

Over the points $\mathfrak{p}'$ and $\mathbf{q}'$, we will see that the relative critical locus has two branches in each of the four nodes of Σ over one of these points, and the abelianized Hecke is isomorphic to the normalization.

In view of the previous paragraphs, we can now assume that the point of Y_0 is not on an exceptional divisor, so it corresponds to a line bundle L of degree 2 on $\widetilde{C}$ whose norm to C is ω_C. The corresponding vector bundle on C is $E = \pi_* L$. Our Hecke transformation is given by a pair (A, x) where $x \in C$ and $A^{\otimes 2} = \mathcal{O}_C(x - \mathbf{p})$. The Hecke transformation is centered at a line $V \subset E_x$ and we let E' be the kernel of the map $E \to E_x/V$. This is a bundle of degree -1 with determinant $\mathcal{O}_C(-x)$. Then $E' \otimes A(\mathbf{p})$ has degree 1 and determinant $\mathcal{O}_C(\mathbf{p})$ so it is a point in X_1.

In turn we have a line $V' \subset E'(x)_x$ and the kernel of $E' \to E'(x)_x/V'$ is E. The line V' corresponds to a line in $E' \otimes A(\mathbf{p})$ and the opposite Hecke transform involving tensoring again with a square-root, gets us back to E. The Hecke line through the point $[E]$ consists of all the bundles obtained as kernels of $E' \to E'(x)/W$ where W is another line $W \subset E'(x)_x$. Letting W be an infinitesimal deformation of V' we get a tangent vector to X_0 at the point $[E]$, represented by a class $\eta \in H^1(\mathrm{End}_0(E))$. We'll give an expression for this class using an exact sequence, below.

The tautological 1-form on Y_0 is a section of $f_0^* T^\vee_{X_0}$ At the point $[L] \in Y_0$ corresponding to the line bundle L that projects to $[E] \in X_0$, $\alpha([L]) \in \left(f_0^* T^\vee_{X_0}\right)_{[L]} = T^\vee_{X_0,[E]}$ can be paired with a tangent vector of the form η. The condition that we are at a point of the upper critical locus means that the value of the pairing is zero.

Our point on Y_0 actually represents a Higgs bundle over C, because Y_0 was the blow-up of $\mathcal{P}_2 \subset \mathsf{Higgs}_0$. The underlying bundle is E (since we are assuming that we are not over the wobbly locus), and the Higgs field $\theta : E \to E \otimes \omega_C$ is induced by the tautological 1-form of $\widetilde{C}$. In these terms, the pairing is just the pairing between $\theta \in H^0(\mathrm{End}_0(E) \otimes \omega_C)$ and $\eta \in H^1(\mathrm{End}_0(E))$. The upper critical locus condition is that this pairing is 0.

Let us now look more closely at the deformation class η. The deformation of V' is given by an element of $\mathrm{Hom}(V', E'(x)_x/V')$. Tensor

$$0 \to E \to E'(x) \to E'(x)_x/V' \to 0$$

with $E^\vee$ to get

$$0 \to \mathrm{End}(E) \to E^\vee \otimes E'(x) \to E^\vee_x \otimes E'(x)_x/V' \to 0.$$

Note that $E_x \to V'$ so $(V')^\vee \to E^\vee_x$. Compose with the connecting map for the previous exact sequence as follows:

$$(V')^\vee \otimes (E'(x)_x/V') \to E^\vee_x \otimes (E'(x)_x/V') \to H^1(\mathrm{End}(E)).$$

The image of an element of $\mathrm{Hom}(V', (E(x)_x/V'))$ is the deformation class of E generated by doing the Hecke operation back using the infinitesimally close W. The determinant of the new bundle is the same as that of E, so the deformation class lies in the trace-free part $H^1(\mathrm{End}_0(E))$.

The exact sequence for the fiber of the Hecke transformation is

$$0 \to E'_x/V'(-x) \to E_x \to V' \to 0.$$

The exact sequence

$$0 \to \mathrm{End}(E) \to E^* \otimes E'(x) \to E_x^\vee \otimes (E'(x)_x/V') \to 0.$$

has as dual, taking $\mathrm{Ext}^1(-, \omega_C)$:

$$0 \to E \otimes (E')^\vee(-x) \otimes \omega_C \to \mathrm{End}(E) \otimes \omega_C \to \mathrm{Ext}^1(E_x^\vee \otimes (E'(x)_x/V'), \omega_C) \to 0.$$

We have

$$\mathrm{Ext}^1(E_x^\vee \otimes (E'(x)_x/V'), \omega_C) = \mathrm{Hom}(\mathcal{O}_C, E_x^\vee \otimes (E'(x)_x/V'))^\vee = \mathrm{Hom}((E'(x)_x/V'), E_x).$$

This maps to $\mathrm{Hom}(E'(x)_x/V', V')$. Use the residue identification $\omega_C(x)_x \cong \mathbb{C}$ and hence $V' \otimes \omega_C(x) \cong V'$, to say that

$$\mathrm{Hom}(E'(x)_x/V', V') \cong \mathrm{Hom}(E'(x)_x/V', V' \otimes \omega_C(x)) \cong \mathrm{Hom}(E'_x/V'(-x), V' \otimes \omega_C).$$

The previous map now becomes

$$\mathrm{Ext}^1(E_x^\vee \otimes (E'(x)_x/V'), \omega_C) \to \mathrm{Hom}(E'_x/V'(-x), V' \otimes \omega_C).$$

Altogether, the second map in the dual exact sequence becomes

$$\mathrm{End}(E) \otimes \omega_C \to \mathrm{Hom}(E'_x/V'(-x), V' \otimes \omega_C),$$

and this is just the evaluation map evaluating a Higgs field on the subspace $E'_x/V'(-x) \subset E_x$ and projecting the answer to the quotent V' tensored with ω_C.

The Serre dual of the map $(V')^\vee \otimes (E'(x)_x/V') \to H^1(\mathrm{End}(E))$ is the action of the previous map on global sections:

$$H^0(\mathrm{End}(E) \otimes \omega_C) \to \mathrm{Hom}(E'_x/V'(-x), V' \otimes \omega_C),$$

To say that this vanishes for a Higgs field $\theta : E \to E \otimes \omega_C$ is equivalent to saying that the filtration with subspace $E'_x/V'(-x) \subset E_x$ and quotient $E_x \to V'$ is respected by the Higgs field.

This is now the same thing as saying that our original Hecke transformation was in the abelianized Hecke. $\square$

Lemma 8.2. *For a general ℓ, $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ does not intersect the inverse images in $\overline{\mathcal{H}}(a)_\ell$ of the 32 points in P given by the intersection of P with the trope conics.*

Proof. Recall that $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \to Y_0$ is the blow-up along a curve isomorphic to $\widehat{C}$, this curve intersecting the planes of the exceptional locus in Y_0 in a finite set. The trope conics are conics in these planes, so their inverse image in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has dimension 1. This projects to a dimension 1 subset of X_1, so a general line ℓ does not intersect that. The same holds for the other piece. So, $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})$ intersected with the pullbacks of the trope conics, intersected with the pullback of ℓ, will be empty. $\square$

8.2 Description of the configuration

We look more closely at the details of the configuration inside and above $\overline{\mathcal{H}}(a)_\ell$. Inside Σ, there are 16 lines $L_{\kappa,\Sigma} \subset \Sigma$ lying over the (strict transforms in $\overline{\mathcal{H}}(a)_\ell$ of the) intersections $P \cap \mathsf{Trope}_\kappa$ of P with the trope planes, on which $\Sigma/\overline{\mathcal{H}}(a)_\ell$ has a simple ramification. Lemma 8.2 says that the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ does not meet these $L_{\kappa,\Sigma}$.

Over general points of $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}} \subset \overline{\mathcal{H}}(a)_\ell$ the ramification of Σ consists of four sheets each of which is a simple ramification. We also know that Σ has four ordinary double points over each of $\mathfrak{p}'$ and $\mathfrak{q}'$. There are probably other singularities for example over the intersection of P with the trope conics, however the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ does not touch those.

There is a spectral line bundle $\mathscr{L}_0$ on Y_0, which pulls back to a line bundle $\mathscr{L}_\Sigma$ on Σ, such that $\mathcal{E}_{\overline{\mathcal{H}}(a)_\ell}$ is the pushforward of $\mathscr{L}_\Sigma$ from Σ to $\overline{\mathcal{H}}(a)_\ell$. The Higgs field of $\mathcal{E}_{\overline{\mathcal{H}}(a)_\ell}$ is given by a holomorphic 1-form α on the smooth locus of Σ. It projects to a section of $\Omega^1_{\overline{\mathcal{H}}(a)_\ell/\ell}(\log D)|_\Sigma$ where the divisor D is the union of $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\mathcal{N}}$ and the strict transforms in $\overline{\mathcal{H}}(a)_\ell$ of the 16 trope lines $P \cap \mathsf{Trope}_\kappa$.

Along a point where $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$ is horizontal over ℓ, the residue of the Higgs field consists of a sum of four nonzero nilpotent transformations, since the Jordan type of the residue has to stay fixed along the divisor because it corresponds to monodromy. Furthermore, over $\boldsymbol{\mathcal{N}}$ at least, the inverse image of $\boldsymbol{\mathcal{N}}$ in Σ is a union of four lines along which the map has simple ramification. Thus, the form α is nonvanishing in the transverse direction to these lines so as a section of $\Omega^1_{\overline{\mathcal{H}}(a)_\ell/\ell}(\log D)|_\Sigma$ it is nonvanishing along these lines except at the points over $\mathfrak{p}'$. On the other hand, the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ is the vanishing locus of the section of $\Omega^1_{\overline{\mathcal{H}}(a)_\ell/\ell}(\log D)|_\Sigma$. It follows that $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \subset \Sigma$ does not intersect the four lines over $\boldsymbol{\mathcal{N}}$

except at points over $\mathfrak{p}'$. In sum, the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ in $\overline{\mathcal{H}}(a)_\ell$ only intersects $\boldsymbol{\mathcal{N}}$ at $\mathfrak{p}'$. Similar considerations show that this also holds for the horizontal part of $\boldsymbol{\mathcal{T}} - \{\mathfrak{p}', \mathsf{q}'\}$.

We would like to know how many branches of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ pass through the point $\mathfrak{p}'$. We will see by the local calculations below that each double point of Σ over $\mathfrak{p}'$ corresponds to two branches of the polar curve of zeros of the relative Higgs field. This says that the full curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ has 2 branches in each of the four double points, thus it has 8 branches over $\mathfrak{p}'$. A monodromy argument says that these have to be evenly distributed, so $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has 4 branches passing through $\mathfrak{p}'$. The local argument will also show that they intersect $\boldsymbol{\mathcal{N}}$ transversally. Thus, if we denote by $[\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell]$ the image of this curve in $\overline{\mathcal{H}}(a)_\ell$, we have

$$[\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell] \cdot \boldsymbol{\mathcal{N}} = 4.$$

We will discuss later the distribution of these 4 branches among the 4 double points of Σ.

We next consider intersection numbers inside the $\mathbb{F}_1$ surface $\overline{\mathcal{H}}(a)_\ell$. Let fib denote a general fiber of $b : \overline{\mathcal{H}}(a)_\ell \to \ell$. Then the divisors fib and $\boldsymbol{\mathcal{N}}$ generate the Picard group of $\overline{\mathcal{H}}(a)_\ell$, with $\mathsf{fib}^2 = 0$, $\boldsymbol{\mathcal{N}}^2 = -1$ and $\mathsf{fib} \cdot \boldsymbol{\mathcal{N}} = 1$.

The curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has degree 8 over ℓ, so

$$[\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell] \cdot \mathsf{fib} = 8.$$

The intersection numbers with $\boldsymbol{\mathcal{N}}$ and fib uniquely determine the class of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ inside $\overline{\mathcal{H}}(a)_\ell$:

$$[\widehat{\mathcal{H}}^{\mathrm{ab}}(\tilde{a})_\ell] \sim 8\boldsymbol{\mathcal{N}} + 12\mathsf{fib}.$$

On the other hand, we also have that $\boldsymbol{\mathcal{T}}$ has degree 3 over ℓ since the intersection with a fiber is all the points in that line intersected the Kummer, except the point $\mathfrak{p}'$ which corresponds to the point of $\boldsymbol{\mathcal{N}}$ intersected that fiber. We have $\boldsymbol{\mathcal{T}} \cdot \mathsf{fib} = 3$ and $\boldsymbol{\mathcal{T}} \cdot \boldsymbol{\mathcal{N}} = 1$ so

$$\boldsymbol{\mathcal{T}} \sim 3\boldsymbol{\mathcal{N}} + 4\mathsf{fib}.$$

Combining these we get

$$[\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell] \cdot \boldsymbol{\mathcal{T}} = (8\boldsymbol{\mathcal{N}} + 12\mathsf{fib})(3\boldsymbol{\mathcal{N}} + 4\mathsf{fib}) = 36 + 32 - 24 = 44.$$

We already know the intersections over the point $\mathfrak{p}'$, there are 4 branches. Also over q' there will similarly be 4 branches, but q' is a double point of $\boldsymbol{\mathcal{T}}$ (whereas the double point of D at $\mathfrak{p}'$ involved both $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{N}}$). Thus, the intersection counts for 8 points over q'. This leaves 32 intersection points over the ramification points of $\boldsymbol{\mathcal{T}}/\ell$. We note that there are 8 such points, mapping to the 8 points in ℓ where $\ell \subset X_1$ intersects Wob_1. We get an intersection number of 4 at each ramification point (a monodromy argument shows that they need to be evenly distributed). This will come from two tacnodes between the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ and the divisor $\boldsymbol{\mathcal{T}}$, in accordance with our calculations for the local contribution of the direct image at a simple ramification point of the horizontal divisor.

The curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has to meet the ramified locus of Y_0 over Kum in two of the four sheets, as each one will contribute a tacnode and from the above discussion there should not be more than 2 tacnodes. The curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$ meets the ramified locus in the other two sheets. On the other hand, for the nodal points we'll see that $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ has one branch in each sheet and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$ has the other branch.

For each of the direct summands of the direct image Higgs bundle, we will get two ramification points of the spectral variety over each point of $\ell \cap \mathrm{Wob}_1$. This agrees with the ramification pattern of $Y_1 \to X_1$ over Wob_1.

We need to verify the part of Hypothesis 3.16 about points of type 3.11.1(d), namely the ramification points of $\boldsymbol{\mathcal{T}}$ over ℓ.

Lemma 8.3. *For a general line ℓ, the value of the spectral 1-form α on the vertical direction in the tangent space of Y_0 at any of the ramification points of $\boldsymbol{\mathcal{T}}/\ell$, is nonzero.*

Proof. As ℓ varies, each of these ramification points varies and constitutes a general point of Kum. So it suffices to note that the value of α on the vertical direction is nonzero. This is because the nilpotent residue of the Higgs field is nonzero along Kum, which in turn is a consequence of the Bogomolov-Gieseker inequality as explained in the proof of Lemma 8.9 below. $\square$

8.3 Apparent singularities

The discriminant of $(\overline{\mathcal{H}}(a)_\ell, \boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}})$ relative to ℓ consists of the following kinds of points:

- images of simple ramification points of $\boldsymbol{\mathcal{T}}$ over ℓ, these are the points of $\ell \cap \mathrm{Wob}_1$ and we expect singularities of the local system at these points;

- images of points where the Hecke line goes through a trope conic— by Lemma 8.2 these points are not contained in the lower critical locus, so by Proposition 12.2 they do not contribute singularities of the higher direct image local system; and

- images of the points $\mathfrak{p}'$ and $\mathfrak{q}'$ where $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$ has a node—we'll see for topological reasons below that these do not contribute singularities, and indeed the Dolbeault higher direct image consideration will also show that.

The following lemma gives the topological proof for the third part.

Lemma 8.4. *Suppose given a point where the horizontal divisor has a simple normal crossing with both branches étale over the base, and suppose that the monodromy transformations around each branch are direct sums of identical size* 2 *unipotent Jordan blocks. Then such a point does not contribute to the monodromy of the higher direct image local system.*

Proof. This description characterizes the nontrivial local rank 2 pieces of the local system upstairs. However, we also obtain the same type of situation if we start with a unipotent rank 2 local system having singularities along a horizontal divisor that is simply ramified over the base, then pull back to a simply ramified double cover of the base. We have seen in subsection 12.4.3 that the monodromy for a simply ramified horizontal divisor with unipotent local system, is a transformation of order 2. Therefore, its pullback by a ramified double cover has trivial monodromy transformation. □

Even though the topological monodromy transformation for such a point is trivial, we still need to use the description of [DPS16] to get the description of the spectral line bundle for the higher direct image at such a point, since the map is singular.

Corollary 8.5. *The singularities of the higher direct image local system over ℓ are located at the points of $\ell \cap \mathrm{Wob}_1$.*

Remark 8.6. In the situation of Lemma 8.4, if we had monodromy over the nodal horizontal divisor decomposing into size 2 blocks with a reflection instead of a nilpotent monodromy transformations, then the higher direct image would have the square of a nilpotent transformation (subsection 12.4.1), so again a nontrivial nilpotent transformation. Applied to the Hecke situation, this means that if we do the Hecke transformation starting with a local system with parabolic weights 1/2 over the Kummer, we will get singularities at the images of $\mathfrak{p}'$ and $\mathfrak{q}'$ in ℓ. These are probably the intersection of ℓ with a singular K3 surface inside X_1, the image of the K3 inside the Hecke variety, and which is the intersection of X_1 with another quadric in $\mathbb{P}^5$.

8.3.1 General position arguments

Let us now look locally at the point $\mathfrak{p}'$. The horizontal divisor $\boldsymbol{\mathcal{T}} \cup \boldsymbol{\mathcal{N}}$ has a node there, and the spectral variety Σ decomposes as a union of four ordinary double points.

The point $\mathfrak{q}'$ is a place where $\boldsymbol{\mathcal{T}}$ has a node, with both branches being étale over the base ℓ. This corresponds to a point where the plane P is tangent to Kum. Again, Σ decomposes into 4 pieces, and each piece restricted to the tangent plane gives an ordinary double point.

In order to apply the construction of Theorems 3.17 and 12.12, we need the following result.

Theorem 8.7. *The points $\mathfrak{p}'$ and $\mathfrak{q}'$ are points of type 3.11.1(e) in the classification of Subsection 3.11.*

Proof. The geometric picture shows that the inverse image of $\mathfrak{p}'$ resp. $\mathfrak{q}'$ in Σ decomposes into four ordinary double points. Recall that over a general point of Kum, the covering $Y_0 \to X_0$ decomposes into four local pieces each of which is a double cover simply ramified over the local piece of Kum. Each of these four pieces leads to a simple double point in the fiber of Σ, either over $\mathfrak{p}'$ or over $\mathfrak{q}'$.

Over $\mathfrak{p}'$, we are restricting to a transverse plane where it is again a smooth double cover ramified over a smooth curve, then blowing up the origin. The point $\mathfrak{p}'$ corresponds to the place where the exceptional divisor $\boldsymbol{\mathcal{N}}$ meets the strict transform $\boldsymbol{\mathcal{T}}$ of Kum, and the double cover gives there an ordinary double point.

Over $\mathfrak{q}'$, we have a plane that is tangent to Kum with nondegenerate second fundamental form, such that the two branches of the intersection between the plane and Kum correspond to the two branches of $\boldsymbol{\mathcal{T}}$ at $\mathfrak{q}'$. The restriction of the local double cover to the tangent plane is again an ordinary double point.

To complete the verification of the conditions required for the classification of our points in the category "type 3.11.1(e)", we need to show that the relative critical locus has two branches at each node of Σ, with the branches having distinct tangent vectors in the tangent cone of Σ.

A degree and monodromy calculation shows that the relative critical locus, that we have identified also with $\widehat{\mathcal{H}}^{\rm ab}(\widehat{a}, \tau\widehat{a})_\ell$, has two branches at each node of Σ. We need to show that they are smooth with distinct tangent vectors. This statement amounts to a general position argument concerning the relationship between the local geometry at a general point of Kum, and the tautological 1-form on the covering Y_0/X_0.

The cases of $\mathfrak{p}'$ and $\mathfrak{q}'$ are different although similar. The basic idea is to find three (or more) directions in a plane that come from the geometry of the moduli spaces, independent of the point in the Hitchin base; then the tautological 1-form is a fourth direction. If the cross-ratio between these directions did not move, it means that the data coming from the tautological 1-form would be constant, and we try to get a contradiction. That will show that the cross-ratio is general for general points, which in turn gives the required general position property.

The proofs for $\mathfrak{p}'$ and $\mathfrak{q}'$ will be completed in detail in the following subsections. $\square$

Let $Y_{\mathsf{Kum}} \subset Y_0$ be the reduced inverse image of Kum in Y_0. We saw in Proposition 5.12 that $Y_{\mathsf{Kum}} \to \mathsf{Kum}$ is a 4-sheeted covering with an explicit description.

Let H_{Kum} be the set of points in the Hecke correspondence $\overline{\mathcal{H}}(a)$ which are intersection points of two lines in the Hecke fiber over points of Kum. Thus, H_{Kum} is the Kummer K3-surface obtained by blowing up the 16 nodes of Kum, with a map $H_{\mathsf{Kum}} \to \mathsf{Kum}$ that is an isomorphism outside of the nodes. We obtain a map $H_{\mathsf{Kum}} \to X_1$ that we'll view as a rational map $\nu : \mathsf{Kum} \dashrightarrow X_1$, whose image is the Kummer K3 surface in X_1 [Bea96, Dol20, GH94, Hud05, Keu97].

Lemma 8.8. *Suppose $x \in \mathsf{Kum}$, not a node. Let $z := \nu(x) \in X_1$ be the image of the point of H_{Kum} lying over x. Let $V_x \subset X_0$ be the Hecke line over z. Then V_x is tangent to Kum at x.*

Proof. Let $d^{-1}(z) \subset \overline{\mathcal{H}}(\widehat{a})$ be the inverse image, which is a $\mathbb{P}^1$ mapping isomorphically to the Hecke line V_x. If $x' \in H_{\mathsf{Kum}} \subset \overline{\mathcal{H}}(\widehat{a})$ is the point over x, so that $z = d(x')$, then $d^{-1}(z)$ passes through x'. The divisor $b^{-1}(\mathsf{Kum}) \subset \overline{\mathcal{H}}(\widehat{a})$ is, locally near x', a union of two branches each of which is a part of the $\mathbb{P}^1$-bundle over the Jacobian of C. The curve $d^{-1}(z)$ meets each of the branches, so its intersection with the divisor $b^{-1}(\mathsf{Kum})$ has multiplicity ≥ 2. This intersection number is the same as the local intersection number of V_x with Kum at x. The fact that it is ≥ 2 means that the line V_x is tangent to Kum at x. $\square$

The inverse image of Kum in Y_0 is $2Y_{\mathsf{Kum}}$ as a divisor, since the map $Y_0 \to X_0$ is fully simply ramified along Y_{Kum}. At points where $Y_{\mathsf{Kum}}/\mathsf{Kum}$ is etale, there is a specified tangent direction to Y, normal to Y_{Kum}, namely the directions that map to zero in the tangent space of X_0. Call this subsheaf

$$N^v \hookrightarrow T(Y_0)|_{Y_{\mathsf{Kum}}}.$$

Lemma 8.9. *The restriction of the spectral* 1*-form* α *of* Y_0 *to* N^v *is nonzero.*

Proof. If it were zero everywhere, this would imply that the Higgs field on the Higgs bundle $\pi_*(\mathscr{L}_0)$ does not have singularities over Kum. Recall from Corollary 5.11 that for an appropriate choice of $\mathscr{L}_0$, the second Chern character violates the Bogomolov-Gieseker inequality. But, if there were no singularities of the Higgs field along Kum, then the Higgs field would be logarithmic along just the trope part of the wobbly divisor. From Corollary 5.5, the trope part of the wobbly divisor has normal crossings up to codimension 2. Since the spectral variety Y is irreducible, the Higgs bundle is stable. This would contradict Mochizuki's Bogomolov-Gieseker inequality [Moc06]. Therefore, the restriction of α to N^v is nonzero.

In fact, it is nonzero at all the points where it is well-defined over the smooth locus of Kum minus the trope conics, because the residue of the Higgs field has to have constant Jordan type along the smooth points of the wobbly divisor. □

Lemma 8.10. *Consider the subspace* $\ker(\alpha) \cap T(Y_{\mathsf{Kum}})$ *varying as a function of the point in* Y_{Kum}*. Over a general point* $x \in \mathsf{Kum}$*, then the projections of these subspaces to* $T_x(\mathsf{Kum})$ *vary as a function of the point in the Hitchin base.*

Proof. If the subspaces depend only on C and $x \in \mathsf{Kum}$ then, in particular, their values on different sheets of Y_{Kum} would be the same. We'll see that this is not the case.

Points of Y_{Kum} are represented by $(L, \tilde{u} + \tilde{v})$ where $\tilde{u}, \tilde{v} \in \widetilde{C}$ are points lying over their images denoted $u, v \in C$, and $L^{\otimes 2} = \mathcal{O}_C(u + v - 2\mathbf{p})$. The restriction of α to Y_{Kum} is given by adding the values of the tautological form on $\widetilde{C}$ evaluated on $\tilde{u}$ and $\tilde{v}$. Specialize near a point where $\tilde{u}$ is not a ramification of $\widetilde{C}/C$ but $\tilde{v}$ is a ramification point. As v moves around the branch point, $\tilde{v}$ changes branches and the tautological form α changes sign when viewed as a dual element of $T_v(C)$. Then, α is the sum of a fixed part namely the tautological form at $\tilde{u}$, plus a part that changes sign namely the value at $\tilde{v}$. Thus, the directions of $\ker(\alpha)$ over these two points in different sheets of Y_{Kum} are different. □

8.3.2 Arguments for $\mathfrak{p}'$

Look first at the point $\mathfrak{p}'$. View it as a general point $\mathfrak{p} \in \mathsf{Kum} \subset X_0$. The Hecke fiber over x has two lines that meet in a point, and the image of this point in X_1 is a point z where the two lines meet. The line ℓ in our picture is one of these two. Going back, the line ℓ corresponds to a plane $P \subset X_0$, transverse to Kum at $\mathfrak{p}$ and tangent to it at q. On the other hand, the Hecke fiber over z is a line in X_0, and this line is tangent to $P \cap \mathsf{Kum}$ at $\mathfrak{p}$. Indeed, after blowing up the Hecke line and the strict transform of Kum both meet the exceptional divisor $\boldsymbol{\mathcal{N}}$ at the same point, meaning that the Hecke line was tangent to Kum before blowing up.

The double cover $Y_P \to P$ ramified along Kum has a point y over $\mathfrak{p}$, and its tangent space at y is a 2-dimensional space containing the following subspaces: the vertical space $N^v(y)$ of the ramification; the tangent space $T_y(Y_{P,\mathsf{Kum}})$ of the ramification divisor $Y_{P,\mathsf{Kum}} := Y_P \cap Y_{\mathsf{Kum}}$; and the two directions of the pullback of the Hecke line $V_{\mathfrak{p}}$ which pulls back to a pair of crossed lines in the double cover since it is tangent to Kum (Lemma 8.8). The cross-ratio of these four points is fixed, because the 4 directions are symmetric for the involution of the double cover. Furthermore, Lemma 8.9 implies that the restriction of the spectral 1-form to Y_P is nonzero. It therefore defines a 1-dimensional subspace $A \subset T_y(Y_P)$.

Proposition 8.11. *For general global parameters $C, \widetilde{C}$, as the point $\mathfrak{p}$ moves around in* Kum*, the subspace A moves with respect to the framing of $T_y(Y_P)$ given by the four previously discussed directions.*

Proof. Proceed by contradiction: suppose that the subspace A has a fixed direction with respect to the framing. We first show that it must then be the direction $T_y(Y_{P,\mathsf{Kum}}$. By Lemma 8.9, A is not the direction $N^v(y)$. It seems likely that the two directions coming from $V_{\mathfrak{p}}$ should interchange under a global symmetry or monodromy operation, and if A were in a fixed direction it would have to be invariant under that operation, which would imply that it is the direction $T_y(Y_{P,\mathsf{Kum}}$. However, we have not been able to specify such an interchange operation. So, instead, let us calculate at a special point.

Recall from Proposition 5.12 that the points of Y_{Kum} are represented by $(L, \tilde{u} + \tilde{v})$ where $\tilde{u}, \tilde{v} \in \widetilde{C}$ are points lying over their images denoted $u, v \in C$, and $L^{\otimes 2} = \mathcal{O}_C(u + v - 2\mathbf{p})$. Specialize near a point on Kum where both u and v are (different) branch points of $\widetilde{C}/C$. This corresponds to a place where two other pieces of the movable ramification locus of Y/X

meet the ramification over Kum. Both $\tilde{u}$ and $\tilde{v}$ then branch. We can write this in local coordinates y_1, y_2, y_3 for Y over coordinates x_1, x_2, x_3 for X, such that $x_1 = 0$ is the equation of Kum, $y_1 = 0$ is the equation of Y_{Kum}, and the map is given by $x_i = y_i^2$. The spectral 1-form along Y_{Kum} is obtained from the tautological 1-form on $\widetilde{C}$ by adding the values at $\tilde{u}$ and $\tilde{v}$. The tautological 1-form on $\widetilde{C}$ looks locally like z^2dz if z is the coordinate on $\widetilde{C}$. We can therefore write the leading term as

$$\alpha = a(y)d(y_1) + y_2^2 d(y_2) + y_3^2 d(y_3).$$

Since α comes from a linear form on the abelian variety, it does not vanish at any point, thus $a(0)$ has to be nonzero.

Consider a point $x = (x_1, x_2, x_3)$ near to $\mathfrak{p} = (0, 0, 0)$. The Hecke line V_x becomes, in these coordinates, a curve tangent to $(x_1 = 0)$. Let us write it (again looking at the highest order term) as

$$(x_1, x_2, x_3) = (b_1 t^2, \epsilon_2 + b_2 t, \epsilon_3 + b_3 t)$$

for some coefficients $b_1 \neq 0$ and $(b_2, b_3) \neq (0, 0)$, and small (ϵ_2, ϵ_3). Note that as $(\epsilon_2, \epsilon_3) \to (0, 0)$, the second order term $b_1 t^2$ approaches a nonzero limit ($(0, 0, 0)$ being itself a point of Kum that is general with respect to C although not necessarily with respect to $\widetilde{C}$).

Specifying a lifting of $(0, \epsilon_2, \epsilon_3)$ in Y i.e. extracting $\epsilon_2^{1/2}$ and $\epsilon_3^{1/2}$, the curve lifts into Y in two branches corresponding to $\pm b_1^{1/2}$. One of these branches is

$$(y_1, y_2, y_3) = \left(b_1^{1/2} t, \epsilon_2^{1/2}(1 + b_2 t/2\epsilon_2 + \ldots), \epsilon_3^{1/2}(1 + b_3 t/2\epsilon_3 + \ldots)\right).$$

Its tangent vector is $(b_1^{1/2}, b_2 \epsilon_2^{-1/2}/2, b_3 \epsilon_3^{-1/2}/2)$.

The vertical tangent vector is $(1, 0, 0)$. We use the middle direction to normalize the horizontal tangent vector with respect to the vertical one, in other words the middle vector should be the sum of the same multiple of the vertical and normalized horizontal vectors. The multiple is $b_1^{1/2}$, so the normalized horizontal vector is

$$\left(0, b_1^{-1/2} b_2 \epsilon_2^{-1/2}/2, b_1^{-1/2} b_3 \epsilon_3^{-1/2}/2\right).$$

The values of α on the vertical tangent vector is $a(0)$. On the normalized horizontal vector, evaluating at the point $(y_1, y_2, y_3) = (0, \epsilon_2^{1/2}, \epsilon_2^{1/2})$, it is

$$\alpha(0, \epsilon_2^{1/2}, \epsilon_2^{1/2}) \cdot \left(0, b_1^{-1/2} b_2 \epsilon_2^{-1/2}/2, b_1^{-1/2} b_3 \epsilon_3^{-1/2}/2\right) = b_1^{-1/2} b_2 \epsilon_2^{1/2}/2 + b_1^{-1/2} b_3 \epsilon_3^{1/2}/2.$$

This approaches 0 as $(\epsilon_2, \epsilon_3) \to (0, 0)$. This tells us that the kernel line A of α on $T(Y_P)$ approaches the horizontal direction as we approach the special point.

This shows that if A is some fixed direction with respect to the framing, it must be the horizontal direction. Let's now take up that possibility. In that case, it means that the line $T_y(Y_{P,\mathsf{Kum}})$ is always in the kernel of α. In particular, the restriction of α to Y_{Kum} has this fixed foliation as a kernel. But Lemma 8.10 says that this does not happen, completing the contradiction. $\square$

The local piece of Σ corresponds to a locally defined rank 2 Higgs bundle on X_0. It comes originally from a rank 2 piece of $\mathscr{F}_0$ whose spectral variety is a double cover of X_0 simply ramified at the point of Kum. We'll use the Higgs bundle to calculate the lower critical locus, knowing that it is the image of the upper critical locus so if we separate the two branches of the lower critical locus that will separate them upstairs too.

Use a coordinate system (x, t) on $P \subset X_0$ with $P \cap \mathsf{Kum}$ given by $t = 0$. The double cover looks locally like $t = w^2$. We can write the form α as

$$\alpha = (a^+(x,t) + wa^-(x,t))dx + (b^+(x,t) + wb^-(x,t))dw$$

$$= (a^+(x,t) + wa^-(x,t))dx + (wb^+(x,t) + tb^-(x,t))(dt/t).$$

Locally we assume that the line bundle is trivial $\mathscr{L}_0 = \mathcal{O}_P$. Then the direct image down to P, which is the rank 2 piece of our Higgs bundle $\mathscr{F}_0|_P$, has basis $1, w$. In these terms, the form α leads to a Higgs field in matrix form, with coefficients being functions of (x, t):

$$\varphi = \begin{pmatrix} a^+dx + b^-dt & ta^-dx + b^+dt \\ a^-dx + b^+(dt/t) & a^+dx + tb^-dt \end{pmatrix}.$$

We next blow up the point $(0, 0)$ with coordinates (x, v) with $t = xv$. This gives $dt = vdx + xdv$. The Higgs field becomes

$$\varphi = \begin{pmatrix} (a^+ + vb^-)dx + xb^-dv & (xva^- + vb^+)dx + xb^+dv \\ a^-dx + b^+(dx/x + dv/v) & (a^+ + vb^-)dx + x^2vb^-dv \end{pmatrix}.$$

This is the formula for the Higgs field on $\overline{\mathcal{H}}(a)_\ell$. We may assume that the map $\overline{\mathcal{H}}(a)_\ell \to \ell$ is given by the function $u - v$, so for the relative differentials it induces the relation $dx = dv$. The relative Higgs field becomes

$$\varphi_{\overline{\mathcal{H}}(a)_\ell/\ell} = \begin{pmatrix} (a^+ + vb^- + xb^-)dv & (xva^- + vb^+ + xb^+)dv \\ (a^- + b^+/x + b^+/v)dv & (a^+ + vb^- + x^2vb^-)dv \end{pmatrix}.$$

The support of the cokernel of this matrix is given by its determinant

$$\det(\varphi_{\overline{\mathcal{H}(\overline{a})}_\ell/\ell}) = (a^+ + vb^- + xb^-)(a^+ + vb^- + x^2vb^-) - (xva^- + (x+v)b^+)(a^- + ((x+v)/xv)b^+)$$
$$= (xv)^{-1}\left[xv(a^+)^2 - (x+v)^2(b^+)^2 + \dots\right]$$

where the next terms have degree ≥ 3 in x, v. The vanishing locus, away from the axes, is therefore given to first order by the equation

$$(b^+)^2x^2 + (2(b^+)^2 - (a^+)^2)xv + (b^+)^2v^2 = 0.$$

Dividing by $(b^+)^2$ and setting $c := (a^+/b^+)^2$ this becomes

$$x^2 + (2-c)xv + v^2.$$

Its discriminant is $(2-c)^2 - 4 = c^2 - 4c$ so if $c \neq 0, 4$, in other words the quotient is not 0, -2 or 2, then there are two branches.

Corollary 8.12. *If $\mathfrak{p}$ was a general point on* Kum *then the quotient $a^+(0,0)/b^+(0,0)$ is general.*

Proof. This restates the result of Proposition 8.11. $\square$

Therefore, at a general point on Kum, the upper critical locus has two branches meeting the exceptional divisor at two distinct points. This completes the proof of Theorem 8.7 for the point $\mathfrak{p}'$.

8.3.3 Arguments for $\mathbf{q}'$

At the point $\mathbf{q} = \mathbf{q}'$ the plane P is tangent to Kum. Introduce coordinates x, y for the plane and z in the transverse direction so that Kum is given by $z = 0$ and the plane is given by $z = xy$. The covering Y has coordinates x, y, w with $w^2 = z$, and Y_{Kum} has equation $w = 0$.

We may scale the coordinates such that the map from the plane to the line ℓ is given by $t = x - y$. Notice that the fibers of the map are not tangent to the two principal directions of $P \cap \mathsf{Kum}$, as we have seen above that the two branches of $\boldsymbol{\mathcal{T}}$ at the point $\mathbf{q}'$ are etale over ℓ.

Write the spectral differential form on Y as

$$\alpha = a(x, y, w)dx + b(x, y, w)dy + c(x, y, w)dw.$$

We have $c(0, 0, 0) \neq 0$ by Lemma 8.9. Also, α does not vanish identically on Y_{Kum}, by Lemma 8.10, and the kernel of α is not generically given by any construction depending only on C (such as the lines $x = 0$, $y = 0$ or $x = y$ in our coordinate system). Thus, at our general point we can assume that $a(0), b(0), a(0) + b(0)$ are nonzero.

Proceed now with the calculation of the directions of the upper critical locus. The covering Σ has equation $w^2 = xy$ mapping to the plane P with coordinates x, y. Blow up the origin in the x, y plane, so we introduce a coordinate v with $y = xv$. The exceptional divisor is $x = 0$ and v is the coordinate along it. Now the form is $\alpha = adx+b(xdv+vdx)+cdw$ and Σ has equation $w^2 = x^2v$. This normalizes to two branches $w = xv^{1/2}$ and $w = -xv^{1/2}$. Let $u = v^{1/2}$ be the coordinate at a general point of the covering of the exceptional divisor, so we have $w = xu$ (there are two branches as $u, -u$ go to the same point v). We have $v = u^2$ so $dv = 2udu$. We have $y = xu^2$.

This normalization is smooth over $u \neq 0$ (actually everywhere) and the form becomes

$$\alpha = adx + b(2xudu + u^2dx) + c(xdu + udx).$$

The mapping function from the plane to the line is $t = x - y$ so $dt = dx - dy$. Relative differentials are obtained by working modulo dt, identifying

$$dx \sim dy = 2xudu + u^2dx.$$

Modulo dt we get

$$dx = \frac{2xu}{1 - u^2}du.$$

Thus

$$\frac{\alpha^{\mathrm{rel}}}{du} = a\frac{2xu}{1 - u^2} + b\left(2xu + \frac{2xu^3}{1 - u^2}\right) + c\left(x + \frac{2xu^2}{1 - u^2}\right).$$

The zeros of α^{rel} are given by the equation

$$a \cdot 2xu + b \cdot (2xu - 2xu^3 + 2xu^3) + c \cdot (x - xu^2 + 2xu^2)$$

$$= (a + b)(2xu) + c(x + xu^2).$$

One can factor x out of this expression.

We notice that $a(0)+b(0)$ is the value of α on the tangent direction of the Hecke line $t=0$, and $c(0)$ is the value of α in the vertical direction. In our current coordinate system, the unit vectors of these directions are normalized to be related by the lines described in the previous subsection: the Hecke line is $y=x, z=x^2$ and its two lifts are $y=x, w=\pm x$.

Therefore, curiously enough, we are now in the same general position setting as in the previous subsection. Proposition 8.11 implies that the ratio $(a(0)+b(0))/c(0)$ moves as a function of the point, so we may assume that it is general. The factored expression for the zeros of the relative 1-form $\alpha^{\mathrm{rel}}/xdu$ along the exceptional curve becomes

$$1+\frac{a(0)+b(0)}{c(0)}u+u^2=0.$$

There are two distinct points u in the exceptional divisor corresponding to limits of zeros of α^{rel}. This gives two branches of the upper critical locus with distinct tangent vectors in the nodal point of our local piece of Σ.

This completes the proof of the $\mathfrak{q}'$ part of Theorem 8.7.

8.4 Spectral line bundle for the higher direct image

Recall the ***spectral line bundle***, on Y_0 or Y_1, is the line bundle $\mathscr{L}_0$ or $\mathscr{L}_1$ whose direct image down to X_0 or X_1 respectively is the level 0 piece of the parabolic Higgs bundles $(\mathscr{F}_{0,\bullet},\Phi_0)$ or $(\mathscr{F}_{1,\bullet},\Phi_1)$ respectively. For X_0 the parabolic Higgs bundle has trivial parabolic structure (away from the codimension 2 tacnode points). On X_1 the parabolic structure has levels 0 and $-1/2$, so this definition of spectral line bundle involves a choice.

We would like to calculate the spectral line bundle $\mathcal{U}$ on $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a},\tau\widehat{a})_\ell$ corresponding to the higher direct image of the pullback Higgs bundle $\mathscr{F}_{0,\overline{\mathcal{H}}(a_\ell}$ down to ℓ, in terms of the spectral line bundle $\mathscr{L}_0$ over Y_0 and its restriction $\mathscr{L}_\ell$ to $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a},\tau\widehat{a})_\ell$.

For this, we use the statement of Theorem 3.17, in particular the part proven in Theorem 12.12. The previous discussion shows that the points $\mathfrak{p}'$ and $\mathfrak{q}'$ are of type 3.11.1(e), and the remaining points are covered by the other parts of the classification in Subsection 3.11. To use the theorem in the presence of the points $\mathfrak{p}'$, $\mathfrak{q}'$, we should take the answer

$$\mathcal{U}=\mathscr{L}_\ell\otimes\omega_{\overline{\mathcal{H}}(\overline{a})_\ell/\ell}.$$

The normalization of the relative critical locus separates the two branches at each of the nodes. By Theorem 7.12 and Corollary 7.13, the two components of the abelianized Hecke are disjoint away from $\mathfrak{p}'$, $\mathfrak{q}'$, so the normalization that was denoted by G in Subsection 3.11 is the same as the disjoint union

$$G = \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell = \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \sqcup \widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell.$$

The prescription of Theorem 3.17 gives a parabolic sheaf $F'_\cdot$ such that

$$F'_0 = f_*(\mathcal{U}|_G)$$

and $F'_{-1/2}$ is the standard subsheaf coming from the ramification points of $f : G \to \ell$ over $\ell \cap \mathrm{Wob}_1$. Theorem 3.17 says that $F'_\cdot$ is a parabolic subsheaf of the parabolic Higgs bundle $F_\cdot$ we are looking for. On the other hand, we know from general principles that $F_\cdot$ has parabolic degree 0. Also, F and F' agree except possibly over the points $f(\mathfrak{p}'), f(\mathfrak{q}') \in \ell$ and these are different from the points where there is a parabolic structure.

So, if we can show that the parabolic degree of $F'_\cdot$ is zero, this will imply that $F_\cdot = F'_\cdot$ and we get the computation of $F_\cdot$. The degree will be calculated in the next subsection.

8.5 Degree calculation

Recall our notation that inside Y_0 that $\boldsymbol{E}_0$ denotes the exceptional divisor of the blow-up $\boldsymbol{\varepsilon}_0 : Y_0 \to \mathcal{P}_2$, and $\boldsymbol{F}_0$ denotes the inverse image of the hyperplane class of X_0.

Lemma 8.13. *For the relative canonical class over the degree 1 moduli space, we have*

$$\omega_{\overline{\mathcal{H}(a)}/X_1} = \mathcal{O}_{X_0}(-2)|_{\overline{\mathcal{H}(a)}} \otimes \mathcal{O}_{X_1}(1)|_{\overline{\mathcal{H}(a)}}.$$

If Ξ denotes the theta divisor on the Prym, or its pullbacks to Y_0, Y_1 or $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$, we have $\boldsymbol{F}_0 = \Xi - \boldsymbol{E}_0$ and $\boldsymbol{F}_1 = 2\Xi - \boldsymbol{E}_1$. We note that $\Xi^3 = 24$, concording with the calculations in Propositions 4.8 and 5.8.

Proof. The canonical bundle $\omega_{\overline{\mathcal{H}(a)}}$ restricts to $\mathcal{O}_{\mathbb{P}^1}(-2)$ on the fibers of the projection to X_1 and to $\mathcal{O}_{\mathbb{P}^1}(2)$ on the fibers of the projection to X_0. It means that it restricts to the pullback of $\mathcal{O}_{X_0}(2)$ on the Hecke lines over points of X_1, and to the pullback of $\mathcal{O}_{X_1}(1)$ on the Hecke conics over points of X_0. The Picard group of $\overline{\mathcal{H}}(a)$ is generated by these two things, and the pullbacks restrict to trivial bundles on their own Hecke fibers, so we conclude

$$\omega_{\overline{\mathcal{H}(a)}} = \mathcal{O}_{X_0}(-2)|_{\overline{\mathcal{H}(a)}} \otimes \mathcal{O}_{X_1}(-1)|_{\overline{\mathcal{H}(a)}}.$$

On the other hand, as X_1 is an intersection of two quadrics in $\mathbb{P}^5$, its canonical class is $\mathcal{O}_{X_1}(-6+2+2) = \mathcal{O}_{X_1}(-2)$. We conclude the stated formula.

For the last parts, we note that the linear system $|\Xi|$ that produces the rational map $\mathcal{P}_2 \dashrightarrow \mathbb{P}^3$ has base points on the 16 points that we blow-up to get Y_0 so on the resulting map on the blow-up, we subtract $\boldsymbol{E}_0$. The 6-dimensional subsystem (anti-invariant part) of $|2\Xi|$ that provides the rational map $\mathcal{P}_3 \dashrightarrow \mathbb{P}^5$ has $\widehat{C}$ as base locus so we subtract $\boldsymbol{E}_1$. For the verification, the formulas of Propositions 5.8 and 4.8 give

$$(\boldsymbol{F}_0 + \boldsymbol{E}_0)^3 = 24 \quad \text{and} \quad (\boldsymbol{F}_1 + \boldsymbol{E}_1)^3 = 192$$

compatible with $\Xi^3 = 24$. $\square$

Recall that $\boldsymbol{Exc}_0$ and $\boldsymbol{Exc}_1$ are the exceptional divisors in $\widehat{\mathcal{H}}^{\mathrm{ab}}$. We could denote their restrictions over the point $\widehat{a} = (A, \tilde{t}) \in \widehat{C}$ by $\boldsymbol{Exc}_0(\widehat{a})$ and $\boldsymbol{Exc}_1(\widehat{a})$. Those are thus the divisors in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ that are strict transforms of the divisors $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$.

We get that

$$\boldsymbol{E}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = \boldsymbol{Exc}_0(\widehat{a})$$

whereas

$$\boldsymbol{E}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = \boldsymbol{Exc}_0(\widehat{a}) + \boldsymbol{Exc}_1(\widehat{a})$$

because starting from Y_1 we blow up lines contained in $\boldsymbol{E}_1$ to get $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$. This gives

$$\boldsymbol{F}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = 2\Xi - \boldsymbol{Exc}_0(\widehat{a})$$

and

$$\boldsymbol{F}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = 4\Xi - \boldsymbol{Exc}_0(\tilde{a}) - \boldsymbol{Exc}_1(\tilde{a}).$$

This gives for the relative differentials

$$\begin{aligned}\omega_{\overline{\mathcal{H}}(a)/X_1}|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} &= \mathcal{O}(-2\boldsymbol{F}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} + \boldsymbol{F}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}) \\ &= \mathcal{O}(\boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a})).\end{aligned}$$

Recall that—up to tensoring with degree 0 line bundles $(\boldsymbol{\varepsilon}_0^*\mathfrak{L}_0)$ resp. $(\boldsymbol{\varepsilon}_1^*\mathfrak{L}_1)$—the spectral line bundle on Y_0 is $\mathscr{L}_0 = \mathcal{O}_{Y_0}(2\boldsymbol{F}_0 + \boldsymbol{E}_0)$ and the spectral line bundle on Y_1 is $\mathscr{L}_1 = \mathcal{O}_{Y_1}(\boldsymbol{F}_1)$. It follows that up to numerical equivalence

$$\mathscr{L}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = \mathcal{O}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}))$$

and

$$\mathscr{L}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = \mathcal{O}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a})).$$

Proposition 8.14. *Define the line bundle $\mathcal{U}$ over $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ as the pullback of the spectral line bundle $\mathscr{L}_0$ on Y_0, tensored with $\omega_{\overline{\mathcal{H}}(a)_\ell/\ell}$. Then $\mathcal{U}$ is the spectral line bundle for the L^2 Dolbeault higher direct image Higgs bundle on ℓ. Over each of the two pieces of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$, $\mathcal{U}$ is isomorphic to the restriction of the spectral line bundle that we construct for X_1, from Y_1 to $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ (respectively $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$).*

Proof. It will be convenient to prove the second part first. Let us use the expressions for the spectral line bundles that we are constructing on Y_0 and Y_1, as calculated prior to the statement of the proposition. We get

$$\begin{aligned}\mathcal{U}|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} &= \mathscr{L}_0 \otimes \omega_{\overline{\mathcal{H}}(\widehat{a})_\ell/\ell}|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \\ &= \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}((4\Xi - \boldsymbol{Exc}_0(\widehat{a})) + (\boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a}))) \\ &= \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(4\Xi - \boldsymbol{Exc}_1(\widehat{a})).\end{aligned}$$

We would like to compare this with the spectral line bundle coming from Y_1 which is, as we have seen above,

$$\mathscr{L}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} = \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a})).$$

These two differ by $\boldsymbol{Exc}_0(\widehat{a})$. However, remember that we are looking at the restriction over a line $\ell \subset X_1$. The divisor components of $\boldsymbol{Exc}_0(\widehat{a})$ are obtained by blowing up 16 disjoint $\mathbb{P}^1$'s in Y_1. These $\mathbb{P}^1$'s are sixteen fibers of $\boldsymbol{E}_1 = \widehat{C} \times \mathbb{P}^1$, so they map to 16 lines in the wobbly locus $\mathrm{Wob}_1 \subset X_1$. A general line ℓ will miss these. Thus, as restricted over ℓ, the two expressions are the same.

The same discussion holds on $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$. Thus, $\mathscr{L}_1|_G \cong \mathcal{U}|_G$.

Let $F_\cdot$ denote the L^2 Dolbeault pushforward (on ℓ) with its parabolic structure, and let $F'_\cdot$ be the parabolic bundle on ℓ whose spectral line bundle is $\mathcal{U}$ over G. The parabolic degree of $F_\cdot$ is zero since it corresponds to a harmonic bundle.

The spectral line bundle $\mathscr{L}_1$ pushes forward to a parabolic Higgs bundle with first parabolic Chern class equal to 0, over X_1 and hence over a line $\ell \subset X_1$. The parabolic structure here is also the standard one coming from the ramification over Wob_1. As we have identified the two line bundles $\mathscr{L}_1|_G$ and $\mathcal{U}|_G$, and since the parabolic structures are standard coming from ramification points of G over $\ell \cap \mathrm{Wob}_1$, it follows that the parabolic degree of $F'_\cdot$ is zero. Thus, $F_\cdot = F'_\cdot$. We conclude that $\mathcal{U}$ is the spectral line bundle for $F_\cdot$. $\square$

8.6 Restriction to a line

We have been considering the local system corresponding to the parabolic Higgs bundle $(\mathscr{F}_{1,\bullet}, \Phi_1)$ via the non-abelian Hodge and Riemann-Hilbert correspondences. So far we focused on the restriction of this local system to a general line $\ell \subset X_1$. We show in this subsection that knowing this restriction suffices in order to identify the local system on $X_1 - \mathrm{Wob}_1$. Throughout the subsection, ℓ denotes a general line in X_1. Let $\ell^\circ := \ell - (\ell \cap \mathrm{Wob}_1)$.

Lemma 8.15. *The local system on ℓ° constructed in Section 4, is irreducible.*

Proof. It suffices to show that the spectral covering is irreducible. Recall from Lemma 4.15 that the spectral covering has the following description. The line ℓ has a natural trigonal covering $k : C \to \ell$ whose fiber over $x \in \ell$ is identified with the set of three lines through x that are different from ℓ. The 8 branch points are the points of $\ell \cap \mathrm{Wob}_1$.

We have the spectral cover $\widetilde{C}/C$. For a point $x \in \ell$ we can make the following set with 8 elements: it is the set of liftings of the subset $k^{-1}(x)$ to a subset of three elements of $\widetilde{C}$. This family determines a covering of ℓ of degree 8, and that is the same as $Y_1 \times_{X_1} \ell \to \ell$.

For ℓ general, the set of 4 branch points of $\widetilde{C}/C$ maps to a subset of 4 distinct points in ℓ. Indeed, the set of four branch points is the inverse image of a general pair of points in $\mathbb{P}^1$ under the hyperelliptic map $\mathsf{h}_C : C \to \mathbb{P}^1$, and a general trigonal map $C \to \mathbb{P}^1$ does not identify opposite points under the hyperelliptic involution on C, so the images of the four branch points are distinct. Thus, as x moves around in ℓ°, we can change individually the parity of any one of the liftings of the three points. This shows that the monodromy action on the 8 points is transitive, so $Y_1 \times_{X_1} \ell$ is irreducible. That was the spectral covering of the Higgs bundle on ℓ corresponding to the restricted local system, so the local system is irreducible. $\square$

Proposition 8.16. *Suppose V and V' are two local systems on $X_1 - \mathrm{Wob}_1$, such that*

$$V|_{\ell^\circ} \cong V'|_{\ell^\circ}$$

and such that this local system on ℓ° is irreducible. Then $V \cong V'$.

Proof. Let $\rho : \pi_1(X_1 - \mathrm{Wob}_1, o) \to GL_N(\mathbb{C})$ and $\rho'\pi_1(X_1 - \mathrm{Wob}_1, o) \to GL_N(\mathbb{C})$ denote the monodromy representations of V and V' with respect to a basepoint $o \in \ell^\circ$ chosen in ℓ°. Let $\zeta : \pi_1(\ell^{circ}, o) \to GL_N(\mathbb{C})$ denote the monodromy representation of $V|_{\ell^\circ}$ and $V'|_{\ell^\circ}$, assuming we choose framings making these representations the same.

Suppose $a \in \pi_1(X_1 - \mathrm{Wob}_1, o)$. Then $\rho(a)$ and $\rho'(a)$ are morphisms of representations of $\pi_1(\ell^\circ, o)$ between ζ and its conjugate ζ^a. Since ζ is irreducible, these two morphisms differ by a scalar. This gives a rank 1 character $\chi : \pi_1(X_1 - \mathrm{Wob}_1, o) \to \mathbb{C}^\times$ such that $\rho \otimes \chi \cong \rho'$, and χ is trivial on $\pi_1(\ell^\circ, o)$.

To finish the proof, we need to note that χ is trivial. This is because the map

$$H_1(\ell^\circ, \mathbb{Z}) \to H_1(X_1 - \mathrm{Wob}_1, \mathbb{Z}) \tag{56}$$

is surjective. Indeed, this follows from the following

Claim 8.17. *The fist homology group $H_1(X_1 - \mathrm{Wob}_1, \mathbb{Z})$ is cyclic and generated by the linking loop in X_1 going around some smooth point of* Wob_1*.*

Proof. Note first that the statement of the claim makes sense since any two linking loops at smooth points of Wob_1 are conjugate in $\pi_1(X_1 - \mathrm{Wob}_1)$ and hence are homologous. This follows immediately since Wob_1 is irreducible and hence the smooth locus of Wob_1 is connected. Using the tubular neighborhood theorem we can view the linking loops as two different fibers of the circle bundle in the normal bundle $N_{\mathrm{Wob}_1^{\mathrm{smooth}}/X_1}$ and hence they are homotopy equivalent up to a conjugation via a path connecting the two points in $\mathrm{Wob}_1^{\mathrm{smooth}}$ over which these circle fibers sit.

Next recall that if M is a connected, oriented, not necessarily compact, C^∞-manifold and if $Z \subset M$ is a connected, oriented C^∞ submanifold of dimension d, which is closed in the Eucledian topology, then Z defines a d-dimensional cohomology class $[Z] \in H^d(M;\mathbb{Z})$. Indeed, let $Z \subset \mathfrak{T} \subset M$ be a tubular neighborhood of Z. Then for every k we have canonical identifications $H^k(M, M - Z; \mathbb{Z}) = H^k(\mathfrak{T}, \partial\mathfrak{T}; \mathbb{Z})$ (by excision) and also $H^k(\mathfrak{T}, \partial\mathfrak{T}; \mathbb{Z}) = H^{k-d}(Z;\mathbb{Z})$ (by the Thom isomorphism theorem). In particular we get that

$$H^d((M, M - Z; \mathbb{Z}) = H^d(\mathfrak{T}, \partial\mathfrak{T}; \mathbb{Z}) = H^0(Z;\mathbb{Z}) = \mathbb{Z},$$

where the last equality holds since Z is connected. But from the long exact cohomology sequence of the pair $(M, M - Z)$ we have a canonical map

$$H^d(M, M - Z; \mathbb{Z}) \to H^d(M;\mathbb{Z}) \tag{57}$$

and we define $[Z] \in H^d(M;\mathbb{Z})$ to be the image of $1 \in \mathbb{Z} = H^d(M, M-Z;\mathbb{Z})$ under the map (57). The cap product

$$[Z] \cap (-) : H_k(M;\mathbb{Z}) \to H_{k-d}(Z;\mathbb{Z}) \to H_{k-d}(M;\mathbb{Z})$$

fits in the classical ***tube exact sequence*** in homology

$$\cdots \longrightarrow H_{k-d+1}(Z;\mathbb{Z}) \xrightarrow{\mathsf{tube}_{Z/M}} H_k(M-Z;\mathbb{Z}) \longrightarrow H_k(M;\mathbb{Z}) \xrightarrow{[Z]\cap(-)} H_{k-d}(Z;\mathbb{Z}) \longrightarrow \cdots,$$

where the middle map is induced from the inclusion $M - Z \subset M$ and the tube map $\mathsf{tube}_{Z/M}$ sends a $(k-d+1)$-cycle A in Z to the k-cycle in $M-Z$ which is the total space of the S^{d-1}-bundle $\partial\mathfrak{T}_{|A} \subset \partial\mathfrak{T} \subset M - Z$.

Taking this into account, consider $Z \subset M$ to be the C^∞ manifolds underlying the smooth complex varietties $(\mathrm{Wob}_1 - \mathrm{Sing}(\mathrm{Wob}_1)) \subset (X_1 - \mathrm{Sing}(\mathrm{Wob}_1))$. Thus Z and M are oriented, Z is of real codimension 2, and since X_1 and Wob_1 are irreducible, both Z and M are connected. Furthermore $M - Z = X_1 - \mathrm{Wob}_1$ as topoloical spaces and so the piece of the tube sequence corresponding to $k = 1$ reads

$$\cdots \longrightarrow H_0(Z;\mathbb{Z}) \xrightarrow{\mathsf{tube}_{Z/M}} H_1(X_1 - \mathrm{Wob}_1;\mathbb{Z}) \longrightarrow H_1(M;\mathbb{Z}) \xrightarrow{[Z]\cap(-)} 0 \longrightarrow \cdots.$$

Since $\mathrm{Sing}(\mathrm{Wob}_1)$ is a compact subvariety of complex codimension 2 in X_1 we have that $H_1(M;\mathbb{Z}) = H_1(X_1 - \mathrm{Wob}_1;\mathbb{Z}) = H_1(X_1;\mathbb{Z}) = 0$. Therefoe

$$\mathsf{tube}_{Z/M} : H_0(Z:\mathbb{Z}) \to H_1(X_1 - \mathrm{Wob}_1;\mathbb{Z})$$

is surjective. But the $H_0(Z;\mathbb{Z}) = \mathbb{Z}$ is generated by the class of of a point in Z, and by the definition of $\mathsf{tube}_{Z/M}$ for any point $pt \in Z$, we have that $\mathsf{tube}_{Z/M}(pt)$ is the class of the circle in $X_1 - \mathrm{Wob}_1$ linking to $Z = \mathrm{Wob}_1 - \mathrm{Sing}(\mathrm{Wob}_1)$ this point. This proves the claim. □

Finally, observe that a general line ℓ will intersect Wob_1 transversally at a set of 8 smooth points, and so any simple loop in ℓ that goes once around a point $x \in \ell \cap \mathrm{Wob}_1$ will be a a loop in X_1 that links to Wob_1 at x. Thus the image of $H_1(\ell^\circ, \mathbb{Z})$ in $H_1(X_1 - \mathrm{Wob}_1, \mathbb{Z})$ contains a linking loop and so the map (56) is surjective. This implies that χ is trivial, and hence $V \cong V'$ which completes the proof of the proposition. □

Apply this now to the Hecke transform, a rank 16 local system on $X_1 - \mathrm{Wob}_1$.

Lemma 8.18. *The rank* 16 *Hecke transform local system on* $X_1 - \mathrm{Wob}_1$ *decomposes as a direct sum of two rank* 8 *local systems which are isomorphic, when restricted to a general line, to the local system constructed in Section 4.*

Proof. The arguments of this section show that the restriction of the rank 16 local system to a general line decomposes as a direct sum of two copies of the rank 8 local system constructed in Section 4. Furthermore, such a decomposition can be obtained by considering the two pieces $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})$. We obtain a family of decompositions over the lines, with the property that when two lines intersect the decompositions correspond. Now, a complete intersection of X_1 with two general hyperplanes is an elliptic curve that can degenerate into a cycle of 4 lines $Z = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$. We get the decomposition on each of these lines and the decompositions coincide on the intersection points. This gives a decomposition of the rank 16 local system over the singular curve Z, into two pieces of rank 8. The map $\pi_1(Z - Z \cap \mathrm{Wob}_1) \to \pi_1(X_1 - \mathrm{Wob}_1)$ is surjective, so we get a decomposition of local systems on $X_1 - \mathrm{Wob}_1$. □

Corollary 8.19. *The rank* 16 *Hecke transform local system on* $X_1 - \mathrm{Wob}_1$ *decomposes as a direct sum of two rank* 8 *local systems which are isomorphic to the local system constructed in Section 4.*

Proof. The lemma gives the decomposition, and the pieces restrict to a general line to the given rank 8 local system. By Lemma 8.15 and Proposition 8.16, the pieces of the decomposition are globally isomorphic to the rank 8 local system constructed in Section 4. □

9 Hecke transformation from X_1 to X_0

The standard diagram for the direction of the Hecke transformation ($X_1 \to X_0$) shows the big Hecke correspondence fitting into

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0 \times \overline{C}. \end{tikzcd}$$

Fix a point $a = (A, t) \in \overline{C}$ where $A^{\otimes 2} = \mathcal{O}_C(t - \mathbf{p})$ and consider the Hecke correspondence $\overline{\mathcal{H}}(a)$ with its diagram

$$\begin{tikzcd} & \overline{\mathcal{H}}(a) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0. \end{tikzcd}$$

The objective in this section is to pull-back the constructed Higgs bundle from X_1 to $\overline{\mathcal{H}}(a)$ and take the higher direct image along q to X_0.

The big abelianized Hecke fits into a diagram of the form

$$\begin{tikzcd} & \widehat{\mathcal{H}}^{\mathrm{ab}} \arrow[dl, "p^{\mathrm{ab}}"'] \arrow[dr, "q^{\mathrm{ab}}"] & \\ Y_1 & & Y_0 \times \widehat{C}. \end{tikzcd}$$

The two points $\widehat{a}$ and $\tau\widehat{a}$ over $a \in \overline{C}$ give two abelianized Hecke varieties $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})$. These are the same as the varieties with the same notation in the previous section, and recall that their disjoint union is denoted by

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a}) := \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \sqcup \widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a}).$$

The map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to Y_1$ is the blow-up along 16 lines contained in the wobbly locus, while the map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}) \to Y_0$ is the blow-up along a curve isomorphic to $\widehat{C}$ (and that will be called by the same name) inside Y_0.

9.1 Restriction to a line

As before, fix a general line $\ell \subset X_0$. Let $\overline{\mathcal{H}}(a)_\ell$ be the inverse image of ℓ in the Hecke variety $\overline{\mathcal{H}}(a)$. As ℓ is general, it does not meet the image in X_0 of the curve $\widehat{C} \subset Y_0$, so if $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ denotes the inverse image of ℓ in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$, the projection induces an isomorphism

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \xrightarrow{\cong} Y_{0,\ell}.$$

The same holds for the other piece $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$.

Recall from Corollary 7.10 that for ℓ general, the images of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$ are disjoint in the spectral variety

$$\Sigma := Y_1 \times_{X_1} \overline{\mathcal{H}}(a)_\ell \to \overline{\mathcal{H}}(a)_\ell.$$

Thus, we may treat each piece separately. We will look mainly at $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ with the understanding that the arguments for $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$ are the same.

Lemma 9.1. *The Hecke correspondence $\overline{\mathcal{H}}(a)_\ell$ over ℓ maps by a closed immersion onto a hyperplane section that we will denote by $H_\ell \subset X_1$. The image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ is a curve of genus* 25*. The map*

$$\phi : \widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell \to \ell$$

has degree 8*. It has:* 16 *ramification points over the intersection points of ℓ with the trope planes, and* 4 *ramification points over each of the* 4 *intersection points in $\ell \cap \mathsf{Kum}$. It has* 32 *other 'movable' ramification points not mapping to points in the wobbly* Wob_0*, so there are altogether* 64 *branch points.*

Proof. Recall that one of the quadrics in the pencil is identified with the Grassmanian of lines in $\mathbb{P}^3$, embedded in $\mathbb{P}^5$ by the Plücker coordinates. The line ℓ itself corresponds to a vector in $v(\ell) \in \bigwedge^2 \mathbb{C}^4$. The condition for another line $\mathsf{m} \subset \mathbb{P}^3$ to meet ℓ is that its vector $v(\mathsf{m}) \bigwedge^2 \mathbb{C}^4$ satisfies $v(\mathsf{m}) \wedge v(\ell) = 0$ in $\bigwedge^4 \mathbb{C}^4 \cong \mathbb{C}$. This is a linear condition on $v(\mathsf{m})$ so it corresponds to a hyperplane in $\mathbb{P}^6$. The image of $\overline{\mathcal{H}}(a)_\ell$ in X_1 corresponds to the subset of points in X_1 whose corresponding line meets ℓ, in other words it is this hyperplane intersected with X_1. This yields the hyperplane section H_ℓ.

Since ℓ is general, its inverse image in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ is the same as its inverse image in Y_0 because the abelianized Hecke is the blow-up of Y_0 on a subset that maps to a curve in X_0 that will be missed by a general ℓ. Now, we may proceed to calculate the normal bundle of this curve in Y_0. It is the pullback of the normal bundle of ℓ in $\mathbb{P}^3$ which is to say $\mathcal{O}(1) \oplus \mathcal{O}(1)$, so that has degree 2 on ℓ. Its pullback has degree 16. Now, Y_0 is the blow-up of an abelian variety at 16 points; the exceptional divisors map to the trope planes so our general ℓ meets each of the exceptional divisors once. The canonical bundle of Y_0 is twice the exceptional divisor, so it has degree 32. The canonical bundle of our curve is the canonical of Y_0 restricted to the curve, plus a divisor of degree 16, so it has degree $32 + 16 = 48$. Therefore, the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ in H_ℓ has genus 25.

The ramification of this curve over ℓ is just the restriction to ℓ of the ramification of Y_0/X_0. We know that has fixed pieces including a simple ramification over each trope plane, plus four simple ramifications over general points of Kum. The remaining ramification points are really movable, as was noted in Lemma 5.13 (see also Corollary 4.23 for the corresponding statement in the other direction). $\square$

The ramification points of the horizontal divisor lie over points of $(\bigcup_\kappa \mathsf{Trope}_\kappa) \cap \ell$. They come from 16 lines inside Wob_1, and correspond to the first 16 ramification points mentioned in Lemma 9.1. The locations of these lines in Wob_1 depend on the choice of point $a = (A, t)$ used to make the Hecke correspondence.

We note that the modular spectral covering Σ has two ramification points over each general point of the wobbly locus, so these 16 points correspond to 32 points in the full abelianized Hecke. They are distrubuted as 16 in each of the two pieces $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ and $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$.

Remark 9.2. The horizontal divisor in H_ℓ is $\mathrm{Wob}_1 \cap H_\ell$. It has nodes and cusps coming from the nodes and cusps of Wob_1. However, for a general ℓ these do not meet the image of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$. In particular, they are not going to contribute singularities to the higher direct image – this is another version of the "apparent singularities" encountered in the previous chapter.

The proof of this statement is that the nodal and cuspidal loci of the wobbly locus Wob_1 are curves; they pull back to curves in Y_1. The centers of the blow-up $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ are the 16 lines in Wob_1 that get blown up to form $\boldsymbol{Exc}_{0,\kappa}$. These are transverse to both the nodal and cuspidal loci. Therefore, the pullbacks of the nodal and cuspidal loci in $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ are 1-dimensional. Their 1-dimensional images in X_0 do not meet a general line ℓ so $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ does not meet the nodal and cuspidal loci of $\overline{W}_1$ and we get the statement of the remark.

Among the hypotheses of Subsection 3.11 is the following statement.

Lemma 9.3. *For a general line ℓ, the intersection of the plane H_ℓ with any of the* 16 *lines in* Wob_1 *that provide ramification of the Hecke correspondence, is a general point on that line. At such a point, the value of the spectral* 1*-form α on the vertical direction in the tangent space of Y_1, at either of the two ramification points of Y_1/X_1 over this point, is nonzero.*

Proof. For the first part, we note that one of these 16 lines in Wob_1 gets blown up to an $\mathbb{F}_1$-surface, that then maps to the corresponding trope plane in $X_0 = \mathbb{P}^3$. The exceptional divisor blows down to a point, that we will call the origin, in the trope plane (corresponding to the intersection in Y_0 of the plane with the $\widehat{C}$ curve). The location of the intersection of H_ℓ with the line corresponds to the slope of the line from this origin to the intersection point of ℓ with the trope plane. For ℓ general, this direction is a general point of the line.

Inside $\mathcal{P}_2$, the curve $\widehat{C}$ passes through 16 points that are blown up to get Y_0. The identification $\mathcal{P}_2 \cong \mathcal{P}_3$ depends on a. Instead of blowing up the points, we blow up the curve $\widehat{C} \subset \mathcal{P}_3$ to get Y_1, and this generates 16 lines inside $\boldsymbol{E}_1 \subset Y_1$. Let $\mathfrak{v}$ denote one of these lines. The normal bundle of $\mathfrak{v}$ in Y_1 is $\mathcal{O} \oplus \mathcal{O}(-1)$, with the $\mathcal{O}$ direction being the normal bundle of $\mathfrak{v}$ in $\boldsymbol{E}_1$.

Let us look at how this maps to the normal bundle of the image line $\mathsf{v} \subset \mathrm{Wob}_1$. The normal bundle of a line in $\mathbb{P}^5$ is $\mathcal{O}(1)^{\oplus 4}$, and to get the normal bundle of the line in the intersection of two quadrics we take the kernel of a map $\mathcal{O}(1)^{\oplus 4} \to \mathcal{O}(2)^{\oplus 2}$. For a typical line, this kernel will be $\mathcal{O}^{\oplus 2}$.

However, for a line in the wobbly locus, that we recall counts twice in the set of four lines through each point of X_1, we claim that the kernel is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. This may be seen by recalling that the lines in the wobbly locus are the tangent lines to the copy of $\overline{C}$ that forms the cuspical locus of Wob_1, and when we move to first order along this $\overline{C}$ the tangent line undergoes a deformation that vanishes at that point in the normal direction. Thus, these first order deformations are sections of the normal bundle that vanish. But, the bundle $\mathcal{O}^{\oplus 2}$ does not have any nonzero sections that vanish somewhere. The only other possibility is the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ since it has to be a subbundle of $\mathcal{O}(1)^{\oplus 4}$. This proves the claim.

As we move in the curve $\overline{C}$, the line moves in Wob_1 so the normal direction has sections, and it has to be the $\mathcal{O}(1)$ subbundle. The map $\mathcal{O} \to \mathcal{O}(1)$ has a zero at the points where the line v crosses the cuspidal locus of Wob_1. However, by looking at the local picture of the covering $Y_1 \to X_1$ near such a cuspidal locus, we can see that the map from the full tangent space of Y_1 into the tangent space of X_1 has image of dimension 2, so the map from the full normal bundle of $\mathfrak{v} \subset Y_1$ to the $\mathcal{O}(1)$ piece in the normal bundle of $\mathsf{v} \subset X_1$ is surjective. We conclude that the bundle of vertical directions, which is the kernel of this map

$$\mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O}(1),$$

is $\mathcal{O}(-2)$ sitting in $\mathcal{O} \oplus \mathcal{O}(-1)$ as a saturated subbundle. In particular it is not contained in any subbundle of the form $\mathcal{O}(-1)$. The map from this bundle of vertical normal directions, into the space of normal directions at the original point of $\mathcal{P}_3$, therefore does not have image in a plane. Thus, the tautological 1-form, that is a nonzero linear form on the tangent space of $\mathcal{P}_3$, does not vanish on the vertical normal direction at a general point of $\mathfrak{v}$. This proves the second part of the lemma, in view of the generality statement of the first part. □

Recall that the fiber of $q : \overline{\mathcal{H}}(a)_\ell \to \ell$ over a general point ℓ is a smooth conic, but over points of the Kummer it degenerates to a union of two lines.

Lemma 9.4. *The branch points of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ lying over points of the Kummer, all map to points of the Kummer K3 surface $H_{\mathsf{Kum}} \subset \overline{\mathcal{H}}(\widehat{a})$, that is to say in $\overline{\mathcal{H}}(a)_\ell$ they map to points where the two lines in the fiber meet.*

Proof. The reasoning for this is as follows: the horizontal divisor in $\overline{\mathcal{H}}(a)_\ell$ is the intersection of $\overline{\mathcal{H}}(a)_\ell$ (considered as a hyperplane H_ℓ in X_1) with the wobbly Wob_1. But, the lines in the fibers over points of the Kummer, represent general lines in X_1 since ℓ intersects the Kummer in general points. A general line in X_1 will intersect Wob_1 transversally.

We note that it is necessary to have four Jordan blocks of size two in the monodromy of the Hecke-transformed local system (the rank 8 piece corresponding to our chosen branch of H^{ab} out of two, that we are hoping is our chosen flat bundle of rank 8 on X_0) at each point of the Kummer. Otherwise, the Hecke transform back in the other direction from X_0 to X_1 will not have the right rank.

So, these all have to come from simple ramification points of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ over the Kummer points. However, if those ramification points were to occur on smooth points of the lines, that wouldn't contribute anything to the monodromy in the direct image (i.e. the residue of the Higgs field) since the map is not singular at those locations. This heuristic argument suggests that $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ should have four points over each point where a fiber of ϕ breaks into two lines.

This can be shown in terms of bundles. Suppose $L \in \mathcal{P}_2$ is a line bundle on $\widetilde{C}$ whose direct image $V = \pi_* L$ is stable with trivial determinant, so it is a point in X_0, and suppose that V contains a line subbundle U of degree 0. Then we claim that V also contains U^{-1}.

To see this, we note that the group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts on $\widetilde{C}$, as may be seen for example by expressing

$$\widetilde{C} = C \times_{\mathbb{P}^1} \mathbb{P}^1$$

as a fiber product with the double cover $\mathbb{P}^1 \to \mathbb{P}^1$ ramified at two points (so the ramification of $\widetilde{C}/C$ is the preimage in C of those two points of $\mathbb{P}^1$). Explicitly the non-trivial elements of $\mathbb{Z}/2 \times \mathbb{Z}/2$ are given by the covering involution $\tau : \widetilde{C} \to \widetilde{C}$ for thr map $\pi : \widetilde{C} \to C$, the covering involution $\sigma : \widetilde{C} \to \widetilde{C}$ for the hyperelliptic map $\mathsf{h}_{\widetilde{C}} : \widetilde{C} \to \mathbb{P}^1$, and their composition $\rho = \tau \circ \sigma$.

Now, note that by construction the group element $\rho \in \mathbb{Z}/2 \times \mathbb{Z}/2$ acts trivially on the Prym of $\widetilde{C}/C$ but acts by -1 on $\mathrm{Jac}(C)$. Applying ρ to the map $\pi^*(U) \to L$ we obtain a map $\pi^*(U^{-1}) \to L$ and hence by adjunction we get an injective map $U^{-1} \to V$. This will give by degree considerations $V = U \oplus U^{-1}$. In the Hecke correspondence this will correspond to a point at the intersection of the two $\mathbb{P}^1$ components of the fiber of q over V (we recall that the affine parts of the two lines themselves were bundles that were semistable but not polystable).

We can also note that $L \cong \pi^*(U) \otimes \mathcal{O}_{\widetilde{C}}(a+b)$ for an effective degree 2 divisor $a+b$ on $\widetilde{C}$, amd that the image divisor $\pi(a) + \pi(b)$ is fixed by the determinant condition for V. Thus, there are four choices of (a, b) lifting this divisor to $\widetilde{C}$. We get the four claimed branches of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ going through a crossing point in $\overline{\mathcal{H}}(a)_\ell$. $\square$

Proposition 9.5. *The map $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \to \Sigma$ identifies the abelianized Hecke as the upper critical locus (see Proposition 12.1)*

$$\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell \cong \widetilde{\mathrm{Crit}}\left(\overline{\mathcal{H}}(a)_\ell/\ell, \left(\mathscr{F}_{0,\overline{\mathcal{H}}(a)_\ell}, \Phi_0\right)\right).$$

These are smooth curves in Σ that decompose into a disjoint union of two pieces corresponding to $\widehat{a}$ and $\tau\widehat{a}$.

Proof. This is similar to the proof of Proposition 8.1, using Lemma 9.3 for smoothness of the upper critical locus at ramification points of the horizontal divisor.

For other points, we note that the abelianized Hecke is identified with Y_0 outside of $\boldsymbol{Exc}_1(\widehat{a})$, but $\boldsymbol{Exc}_1(\widehat{a})$ has image equal to a curve in X_0 that does not intersect a general ℓ (as will be pointed out again in Lemma 9.6 below). Thus, the inverse image of a general line is smooth in this open subset of the abelianized Hecke. This shows that $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a},\tau\widehat{a})_\ell$ is smooth away from the ramification points of the horizontal divisor.

The pointwise identification between the abelianized Hecke and the upper critical locus, whose proof is the same as for Proposition 8.1, therefore gives an identification of subschemes (again, neither of them has embedded points). We have seen that one or the other is smooth at every point, so the are both smooth. Disjointness of the two pieces comes from Corollary 7.10 . $\square$

We next turn to the calculation of the direct image. Suppose given a spectral line bundle $\mathscr{L}_1$ over Y_1 whose direct image to X_1 corresponds to our parabolic Higgs bundle. Recall that this means, more precisely, that $f_{1*}(\mathscr{L}_1)$ is the parabolic level 0 piece of the Higgs bundle, with parabolic level $1/2$ piece equal to $f_{1*}(\mathscr{L}_1(\boldsymbol{E}_1))$.

The direct image formula for the holomorphic Dolbeault complex leads to a line bundle over $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$. At points where $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ goes through the crossing points of the fibers over Kummer points, the contribution is just $\mathscr{L}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \otimes \omega_{\overline{\mathcal{H}}(a)_\ell/\ell}$, since the horizontal divisor does not intervene. Recall that $\omega_{\overline{\mathcal{H}}(a)_\ell/\ell}$ may also be viewed as the quotient of the forms with logarithmic singularities along the fiber, modulo logarithmic forms from the base.

In the notations of Subsection 3.11, the present situation is covered by the situation there, and there are no points of type 3.11.1(e). Indeed we are in the "parabolic" case where there is a parabolic structure with weights $0, 1/2$ on the source space, and such points are not allowed as the horizontal divisor does not have nodes.

The direct image calculations were summarized in Theorem 3.17 and proven in Theorem 12.10 based on Proposition 12.7. Near a point $\mathbf{q}$ where the horizontal divisor (which is $\mathrm{Wob}_1 \cap H_\ell$) has a ramification over a point of a trope plane, the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ also passes through that point, and the contribution for the direct image is

$$\mathscr{L}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \otimes \omega_{H_\ell/\ell}(\mathbf{q}).$$

This formula would also hold near a point $\mathbf{q}$ where the upper critical locus intersects the ramification divisor $\mathsf{Ram} \subset \Sigma$ over points where the horizontal divisor is étale over the base.

Indeed, the proof of Theorem 12.10 included a discussion of that possibility. The ramification divisor is the same as the pullback of $\boldsymbol{E}_1$ to Σ. However, in fact, these kinds of points do not occur:

Lemma 9.6. *For general ℓ, the curve $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tilde{a})_\ell$ viewed inside Y_1, only meets the divisor $\boldsymbol{E}_1$ at points $\mathbf{q}$ where the horizontal divisor has a ramification over a point of a trope plane intersected with ℓ.*

Proof. The pullback of $\boldsymbol{E}_1$ to $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ is the sum $\boldsymbol{Exc}_0(\widehat{a}) \cup \boldsymbol{Exc}_1(\widehat{a})$ of the strict transform $\boldsymbol{Exc}_1(\widehat{a})$ and the exceptional divisors over the 16 lines that form $\boldsymbol{Exc}_0(\widehat{a})$. However, $\boldsymbol{Exc}_1(\widehat{a})$ is contracted and maps to a curve inside Y_0, hence also in X_0. A general line does not meet this curve. Thus, taking the inverse image of a general line ℓ inside $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})$ and projecting back to Y_1, gives a curve that only meets $\boldsymbol{E}_1$ at points of the 16 lines that generate ramification over a trope plane. ☐

Either using this lemma, or in any case by the remark of the preceding paragraph, the contribution to the higher direct image coming from points near $\boldsymbol{E}_1$ is the restriction to the curve G of

$$\mathscr{L}_1(\boldsymbol{E}_1)|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \otimes \omega_{H_\ell/\ell}.$$

This globalizes to other points of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$. For the points lying over the Kummer, notice that the relative dualizing sheaf is the same as the relative sheaf of logarithmic differentials that enters into the higher direct image calculations, so the spectral line bundle is obtained by taking $\mathscr{L}_1$ and tensoring with $\omega_{H_\ell/\ell}$ near these points. Since these points are not on $\boldsymbol{E}_1$ (as follows from Lemma 9.4), the contributions for those points are given by the same expression. Putting these all together, we obtain the computation of the spectral line bundle:

Proposition 9.7. *Taking the Hecke transform and restricting to the line ℓ, the spectral line bundle on $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ is the line bundle*

$$\mathscr{L}_1(\boldsymbol{E}_1)|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \otimes \omega_{H_\ell/\ell}.$$

The same holds for the spectral line bundle on $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$, and these two are disjoint in Σ, so this expression gives the spectral line bundle over the disjoint union $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a},\tau\widehat{a})_\ell$.

9.2 Calculation of the pushforward

Calculate in the same way as at the end of the previous chapter. Recall that if we write Ξ for the theta divisor on the Prym, or its pullbacks to Y_0, Y_1 or $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a},\tau\widehat{a})_\ell$, we have $\boldsymbol{F}_0 = 2\Xi - \boldsymbol{E}_0$ and $\boldsymbol{F}_1 = 4\Xi - \boldsymbol{E}_1$. Also

$$\boldsymbol{E}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = \boldsymbol{Exc}_0(\widehat{a})$$

whereas

$$\boldsymbol{E}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = \boldsymbol{Exc}_0(\widehat{a}) + \boldsymbol{Exc}_1(\widehat{a}),$$

and

$$\boldsymbol{F}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = 2\Xi - \boldsymbol{Exc}_0(\widehat{a})$$

$$\boldsymbol{F}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = 4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a}).$$

Lemma 9.8. *For the relative canonical class over the degree* 0 *moduli space, we have*

$$\omega_{\overline{\mathcal{H}}(a)/X_0} = \mathcal{O}_{X_0}(2)|_{\overline{\mathcal{H}}(a)} \otimes \mathcal{O}_{X_1}(-1)|_{\overline{\mathcal{H}}(a)}$$

and this pulls back to $\mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})}(\boldsymbol{Exc}_1(\widehat{a}) - \boldsymbol{Exc}_0(\widehat{a}))$.

Proof. As in Lemma 8.13,

$$\omega_{\overline{\mathcal{H}}(a)} = \mathcal{O}_{X_0}(-2)|_{\overline{\mathcal{H}}(a)} \otimes \mathcal{O}_{X_1}(-1)|_{\overline{\mathcal{H}}(a)}.$$

On the other hand, $\omega_{X_0} = \mathcal{O}_{X_0}(-4)$, giving the first formula. Then

$$\omega_{\overline{\mathcal{H}}(a)/X_0}|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = \mathcal{O}(2\boldsymbol{F}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} - \boldsymbol{F}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})})$$

$$= \mathcal{O}(2(2\Theta - \boldsymbol{Exc}_0(\widehat{a})) - (4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a}))) = \mathcal{O}(\boldsymbol{Exc}_1(\widehat{a}) - \boldsymbol{Exc}_0(\widehat{a})).$$

□

Recall that the spectral line bundle on Y_0 is $\mathscr{L}_0 = \mathcal{O}_{Y_0}(2\boldsymbol{F}_0 + \boldsymbol{E}_0)$ and the spectral line bundle on Y_1 is $\mathscr{L}_1 = \mathcal{O}_{Y_1}(\boldsymbol{F}_1)$. As before, we get

$$\mathscr{L}_0|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = \mathcal{O}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}))$$

and

$$\mathscr{L}_1|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})} = \mathcal{O}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a})).$$

From Proposition 9.7, the spectral line bundle of the Hecke transform restricted over ℓ is the bundle

$$\begin{aligned}
&\mathscr{L}_1 \otimes \mathcal{O}_{Y_1}(\boldsymbol{E}_1)|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \otimes \omega_{H_\ell/\ell} \\
&\quad = \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}) - \boldsymbol{Exc}_1(\widehat{a})) \otimes \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(\boldsymbol{Exc}_0(\widehat{a}) + \boldsymbol{Exc}_1(\widehat{a})) \\
&\qquad\qquad \otimes \mathcal{O}_{H^{\mathrm{ab}}}(\boldsymbol{Exc}_1(\widehat{a}) - \boldsymbol{Exc}_0(\widehat{a}))|_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell} \\
&\quad = \mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(4\Xi + \boldsymbol{Exc}_1(\widehat{a}) - \boldsymbol{Exc}_0(\widehat{a})).
\end{aligned}$$

This compares with the spectral line bundle coming from Y_0 which is $\mathcal{O}_{\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell}(4\Xi - \boldsymbol{Exc}_0(\widehat{a}))$.

As before, the divisor $\boldsymbol{E}_1$ is the exceptional divisor of blowing-up Y_1 along a copy of the curve $\widehat{C}$. The image of the curve in $\mathbb{P}^3$ does not intersect a general line ℓ so $\boldsymbol{Exc}_1(\widehat{a})$ does not intersect $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$. Thus, the spectral line bundle of the Hecke transform coincides with the spectral line bundle coming from Y_0 on $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$.

Proposition 9.9. *Define the line bundle $\mathcal{U}$ over $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$ as the pullback of $\mathscr{L}_1(\boldsymbol{E}_1)$ on Y_1, tensored with $\omega_{H_\ell/\ell}$. Then $\mathcal{U}$ is the spectral line bundle for the L^2 Dolbeault higher direct image Higgs bundle on ℓ. Over each of the two pieces of $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a}, \tau\widehat{a})_\ell$, $\mathcal{U}$ is isomorphic to the restriction of the spectral line bundle that we construct for X_0, from Y_0 to $\widehat{\mathcal{H}}^{\mathrm{ab}}(\widehat{a})_\ell$ (respectively $\widehat{\mathcal{H}}^{\mathrm{ab}}(\tau\widehat{a})_\ell$).*

Proof. The above calculations show this, it is also described in Section 7. Notice that the proof in this direction is significantly less complicated than in the previous section since we do not need to deal with the points of type 3.11.1(e). □

10 The big Hecke correspondences

In this section we will consider the "big" Hecke correspondences fitting into a diagram of the form, for the $(X_1 \to X_0)$ direction:

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_0 \times \overline{C} \end{tikzcd}$$

or similarly in the opposite direction $(X_0 \to X_1)$ pictured below. We would like to show that the local system obtained by pulling back our local system from (X_1, Wob_1) then taking $R^1 q_*$, is an exterior tensor product of our local system on (X_0, Wob_0) by the initially given rank two local system on C, pulled back to $\overline{C}$.

The main part of the proof on the spectral data was given in Subsection 7.4. The objective of this section is to prove some complementary statements designed to deal with possible apparent singularities of the higher direct image operation. For this, we introduce the notion of ***effective discriminant divisor***, this is the part of the discriminant divisor on which the higher direct image local system really does have singularities.

10.1 From X_1 to X_0

For the moment we will work in the direction from X_1 to X_0 as pictured in the previous diagram. Let $\Delta \subset X_0 \times \overline{C}$ be the discriminant divisor of the map q with respect to the pair $(\overline{\mathcal{H}}, p^{-1}\mathrm{Wob}_1)$. This includes points in $X_0 \times \overline{C}$ over which the map q is not smooth, and points over which the horizontal divisor $p^{-1}\mathrm{Wob}_1$ is not étale. The singularities of the $R^1 q_*$ local system are ***a priori*** contained in Δ. Let Δ_{eff} be the ***effective*** singular divisor, namely the divisor over which the local system has singularities. Thus $\Delta_{\mathrm{eff}} \subset \Delta$.

From the previous section we know the following statement:

Proposition 10.1. *If a is a general point of $\overline{C}$ then along the fiber $X_0 \times \{a\}$, the singular divisor Δ_{eff} consists of just $\mathrm{Wob}_0 \subset X_0$.*

This proposition readily implies the following:

Corollary 10.2. *The effective singular divisor consists of* $\mathrm{Wob}_0 \times \overline{C}$*, possibly union with a finite number of fibers of the form* $X_0 \times \{a_i\}$ *for points* $a_i \in \overline{C}$*.*

Proof. In general a divisor such as Δ_{eff} in a product $X_0 \times \overline{C}$ decomposes as

$$\Delta_{\text{eff}} = \Delta_{\text{eff}}^{\text{vert}} + \Delta_{\text{eff}}^{\text{horiz}} + \Delta_{\text{eff}}^{\text{mov}}$$

where $\Delta_{\text{eff}}^{\text{vert}}$ is a sum of vertical components $X_0 \times \{a_i\}$, $\Delta_{\text{eff}}^{\text{horiz}}$ is a sum of divisors of the form $D_i \times \overline{C}$, and $\Delta_{\text{eff}}^{\text{mov}}$ is given by a moving family of divisors parametrized by $\overline{C}$. Over a general point of $\overline{C}$, the intersection of these divisors with the fiber will be: for $\Delta_{\text{eff}}^{\text{vert}}$, empty; for $\Delta_{\text{eff}}^{\text{horiz}}$, the union of the D_i; and for $\Delta_{\text{eff}}^{\text{mov}}$, a divisor that moves as a function of the point. The proposition says that these all consist of just $\mathrm{Wob}_0 \subset X_0$, not moving as a function of the point of $\overline{C}$. It follows that $\Delta_{\text{eff}}^{\text{mov}} = \varnothing$ and $\Delta_{\text{eff}}^{\text{horiz}} = \mathrm{Wob}_0 \times \overline{C}$. There remains the possibility of a nonempty $\Delta_{\text{eff}}^{\text{vert}}$. □

In order to rule out the possibility of having a vertical piece in Δ_{eff} we'll just rule that out for the full discriminant divisor.

Proposition 10.3. *The discriminant* Δ *of the map* q *from* $(\overline{\mathcal{H}}, p^{-1}\mathrm{Wob}_1)$ *to* $X_0 \times \overline{C}$ *does not contain any vertical pieces of the form* $X_0 \times \{a\}$*.*

Proof. We need to show that for any point $a \in \overline{C}$, the full fiber $X_0 \times \{a\}$ is not contained in the discriminant.

The point $a \in \overline{C}$ corresponds to a pair $a = (A, t)$ where A is a line bundle of degree 0 and $t \in C$ such that $A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(t)$ (recall that $\mathbf{p}$ is our fixed Weierstrass point). We need to show that for a general point $\mathcal{F} \in X_0$, the Hecke curve corresponding to $(\mathcal{F}, (A, t))$ is smooth and intersects Wob_1 transversally.

A general $\mathcal{F}$ is stable. A bundle obtained by a Hecke transformation is the kernel in

$$0 \to \mathcal{E} \to \mathcal{F} \otimes A(\mathbf{p}) \to \mathbb{C}_t \to 0$$

where the quotient corresponds to a rank 1 quotient of the fiber $(\mathcal{F} \otimes A(\mathbf{p}))_t$ over the point t. If $\mathcal{F}$ is stable of degree 0 then its maximal degree line subbundles have degree -1, so any line subbundle of $\mathcal{E}$ must have degree ≤ 0. Thus, $\mathcal{E}$ is stable. The Hecke curve is therefore isomorphic to the space of such rank 1 quotients, so it is $\mathbb{P}^1$ and is hence smooth.

A little more precisely, let $\overline{\mathcal{H}}^s \subset \overline{\mathcal{H}}$ be the moduli space of line bundles $B \in \overline{C}$ paired with inclusions $\mathcal{E} \subset \mathcal{F} \otimes A(\mathbf{p})$ of colength 1 such that $\mathcal{E}$ and $\mathcal{F}$ are stable. Let $X_0^s \subset X_0$ denote the open subset of stable bundles. By the previous paragraph, $\overline{\mathcal{H}}^s$ is the inverse image in $\overline{\mathcal{H}}$ of $X_0^s \times \overline{C}$. The projection from here to $X_0 \times \overline{C}$ is a $\mathbb{P}^1$-bundle over the open subset $X_0^s \times \overline{C}$. This was also verified synthetically in section 2.4.

Thus, $\overline{\mathcal{H}}$ is smooth over $X_0^s \times \overline{C}$. This shows that for any $a = (A, t)$ and for a general $\mathcal{F}$ the Hecke curve over $(\mathcal{F}, a)$ is smooth. Next, we would like to understand its intersection with $p^{-1}\mathrm{Wob}_1$. Recall that $\mathcal{E} \in \mathrm{Wob}_1$ if and only if there exists a line subbundle $B \subset \mathcal{E}$ of degree 0 such that $B^{\otimes 2}(\mathbf{p})$ is effective. If $\mathcal{E}$ is a Hecke transformation of $(\mathcal{F}, (A, t))$ and B is such a line subbundle then we get an injection $B \hookrightarrow \mathcal{F} \otimes A(\mathbf{p})$ or equivalently

$$L := B \otimes A^\vee(-\mathbf{p}) \hookrightarrow \mathcal{F}.$$

Here L is a line bundle of degree -1. If $\mathcal{F}$ is stable of degree 0, that is the maximal degree of a locally free subsheaf, in particular $L \subset \mathcal{F}$ is a saturated locally free subsheaf, i.e. a strict subbundle.

In the other direction, given a degree -1 subbundle $L \subset \mathcal{F}$ there is a unique rank 1 quotient over the point t such that $B = L \otimes A(\mathbf{p})$ maps into the kernel of $\mathcal{F} \otimes A(\mathbf{p}) \to \mathbb{C}_t$. Note that

$$B^{\otimes 2}(\mathbf{p}) = L^{\otimes 2} \otimes A^{\otimes 2}(3\mathbf{p}) = L^{\otimes 2}(2\mathbf{p} + t).$$

This makes an isomorphism between the set of intersection points of the Hecke curve with Wob_1, and the set of solutions L of the pair of conditions

(1) $h^0(L^\vee \otimes \mathcal{F}) > 0$

(2) $h^0(L^{\otimes 2}(2\mathbf{p} + t)) > 0$.

We will look at the solutions as a subset of $\mathrm{Jac}^{-1}(C)$ the Jacobian of line bundles of degree -1 on C. Solutions of Condition (1) form a divisor $D_{\mathcal{F}}$ in the linear system $|2\Theta|$. Indeed, $D_{\mathcal{F}}$ is the Narasimhan-Ramanan point corresponding to $\mathcal{F}$ [NR69]. We have

$$\begin{array}{ccll} \mathrm{Jac}^{-1} & \longrightarrow & \mathbb{P}^3 & = \mathbb{P}\left(H^0(\mathrm{Jac}^{-1}(C), \mathcal{O}(2\Theta)^\vee\right) \\ \cup & & \cup & \\ D_{\mathcal{F}} & \longrightarrow & H_{\mathcal{F}} & \end{array}$$

where the $\mathbb{P}^3 = \mathbb{P}\left(H^0(\mathrm{Jac}^{-1}(C), \mathcal{O}(2\Theta)^\vee\right)$ here is the dual of $X_0 = \mathbb{P}\left(H^0(\mathrm{Jac}^{-1}(C), \mathcal{O}(2\Theta)\right)$ and $H_{\mathcal{F}}$ is the hyperplane corresponding to the point $\mathcal{F} \in X_0$. The top map is the mapping given by $|2\Theta|$ that is basepoint-free [Mum07a, BL04b].

The second Condition (2) corresponds to the pullback of $C \hookrightarrow \mathrm{Jac}^1$ by the composed map

$$\mathrm{Jac}^{-1}(C) \xrightarrow{(-)^{\otimes 2}} \mathrm{Jac}^{-2}(C) \xrightarrow{(-)\otimes\mathcal{O}(2\mathbf{p}+t)} \mathrm{Jac}^1(C).$$

Let $\overline{C}^t \subset \mathrm{Jac}^{-1}$ denote this pullback curve.

Solutions of both conditions together correspond to the intersection of these two subspaces. It corresponds to mapping $\overline{C}^t \to \mathbb{P}^3$ by the composed map

$$\overline{C}^t \to \mathrm{Jac}^{-1} \to \mathbb{P}^3$$

and then pulling back a hyperplane section $H_{\mathcal{F}}$. If $\mathcal{F}$ is a general point of X_0 then this is a general hyperplane section.

The map $\overline{C}^t \to \mathbb{P}^3$ is nonconstant, so the pullback of a general hyperplane section is reduced, consisting of a collection of distinct points. The number of points is the degree of the map, which we claim is 16. To prove that, note that our arguments below will show that it can not be > 16 otherwise the scheme-theoretical intersection of the Hecke curve, a conic, with Wob_1 would be too big. The degree is the intersection number of the pushforward of 2Θ by the squaring map. Let us push forward the original Θ divisor, which is just $C \subset \mathrm{Jac}^1$. The square is the set of divisors of the form $2x$ for $x \in C$, translated back to Jac^1 as the set of divisors of the form $2x - t$. We want to know when this is effective, i.e. how many pairs (x, y) solve $2x - t = y$. This may be written as the equation $2x + y' = t + 2\mathbf{p}$, so it is the set of ramification points of the trigonal curve associated to the linear system $|\mathcal{O}(t + 2\mathbf{p})|$. We know that this is 8, that is the intersection of the pushforward of Θ with C has 8 points, so degree ≥ 8. Thus, the intersection of the pushforward of 2Θ with C has degree ≥ 16. This completes the proof that the degree of the map $\overline{C}^y \to \mathbb{P}^3$ is 16.

Thus, there are 16 distinct solutions L of the two conditions. Each point L yields a line bundle $B = L \otimes A(\mathbf{p})$, with $(B, s) \in \overline{C}$ for some $s \in C$, and for distinct $L \in \mathrm{Jac}^{-1}$ the line bundles B are distinct. They correspond therefore to distinct points in $\overline{C}$, hence to distinct points of the normalization of Wob_1 that fibers over $\overline{C}$. We next note that since our 16 points were obtained by intersecting the image $\overline{C} \to \mathbb{P}^3$ with a general hyperplane section, any pair of two points are general with respect to each other. Pairs of points can be glued together under the normalization map for Wob_1, but this generality condition implies that our 16 solutions L, leading to 16 line bundles A, can not contain pairs of points that are glued together. This shows that they give 16 distinct points in Wob_1.

Now, the scheme theoretic intersection of a Hecke curve with Wob_1 has length 16, so if there are 16 distinct points then they have to be reduced points in the scheme theoretic intersection. This implies that $p^{-1}\mathrm{Wob}_1$ is unramified, hence etale over a general points of $X_0^s \times \overline{C}$. This completes the proof of the proposition. $\square$

Corollary 10.4. *The effective singular locus Δ_{eff} of the R^1q_* local system on $X_0 \times \overline{C}$ is $\mathrm{Wob}_0 \times \overline{C}$.*

Corollary 10.5. *The R^1q_* local system of rank 16 on $X_0 \times \overline{C}$ decomposes as an exterior tensor product of the rank 8 local system we have constructed on X_0 with a rank 2 local system on $\overline{C}$ whose spectral curve is $\widehat{C} \to \overline{C}$.*

Proof. We have seen that on $X_0 \times \{a\}$ the Higgs bundle is a direct sum of two copies of the rank 8 Higgs bundle we construct over X_0. Furthermore, the direct sum consists of two copies that are preserved by the Higgs field in the $\overline{C}$ direction, indeed the Higgs field comes from the section of the sheaf of total differentials on the upper relative critical locus which is a disjoint union. The two components vary in a covering given by $\widehat{C} \to \overline{C}$. Also, the spectral 1-form is the canonical 1-form over $\widehat{C}$. This may be seen by restricting to horizontal copies of $\widehat{C}$ inside $Y_0 \times \overline{C}$ which map to translates of standard copies of $\widehat{C}$ in Y_1, on which the spectral 1-form is the canonical one for $\widehat{C}$.

It follows that the Higgs bundle is not a direct sum of two copies of a rank 8 Higgs bundle on $X_0 \times \widehat{C}$. In view of the theorem on irreducible representations of product groups, the only other possibility is that it is an exterior tensor product. The spectral curve of the rank 2 local system on $\overline{C}$ is $\widehat{C}$ with embedding given by the canonical 1-form on $\widehat{C}$, so this identifies the spectral curve as a curve in $T^{\vee}\overline{C}$. $\square$

10.2 From X_0 to X_1

Consider next the diagram

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "d"'] \arrow[dr, "b"] & \\ X_0 & & X_1 \times \overline{C}. \end{tikzcd}$$

In this case, b is a fibration with fibers $\mathbb{P}^1$, so there is no discriminant for the map b. Let $\Delta \subset X_1 \times \overline{C}$ now denote the discriminant for the horizontal divisor $d^{-1}\mathrm{Wob}_0$ over $X_1 \times \overline{C}$. Here again, let $\Delta_{\mathrm{eff}} \subset \Delta$ denote the effective singularities of the local system $R^1 b_*$ of the pullback by b of the local system we construct over (X_0, Wob_0).

As before, the effective discriminant decomposes into potentially nonempty pieces as

$$\Delta_{\mathrm{eff}} = \Delta_{\mathrm{eff}}^{\mathrm{vert}} + \Delta_{\mathrm{eff}}^{\mathrm{horiz}} + \Delta_{\mathrm{eff}}^{\mathrm{mov}}.$$

Lemma 10.6. *We have $\Delta_{\mathrm{eff}}^{\mathrm{mov}} = \emptyset$, and $\Delta_{\mathrm{eff}}^{\mathrm{horiz}} = \mathrm{Wob}_1 \times \overline{C}$.*

Proof. We have seen in Section 8 that the effective discriminant in the fiber over a general point $a \in \overline{C}$ is just $\mathrm{Wob}_1 \times \{a\} \subset X_1 \times \{a\}$. □

We would like to rule out the possibility of a component $X_1 \times \{a\}$ in $\Delta_{\mathrm{eff}}^{\mathrm{vert}}$. As before, for this we show that the full discriminant Δ itself does not contain any vertical components. We would like to show that for any point $a \in \overline{C}$, and for a general $\mathcal{F} \in X_0$, the Hecke line associated to $(\mathcal{F}, a)$ intersects Wob_0 transversally.

Since it is a line in $\mathbb{P}^3$, it will be transverse to the trope planes unless it is contained in one of them. Therefore, we would like to show that the general Hecke line is not contained in a trope plane, and that it intersects the Kummer in 4 distinct points.

The bundle $\mathcal{F}$ is a stable bundle of determinant $\mathcal{O}_C(\mathbf{p})$. If $a = (A, t)$ with $A^{\otimes 2}(\mathbf{p}) = \mathcal{O}_C(t)$, then the bundles in the Hecke line are kernels of the form

$$0 \to \mathcal{E} \to \mathcal{F} \otimes A \to \mathbb{C}_t \to 0$$

noting that the determinant of $\mathcal{E}$ is $A^{\otimes 2}(\mathbf{p} - t) = \mathcal{O}_C$.

Notice that $\mathcal{E}$ is semistable, indeed if it had a subbundle of degree 1 that would give a degree 1 subbundle of $\mathcal{F}$ contradicting stability of $\mathcal{F}$.

We have that $\mathcal{E}$ is in a trope plane if there is a line subbundle $B \subset \mathcal{E}$ of degree -1 such that $B^{\otimes 2}(2\mathbf{p}) = \mathcal{O}_C$. The 16 trope planes correspond to the 16 solutions of this equation. If this is the case, it gives

$$L := B \otimes A^\vee \to \mathcal{F}.$$

The set of possible L's is finite.

If $h^0(L^\vee \otimes \mathcal{F}) \geq 2$ then there is a morphism $L^{\oplus 2} \hookrightarrow \mathcal{F}$, with cokernel a nontrivial skyscraper sheaf. If a point $y \in C$ is in the support of this skyscraper shead, then we will get a map $L(y) \to \mathcal{F}$, showing that $\mathcal{F} \in \mathsf{Kum}$. Thus, for general $\mathcal{F}$ there is at most one map up to scalars from L to $\mathcal{F}$. For similar reasons, the image is a saturated subsheaf of $\mathcal{F}$. Thus, for a given L there is at most one rank 1 quotient of $(\mathcal{F} \otimes A)_t$ whose kernel contains $B = L \otimes A$. As there are finitely many L, the Hecke transformed bundles $\mathcal{E}$ can not all be in the wobbly locus. This shows that, for a general $\mathcal{F}$, the Hecke line is not contained entirely in a trope plane.

Let's now look at the intersection of the trope line with the Kummer. The point of X_0 corresponding to the bundle $\mathcal{E}$ is a point of the Kummer if $\mathcal{E}$ has a subbundle of degree 0. So we need to consider the possibility that there is a subbundle

$$B \hookrightarrow \mathcal{E}$$

with A of degree 0. This gives

$$L := B \otimes A^\vee \hookrightarrow \mathcal{F}$$

with L of degree 0. In that case, note that the inclusion has to be strict, since $\mathcal{F}$ can't contain a subbundle of degree 1. Then, there is a unique rank 1 quotient of $\mathcal{F}_t$ such that this subbundle corresponds to a subbundle of $\mathcal{E}$.

This reduces to our classical situation: such a subbundle $L \subset \mathcal{F}$ corresponds to a line in X_1 through $\mathcal{F}$, and we know from section 2 that for a general $\mathcal{F}$ there are four distinct lines. So, for a general $\mathcal{F}$ there are four distinct subbundles $L \subset \mathcal{F}$ of degree 0.

These in turn correspond to points $(L \otimes A) \oplus (L \otimes A)^{-1}$ of the Kummer.

Lemma 10.7. *Suppose $a = (A, t) \in \overline{C}$ is fixed. Choose $\mathcal{F} \in X_1$ general with respect to a, and let $L_1, L_2, L_3, L_4 \subset \mathcal{F}$ be the four subbundles of degree 0 corresponding to the four lines through $\mathcal{F}$. Then the four points $(L_i \otimes A) \oplus (L_i \otimes A)^{-1}$ of the Kummer are distinct.*

Proof. First the L_i are distinct, so $L_i \otimes A$ is not isomorphic to $L_j \otimes A$ for $i \neq j$. We need to show that $L_i \otimes A$ is not isomorphic to $(L_j \otimes A)^{-1}$ for $i \neq j$. If they were isomorphic we would have

$$L_i \otimes L_j \cong A^{\otimes -2} = \mathcal{O}_C(p - t).$$

Look at a bundle L_1 and choose $\mathcal{F}$ general along the corresponding line. This L_1 is the first subbundle of $\mathcal{F}$, the other three being line bundles corresponding to points in one of the fibers of the trigonal covering $C \to \mathbb{P}^1$. Here, more precisely, if $y \in C$ is a point then $L_1^{-1}(\mathbf{p} - y)$ is the corresponding line, and this for the three points y_1, y_2, y_3 in the fibers of the trigonal cover over the point $\mathcal{F}$ in the line corresponding to L_1.

In particular, $L_1 \otimes L_j$ is among a moving family of line bundles, so for a general point on the line, $L_1 \otimes L_j$ is not equal to the fixed $\mathcal{O}_C(\mathbf{p} - t)$. We conclude that if $\mathcal{F}$ is general, and contained in some line, then none of the other three lines corresponds to the same point of the Kummer. This holds for all the lines through the general point, concluding the proof of the lemma. □

Fixing $a = (A, t)$ and for a general $\mathcal{F} \in X_1$ with respect to a, then the Hecke line corresponding to $(\mathcal{F}, a)$ is transverse to the trope planes and meets the Kummer in four distinct points. We note that the points where the Hecke line meets the Kummer are not on the trope conics. Indeed, for $\mathcal{F}$ general with respect to a, the lines L are general points of the Jacobian with respect to A that is fixed, so the points of the Kummer are general.

This completes the proof of the following proposition. It gives the same corollaries as in the degree 0 case.

Proposition 10.8. *The discriminant Δ does not contain any components of the form $X_1 \times \{a\}$ for $a \in \overline{C}$.*

Corollary 10.9. *The effective singular locus Δ_{eff} of the R^1p_* local system on $X_1 \times \overline{C}$ is $\mathrm{Wob}_1 \times \overline{C}$.*

Corollary 10.10. *The R^1p_* local system of rank 16 on $X_1 \times \overline{C}$ decomposes as an exterior tensor product of the rank 8 local system we have constructed on X_1 with a rank 2 local system on $\overline{C}$ whose spectral curve is $\widehat{C} \to \overline{C}$.*

10.3 Identification of the eigenvalues

Proposition 10.11. *The eigenvalue rank 2 local systems on $\overline{C}$ in Corollaries 10.5 and 10.10 are the same as the original local system Λ associated to the Higgs bundle (E, θ), pulled back to $\overline{C}$.*

Proof. We have seen that the spectral curve of the eigenvalue is $\widehat{C}$. This is the spectral curve of $(E, \theta)|_{\overline{C}}$.

The correspondence of Proposition 7.6, and the analogous statement in the $(Y_0 \to Y_1)$ direction, tell us that the spectral line bundle on $\widehat{C}$ is the same as the one for $(E, \theta)|_{\overline{C}}$, away from the ramification points of $\widehat{C}/\overline{C}$. Our arguments did not apply to those ramification points, because the full abelianized Hecke does not decompose into two pieces, rather it is non-reduced over those points.

The spectral line bundle of the eigenvalue, and the spectral line bundle of $(E, \theta)|_{\overline{C}}$, are two line bundles of degree zero that differ possibly by a divisor of degree 0 supported on this ramification set. But as we move around in the Hitchin base, and in the moduli of genus 2 curves, the set of ramification points is permuted transitively. Thus, the coefficients of each point in the divisor must be the same. As the divisor has degree 0, this implies that it vanishes, and we obtain the identification of spectral line bundles.

It remains to identify the spectral 1-forms. In the $(Y_0 \to Y_1)$ direction, the abelianized Hecke correspondence gives for each point of Y_0 a translated map $\widehat{C} \to Y_1$. The spectral 1-form for the eigenvalue Higgs bundle is the pullback of the spectral 1-form on Y_1 to a 1-form on $\widehat{C}$. This is the same as our original 1-form, the spectral 1-form of (E, θ). The proof in the $(Y_1 \to Y_0)$ direction is identical.

This completes the identification between spectral data for the eigenvalue Higgs bundle and (E, θ), so the associated local systems are isomorphic. $\square$

11 Construction of a third kind of local system

In this section we take note of a different but similar construction of a flat parabolic Higgs bundle over (X_0, Wob_0). Namely, we use the same spectral covering $Y_0 \to X_0$, still putting trivial parabolic structures over the trope planes, but putting a parabolic structure with levels $0, 1/2$ over the Kummer. For this section X denotes X_0 and Y denotes Y_0.

More precisely, let $R \subset Y_0$ be the reduced inverse image of the Kummer surface. Thus the inverse image of the Kummer is $2R$ since $f : Y \to X$ has four simple ramification points over each point of the Kummer. As the Kummer has degree 4 we get $R \sim 2F$.

Let $\mathscr{L}$ be a spectral line bundle over Y. Use this to define a parabolic Higgs bundle $(\mathscr{F}_\bullet, \Phi)$ by

$$\mathscr{F}_a := f_*(\mathscr{L}), \quad 0 \le a < 1/2$$

$$\mathscr{F}_a := f_*(\mathscr{L}(R)), \quad 1/2 \le a < 1.$$

Proposition 11.1. *Let $\mathfrak{L}_0$ be a flat line bundle over $\mathcal{P}_2$. If we choose $\mathscr{L} = (\boldsymbol{\varepsilon}_0^* \mathfrak{L}_0) \otimes \mathcal{O}_Y(\boldsymbol{E}_0 + \boldsymbol{F}_0)$ then*

$$H^2 \cdot \mathrm{ch}_1^{\mathrm{par}}(\mathscr{F}_\bullet) = 0 \quad \textit{and } H \cdot \mathrm{ch}_2^{\mathrm{par}}(\mathscr{F}_\bullet) = 0.$$

Furthermore, there is no need for a correction term at the tacnodes, so we get a Higgs bundle corresponding to a local system on $X_0 - \mathrm{Wob}_0$.

Proof. The computations are left to the reader using the formulae of Propositions 4.7 and 4.8. We note that the parabolic structure may be viewed as a bundle on the root stack $X_0[\frac{1}{2}\mathsf{Kum}]$. Over this root stack, a tacnode of a trope plane with the Kummer pulls back to a normal crossings. So, over the root stack where the parabolic structure over the Kummer disappears, we have a logarithmic Higgs field along a normal crossings divisor, so no additional correction term is needed to $\mathrm{ch}_2^{\mathrm{par}}$. □

Remark 11.2. Let $X' \to X$ be the smooth double cover branched over the Kummer. This may be viewed as the moduli space for parabolic vector bundles on $\mathbb{P}^1$ with parabolic structure over 6 points (the 6 branch points of $C/\mathbb{P}^1$), via the correspondence of Goldman and Heu-Loray [HL19]. The above parabolic Higgs bundle pulls back to a Higgs bundle with trivial parabolic structures over X'. This is probably a Langlands local system over X'. It would go outside our current scope to pursue this here.

We think that the resulting local systems on $X_0 - \mathrm{Wob}_0$ should be the ones corresponding to the PGL_2-local systems of degree 1 by the geometric Langlands correspondence between those and perverse sheaves over Bun_{SL_2}. Notice that since we expect to get a perverse sheaf on Bun_{SL_2}, there isn't an *a priori* corresponding perverse sheaf on X_1.

Nonetheless, we can take the Hecke transform of this local system over to X_1. In this case, Lemma 8.4 does not apply, see Remark 8.6. The Hecke transformed local system will therefore have singularities over $\mathrm{Wob}_1 \cup K_1$ where K_1 is another subvariety of X_1. It is the *Kummer K3 surface* [Bea96, Dol20, GH94, Hud05, Keu97] that, we recall, may be described as follows: inside the Hecke space we have a K3 surface birational to the Kummer, it is the set of points where the two lines meet in the Hecke fiber over a point of the Kummer. Then project this to X_1 to get K_1.

The intersection of K_1 with a general line $\ell \subset X_1$ consists of the two points on ℓ that are the images of the points p and q appearing in the description of Section 8. In particular K_1 has degree 2.

Remark 11.3. Indeed, K_1 is the intersection of X_1 with a third quadric in $\mathbb{P}^5$. It is the K3 surface obtained by resolving the 16 singularities of the Kummer. The embedding $K_1 \hookrightarrow X_1$ depends on the point of $\overline{C}$ over which we make the Hecke transformation, and the third quadric is identified ?? in the synthetic description with the K3 surface denoted by Σ in [GH94].

The Hecke transform of our rank 8 local system on $X_0 - \mathrm{Wob}_0$ is a rank 16 local system over an open subset of $X_1 \times \overline{C}$. This seems to reduce to a direct sum of two copies of a rank 8 local system on $X_1 - \mathrm{Wob}_1 - K_1$ over each point of $\overline{C}$.

Calculation of the rank seems to say that the rank 8 local system on $X_1 - \mathrm{Wob}_1 - K_1$ then Hecke transforms back to a local system of rank 24 on an open subset of $X_0 \times \widehat{C}$. This would be supposed to correspond to the Hecke eigensheaf property expected of the perverse sheaf corresponding to an odd degree PGL_2 local system.

We close this topic for now, with the prospect of further discussion elsewhere.

12 Some pushforward calculations

The objective of this section is to arrive at a proof of Theorem 3.17. We'll do that by applying the theory of [DPS16] in some specific cases. The situation of Theorem 3.17 is fairly specific and tailored to our needs for the Hecke transform calculations. We'll go through some intermediate steps of varying degrees of generality that might be of independent interest as complements to the discussion of [DPS16], for example we extend the general theory of [DPS16] to the case of morphisms with multiple fibers.

For the most part, the notations in this section will be general, not related to our moduli spaces of stable bundles on the genus 2 curve.

12.1 The relative critical locus

Suppose $f : X \to S$ is a projective morphism from a smooth surface to a smooth curve. Suppose $D \subset X$ is a divisor, and let $D_H \subset D$ be the union of components that map surjectively to S. Assume that D_H is reduced. Let D° be the smooth locus of D and let X° be the complement of the singular points of D.

Suppose $\mathcal{E}.$ is a parabolic bundle on (X^*, D^*) with a compatible Higgs field φ, and let $\Sigma^* \to X^*$ be the spectral covering of φ. Extend this to a finite covering $\Sigma \to X$. As Σ^* is the spectral covering of the logarithmic Higgs field φ acting on $\mathcal{E}_0$, there is a natural inclusion

$$\Sigma^* \hookrightarrow T^*(X^*, \log D^*).$$

The tautological section of $T^*(X^*)|_{\Sigma^*}$ restricts to a section of $T^*(X^*/S)|_{\Sigma^*}$ that we'll call the relative tautological 1-form denoted α^{rel}.

Define the *upper critical locus*

$$\widetilde{\mathrm{Crit}}(X/S, \mathcal{E}., \varphi)^* \subset \Sigma^*$$

to be the zero-scheme of α^{rel}. Define the *lower critical locus*

$$\mathrm{Crit}(X/S, \mathcal{E}., \varphi)^* \subset X^*$$

to be its image in X^*. Denote by $\widetilde{\mathrm{Crit}}(X/S, \mathcal{E}., \varphi)$ and $\mathrm{Crit}(X/S, \mathcal{E}., \varphi)$ their closures in Σ and X respectively.

Proposition 12.1. *Consider the relative L^2-Dolbeault complex*

$$Dol_{L^2}(X^*/S, \mathcal{E}., \varphi) \quad \text{over } X^*$$

from [DPS16]. Assume that the (upper or lower) critical locus has dimension 1*. Then the cohomology sheaf of the L^2-Dolbeault complex in degree* 0 *vanishes, and the cohomology sheaf in degree* 1 *(i.e. the cokernel) is supported on* $\mathrm{Crit}(X/S, \mathcal{E}., \varphi)^*$.

Assume that Σ is smooth in codimension ≤ 1. There is a dense Zariski subset $S^o \subset S$ such that, using $()^o$ for the restriction over S^o, the map $\widetilde{\mathrm{Crit}}(X/S, \mathcal{E}., \varphi)^o \to S^o$ is provided *with a section of the pullback of $T^*(S^o)$ making it into the spectral variety for the higher direct image Higgs bundle over S^o.*

Proof. See [DPS16]. The differential in the L^2-Dolbeault complex is a morphism between locally free sheaves of the same rank. Thus, the condition that the critical locus has dimension 1 means that this map has maximal rank at the general point of X, in particular it is injective. This shows the first paragraph of the statement.

For the second part, we may assume that over S^o, the critical locus is relatively 0-dimensional, contained in the smooth points of Σ, the map $X^o \to S^o$ is smooth, and the horizontal divisor is etale. We may also assume that the critical locus does not meet the horizontal divisor. The Higgs bundle $\mathcal{E}$ is the direct image of a spectral line bundle on Σ, and the cokernel of the Dolbeault complex is the spectral line bundle, tensored with the relative differentials and then restricted to $\widetilde{\mathrm{Crit}}(X/S, \mathcal{E}., \varphi)^o$. The higher direct image of the L^2-Dolbeault complex is the usual direct image of this line bundle on the critical locus, down to S^o. The spectral embedding of Σ gives a section $\Sigma \to T^*X$. In view of the exact sequence

$$0 \to f^*T^*S^o \to T^*X^o \to T^*(X^o/S^o) \to 0,$$

the critical locus being the zero set of the projection $\Sigma^o \to T^*(X^o/S^o)$ is provided with a map $\widetilde{\mathrm{Crit}}(X/S, \mathcal{E}., \varphi)^o \to f^*T^*S^o$. This gives the tautological differential making the critical locus into the spectral variety of the higher direct image Higgs bundle [DPS16]. □

Proposition 12.2. *Given a point where the map f is not smooth, or where the horizontal divisor is not etale over the base, if the point is not in the lower critical locus then it does not contribute a singularity of the Dolbeault higher direct image Higgs bundle on S.*

Proof. When we blow up to resolve the singularities of the map and apply the technique of [DPS16], such a point could result in an isolated vertical component of the critical locus for the blown-up map. However, this can't contribute anything to any of the parabolic levels of the higher direct image of the L^2-Dolbeault complex, since we know that the higher direct image parabolic Higgs bundle has level pieces that are locally free over S, in particular they can't have sections supported over a finite set in S. □

12.2 Vertical divisors with multiple components

We would like to frame an extended pushforward setup.

Let $f : X \to S$ be a projective morphism, with S a curve and X a smooth surface. We have a point $o \in S$, the discriminant of f is $\{o\}$. Suppose that $D_V = f^{-1}(o)$ is a simple normal crossings divisor, possibly non-reduced. Write the components as $D_1, \ldots, D_r$ with multiplicities $m_1, \ldots, m_r$.

Suppose given D_H a smooth divisor that is transversal to D_V, not going through a node of D_V. Thus, $D = D_H + D_V$ is a normal crossings divisor.

At points where D_H intersects a component of D_V that has multiplicity > 1, the map $D_H \to S$ will be ramified, with order of ramification equal to the multiplicity of D_V at that point.

Suppose given a logarithmic parabolic Higgs bundle $(E., \varphi)$ on E with respect to the reduced divisor $D^r := D^{\mathrm{red}}$. We assume the usual hypothesis on φ (parabolic with nilpotent graded parts of the residue).

Let E_0 be the bundle obtained by assigning parabolic levels 0 along all components. The Higgs field is

$$\varphi : E_0 \to E_0 \otimes \Omega^1_X(\log D^r).$$

The residue of φ along D_H induces a weight filtration on $E|_{D_H}$. We denote by W_iE the subsheaf of E of sections that restrict to sections of $E|_{D_H}$ that are in $W_i(E|_{D_H})$.

The weight filtration extends over the crossing points as a strict filtration, in the same way as was discussed in the original case. Locally at a point where the horizontal and vertical divisors meet we can take a root of the function t and reduce this question to the original case of reduced vertical divisor.

Define

$$\Omega^1_{X/S}(\log D) := \Omega^1_X(\log D)/f^*\Omega^1_Y(\log o).$$

This is a line bundle over X.

Caution: This is not the same as the relative dualizing sheaf $\omega_{X/S}(D_H)$ if any multiplicities of D_V are > 1. The relation between these two is given by

$$\omega_{X/S}(D_H) = \Omega^1_{X/S}(\log D) \otimes \mathcal{O}_X(D - D^r).$$

The projection of φ to a Higgs field with coefficients in the vertical cotangent bundle provides a map

$$\varphi : W_iE_0 \to W_{i-2}E_0 \otimes \Omega^1_{X/S}(\log D).$$

Theorem 12.3. *Define the relative Dolbeault complex, in parabolic level* 0*, to be*

$$Dol(E_0, \varphi) = \left[W_0E_0 \xrightarrow{\varphi} W_{-2}E_0 \otimes \Omega^1_{X/S}(\log D) \right].$$

This is quasi-isomorphic to the modified relative Dolbeault complex

$$Dol'(E_0, \varphi) = \left[W_1 E_0 \xrightarrow{\varphi} W_{-1} E_0 \otimes \Omega^1_{X/S}(\log D) \right].$$

Define

$$F^i_0 := R^i f_* Dol_0 \cong R^i f_* Dol'_0.$$

Then F^i_0 is the parabolic level 0 piece of the parabolic Higgs bundle on S corresponding to the i-th higher direct image local system of the local system corresponding to $(E_\cdot, \varphi)$.

In order to get the other parabolic level pieces of $F_\cdot$ we proceed as follows. Consider the parabolic line bundle $\mathcal{O}_S(a \cdot o)$ that has a jump at parabolic level $-a$.

We get a pullback parabolic bundle $\mathcal{O}_X(a \cdot D_V) := f^* \mathcal{O}_S(a \cdot o)$. We can tensor this with $E_\cdot$ to get the parabolic Higgs bundle

$$(E(a \cdot D_V), \varphi).$$

One should be careful that the component sheaves along components of D_V of higher multiplicity need to be calculated using the correct notions of pullback and tensor product of parabolic bundles.

Then we can also form the complexes

$$Dol(E(a \cdot D_V)_0, \varphi) \quad \text{and} \quad Dol'(E(a \cdot D_V)_0, \varphi)$$

in the same way as before:

$$Dol(E(a \cdot D_V)_0, \varphi) = \left[W_0 E(a \cdot D_V)_0 \xrightarrow{\varphi} W_{-2} E(a \cdot D_V)_0 \otimes \Omega^1_{X/S}(\log D) \right]$$

and

$$Dol'(E(a \cdot D_V)_0, \varphi) = \left[W_1 E(a \cdot D_V)_0 \xrightarrow{\varphi} W_{-1} E(a \cdot D_V)_0 \otimes \Omega^1_{X/S}(\log D) \right]$$

and again these are quasi-isomorphic.

Then

$$F^i_a = R^i f_* Dol(E(a \cdot D_V)_0, \varphi) \cong R^i f_* Dol'(E(a \cdot D_V)_0, \varphi)$$

is the level a piece of the parabolic Higgs bundle on S corresponding to the i-th higher direct image local system of the local system corresponding to $(E_\cdot, \varphi)$.

12.3 Proof

For any point $p \in D_i \cap D_j$ let $k(p)$ be the LCM of the multiplicities m_i and m_j.

Consider a cover $\widetilde{S} \to S$ given by $t = s^n$. We assume that for any crossings point p we have $k(p)$ divides n.

Let $\widetilde{X}$ be the normalization of $X \times_S \widetilde{S}$.

Proposition 12.4. *Over any crossing point p, the space $\widetilde{X}$ has a singularity of type A_{m-1} where $m = n/k(p)$. Let $\widehat{X} \to \widetilde{X}$ be obtained by applying the minimal resolution to each of these points. Then*

$$\hat{f} : \widehat{X} \to \widetilde{S}$$

is a map from a smooth surface to $\widetilde{S}$ such that $\hat{f}^{-1}(0)$ is a reduced normal crossings divisor.

This will be seen in subsection 12.3.4 below.

Let G be the cyclic group of symmetries of $\widetilde{S}/S$. Let $\eta : \widehat{X} \to X$ be the map. Suppose given a parabolic Higgs bundle (E, φ) on X with respect to a diviror D incuding $D_V = f^{-1}(0)$ and a horizontal part D_H not in our picture.

We have defined above the Dolbeault complex $Dol(X/S, E)_0$ for parabolic level 0 in the case of multiple components.

On the other hand, let η^*E be the pullback parabolic bundle on $\widehat{X}$. We can also form the Dolbeault complex upstairs for parabolic level 0 $Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0$. This is for a case of a reduced divisor.

Theorem 12.5. *With these notations, we have*

$$(R\eta_*(Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0))^G = Dol(X/S, E)_0.$$

This theorem implies Theorem 12.3, using Lemma 12.6 of the next subsection.

12.3.1 A lemma on parabolic structures on S

Suppose given a parabolic bundle F on S, and consider a base change $\xi : \widetilde{S} \to S$ given by $t = s^n$. Let G be the cyclic group acting on $\widetilde{S}/S$.

Lemma 12.6. *We have*

$$\xi_*((\xi^*F)_0)^G \cong F_0.$$

Proof. We may assume that $F = \mathcal{O}_S(a \cdot 0)$ is a parabolic line bundle.

Then

$$\xi^* F = \mathcal{O}_{\widetilde{S}}(na \cdot \tilde{0}).$$

Thus

$$(\xi^* F)_0 = \mathcal{O}_{\widetilde{S}}(\lfloor na \rfloor \cdot \tilde{0}).$$

Let's write things in terms of modules. We have that

$$(\xi^* F)_0 \leftrightarrow s^{-j} k[s], \quad j = \lfloor na \rfloor.$$

We then think of this as a $k[t]$ module.

The G fixed part is the sum of all the monomials $t^i = s^{ni}$ that are contained in here. Being contained in here is equivalent to $ni \geq -\lfloor na \rfloor$, so in terms of monomials in t we have the sum of the t^i whenever

$$i \geq -\lfloor na \rfloor / n.$$

On the other hand, $F_0 = \mathcal{O}_S(\lfloor a \rfloor \cdot 0)$ and this is the sum of monomials t^i for $i \geq -\lfloor a \rfloor$. Thus, the statement of our lemma is equivalent to saying, for integers i, that

$$i \geq -\lfloor na \rfloor / n \Leftrightarrow i \geq -\lfloor a \rfloor.$$

Changing the sign of i this is equivalent to the statement for all i

$$i \leq \lfloor na \rfloor / n \Leftrightarrow i \leq \lfloor a \rfloor,$$

which in turn is equivalent to

$$\lfloor \lfloor na \rfloor / n \rfloor = \lfloor a \rfloor.$$

Write $a = \lfloor a \rfloor + (b/n) + c$ where $0 \leq b < n$ is an integer and $0 \leq c < 1/n$ is a real number. Then

$$\lfloor na \rfloor = n\lfloor a \rfloor + b$$

and

$$\lfloor \lfloor na \rfloor / n \rfloor = \lfloor n\lfloor a \rfloor / n + b/n \rfloor = \lfloor a \rfloor + \lfloor b/n \rfloor = \lfloor a \rfloor.$$

□

12.3.2 The local case when $n = k(p)$

Consider the local picture at a crossings point p for a covering with $n = k(p)$. Let D_1 and D_2 denote the divisor components. The coordinates on X are x, y in our local neighborhood, with D_1 defined by $x = 0$ and D_2 defined by $y = 0$. Let m_i be the multiplicity of D_i in $f^{-1}(0)$, so we can assume that f is given by $t = x^{m_1}y^{m_2}$.

Write $n = m_1 a = m_2 b$ with a and b relatively prime. Let $\widetilde{X} \to X$ be the covering of order ab given in coordinates by

$$x = u^a, \quad y = v^b.$$

The map $\widetilde{X} \to S$ is given in coordinates by

$$t = u^n v^n$$

so it factors through the map $\widetilde{X} \to \widetilde{S}$ given by $s = uv$.

If d is the GCD of m_1 and m_2, the normalization of the fiber product

$$X \times_S \widetilde{S}$$

consists of the disjoint union of d copies of the above covering $\widetilde{X}$.

We can take the relative Dolbeault complex on the normalization of the fiber product, then take the direct image by an etale map down to $\widetilde{X}$, then take the $\mathbb{Z}/d\mathbb{Z}$ invariants; this is the same as the relative Dolbeault complex of $\widetilde{X}/\widetilde{S}$.

So, to show the desired property in this case we can consider the direct image from $\widetilde{X}$ to X and take the $\mathbb{Z}/ab\mathbb{Z}$-invariants.

The pullback of the logarithmic forms on X/S is the same as the logarithmic forms on $\widetilde{X}/\widetilde{S}$.

Lemma 12.6 applied in each coordinate yields the statement that the parabolic level 0 of the pullback parabolic bundle, goes under push-forward and taking invariants to the level 0 piece on X. This yields the required statement for $\widetilde{X} \to X$.

12.3.3 The local case when the divisor is already reduced

Suppose given $f : X \to S$ that locally looks like $(x, y) \mapsto t = xy$.

Consider a covering $\widetilde{S} \to S$ given by $t = s^n$. Then

$$X \times_S \widetilde{S}$$

has a single A_{n-1} surface singularity over the node of $f^{-1}(0)$. Indeed the equation is $xy = s^n$.

Resolve this by the minimal resolution to get

$$\begin{array}{ccccc} \widehat{X} & \to & X \times_S \widetilde{S} & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{S} & = & \widetilde{S} & \to & S \end{array}$$

Claim 1): the fiber of $\hat{f} : \widehat{X} \to \widetilde{S}$ over $s = 0$ is reduced and of the form

$$\widehat{X}_0 \cup E_1 \cup \cdots \cup E_{n-1}$$

where $\widehat{X}_0$ is the strict transform of $f^{-1}(0)$ and $E_1, \ldots, E_{n-1}$ form an A_{n-1} configuration of (-2) curves. This is a classical statement [Val34], see for example [Bri02].

?? found that reference available on the web without buying a book ?

Claim 2): Let G be the cyclic group of symmetries of $\widetilde{S}/S$. Let $\eta : \widehat{X} \to X$ be the map. Suppose given a parabolic Higgs bundle (E, φ) on X with respect to a diviror D incuding $D_V = f^{-1}(0)$ and a horizontal part D_H not in our picture. Form the Dolbeault complex $Dol(X/S, E)_0$ for parabolic level 0.

On the other hand, let $\eta^* E$ be the pullback parabolic bundle on $\widehat{X}$. We can also form the Dolbeault complex upstairs for parabolic level 0 $Dol(\widehat{X}/\widetilde{S}, \eta^* E)_0$.

We claim that the natural map

$$Dol(X/S, E)_0 \to (R\eta_*(Dol(\widehat{X}/\widetilde{S}, \eta^* E)_0))^G$$

is a quasi-isomorphism.

We prove this in the following way. We show that the map induces an isomorphism on the G-invariant $R\eta_*$ of each piece of the Dolbeault complex. The statement is local at the normal crossings point, and in particular it does not concern the horizontal divisor. Thus, we may decompose E into a direct sum of parabolic line bundles.

A parabolic line bundle is determined by the parabolic levels along the two divisor components D_1 and D_2. Any parabolic levels can occur, as may be seen by considering a toy example where the fundamental group of the total space is $\mathbb{Z} \times \mathbb{Z}$.

We can apply the general theory for reduced normal crossings divisors, to both $X \to S$ and to $\widehat{X} \to \widetilde{S}$. We know in both cases that the higher direct image of the Dolbeault complex calculates the parabolic bundle, respectively $F_\cdot$ and $\widehat{F}_\cdot$, corresponding to the higher direct image local systems. We have seen in Lemma 12.6 that taking the G-invariants of the direct image of $\widehat{F}_0$ from $\widetilde{S}$ down to S yields F_0. Therefore, the morphism

$$Dol(X/S, E)_0 \to (R\eta_*(Dol(\widehat{X}/\widetilde{S}, \eta^* E)_0))^G$$

induces a quasi-isomorphism between the higher direct images on S.

But with our current reduction, the Higgs field is zero, so in both cases the higher direct image of the Dolbeault complex is the direct sum of the higher direct images of the two pieces. If we denote the component pieces by $Dol(X/S,E)^i_0$ for $i=0,1$ we get that

$$Rf_*Dol(X/S,E)^i_0 \to Rf_*(R\eta_*(Dol(\widehat{X}/\widetilde{S},\eta^*E)^i_0))^G$$

is an isomorphism for $i=0,1$.

Consider C^i the cone

$$Dol(X/S,E)^i_0 \to (R\eta_*(Dol(\widehat{X}/\widetilde{S},\eta^*E)^i_0))^G \to C^i.$$

It is a complex concentrated at the crossings point $p\in X$. From the above, we conclude that

$$Rf_*(C^i)=0.$$

This implies that C^i is quasi-isomorphic to 0. Indeed, the cohomology sheaves of C^i are coherent sheaves concentrated at p so their higher direct images in strictly positive degrees vanish. The spectral sequence going from the higher direct images of the cohomology sheaves to the cohomology of the higher direct image, therefore starts out with only a single line, so it degenerates right away. If any of the cohomology sheaves were nonzero this would give a nonzero $Rf_*(C^i)$ contradicting the previous statement; thus, C^i is acyclic.

The cone being acyclic implies that the map

$$Dol(X/S,E)^i_0 \to (R\eta_*(Dol(\widehat{X}/\widetilde{S},\eta^*E)^i_0))^G$$

is a quasi-isomorphism.

This holds for the case of a parabolic line bundle with zero Higgs field, and hence for the case of any parabolic bundle with zero Higgs field.

It is a local statement at the point p, and the spectral sequence for $R\eta_*$ of the Dolbeault complex in presence of a Higgs field, implies that the map

$$Dol(X/S,E)_0 \to (R\eta_*(Dol(\widehat{X}/\widetilde{S},\eta^*E)_0))^G$$

is a quasi-isomorphism. This proves Claim 2.

12.3.4 Combining these cases

Suppose now that we are given a projective map $f : X \to S$ such that the vertical divisor $D_v = f^{-1}(0)$ is simple normal crossings, with smooth components having various multiplicities m_i. Let n be a number divisible by all the m_i. Let $\widetilde{S} \to S$ be a cyclic covering of degree n fully ramified over 0.

Let $\widetilde{X}$ be the normalization of $X \times_S \widetilde{S}$.

For $p \in D_i \cap D_j$, let $k(p)$ denote the LCM of m_i and m_j. Let $n/k(p)$.

Let $G^k := \mathbb{Z}/k(p)\mathbb{Z}$ and $G' := \mathbb{Z}/(n/k(p))\mathbb{Z}$. Let X' be the intermediate covering of X of degree $k(p)$, over $S' \to S$ of degree $k(p)$. The map $X' \to X$ is a finite covering with X' smooth and the fiber of $f' : X' \to S'$ has reduced components with normal crossings, as we saw in subsection 12.3.2. Note that p splits into several points p', the number is the GCD of the multiplicities at p.

Then, the map $\widetilde{X} \to X'$ is the covering considered in subsection 12.3.3 of degree $n/k(p)$. Over each point p' we obtain a point $\widetilde{p} \in \widetilde{X}$ with an $A_{n/k(p)-1}$ singularity. This shows Proposition 12.4.

Let $\widehat{X}$ be obtained by taking, at each point $\tilde{p}$, the minimal resolution of the $A_{n/k(p)-1}$ singularity.

Then the map $\hat{f} : \widehat{X} \to \widetilde{S}$ has a reduced normal crossings fiber over $\tilde{0}$.

We show the desired property locally at each point p. The covering $S' \to S$ and $X' \to X$ are covered by the case of subsection 12.3.2. Thus, the map

$$Dol(X'/S')_0 \to (R\eta'_* Dol(\widehat{X}/\widetilde{S})_0)^{G'}$$

is a quasi-isomorphism,

The map $X' \to S'$ has normal crossings fibers, so the covering $\widetilde{S} \to S'$ and upstairs $\widehat{X} \to \widetilde{X} \to X'$ are covered by the already-reduced case of subsection 12.3.3. Therefore

$$Dol(X/S)_0 \to (R\eta^k_* Dol(X'/S')_0)^{G^k}$$

is a quasi-isomorphism. Putting these together we conclude the desired statement that

$$Dol(X/S)_0 \to (R\eta_* Dol(\widehat{X}/\widetilde{S})_0)^G$$

is a quasi-isomorphism.

12.3.5 Application to the higher direct image

Let F be the parabolic Higgs bundle on S corresponding to the higher direct image local system from X. Then ξ^*F is the parabolic Higgs bundle on $\widetilde{S}$ corresponding to the higher direct image local system coming from the local system on $\widehat{X}$ that corresponds to the pullback Higgs bundle η^*E.

On the other hand, since the fiber $\hat{f}^{-1}(\hat{0})$ is reduced normal crossings and the horizontal divisor meets it transversally, we know that this is the same as the parabolic bundle on $\widetilde{S}$ corresponding to the higher direct image of the Dolbeault complex $Dol(\widehat{X}/\widetilde{S}, \eta^*E)$.. In particular this gives, for the level 0 pieces, that

$$(\xi^*F)_0 = R\hat{f}_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0.$$

Lemma 12.6 says that

$$\begin{aligned} F_0 &= \xi_*((\xi^*F)_0)^G. \\ &= \left[\xi_*\left(R\hat{f}_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right)\right]^G. \end{aligned}$$

We can write

$$\xi_*\left(R\hat{f}_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right) = Rf_*\left(R\eta_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right).$$

Thus

$$F_0 = \left[Rf_*\left(R\eta_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right)\right]^G.$$

We can put the G-invariants inside before taking Rf_*, so this becomes

$$F_0 = Rf_*\left[\left(R\eta_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right)^G\right].$$

Now the theorem says

$$\left(R\eta_*Dol(\widehat{X}/\widetilde{S}, \eta^*E)_0\right)^G = Dol(X/S, E)_0.$$

Thus we get

$$F_0 = Rf_*Dol(X/S, E)_0.$$

This is the statement we want.

The similar statement holds with Dol replaced by Dol'.

12.4 Applications

Consider the following situation: $f : X \to Y$ is a map from a surface to a curve, with a divisor $D \subset X$ such that in local coordinates (x, y) with y the coordinate of Y, D is given by $y = x^2$.

Let $E.$ be a parabolic logarithmic Higgs bundle on X with singularities along D. Although, in the end we need to apply the theory to Higgs bundles of higher rank (8 to be exact), the local picture involves a piece of rank 2 so we'll suppose here that $E.$ has rank 2.

We would like to calculate a complex on X that calculates the higher direct image of the L^2 Dolbeault complex on a resolution. For this, let $\beta : \widetilde{X} \to X$ be obtained by blowing up twice, first at the origin and then at the resulting triple intersection point of the exceptional line with the strict transforms of the fiber and D.

Notationally, call $\widetilde{D}$ the strict transform of D in $\widetilde{X}$. Let A be the strict transform of the first exceptional divisor, and B the second exceptional divisor. Thus, $\widetilde{D}$ meets B, and A meets B, and the map $\tilde{f} : \widetilde{X} \to Y$ has fiber over the origin equal to $A + 2B + \widetilde{D}$.

After the second blowing up, the fiber is a normal crossings divisor, but with nontrivial mulitplicity 2 on the middle component. The higher direct image statement of Theorem 12.3, for maps whose fibers have nontrivial muliplicity, yields the Dolbeault complex on $\widetilde{X}$ that calculates the higher direct image to Y. Then take its higher direct image down to X, this is the Dolbeault complex on X to be identified.

We'll consider two cases, both for the local situation when $E.$ has rank 2. One case is for parabolic levels $0, -1/2$, the other is for trivial parabolic structure but a nontrivial nilpotent residue of the Higgs field along D.

One main idea for doing the calculations is to write the rank 2 bundles as a direct sum of two rank 1 pieces, even though such a decomposition is not compatible with the Higgs field. This allows for computation of the pieces in holomorphic Dolbeault complexes, which are then put back together before inputting the Higgs field.

12.4.1 Parabolic levels $0, -1/2$

In this case the level zero piece is a rank 2 bundle E on X, provided with $U \subset E_D$ of rank 1. Let $Q := E_D/U$ be the quotient line bundle over D.

The parabolic structure is given by $E_0 = E$ and $E_t = \ker(E \to Q)$ for $-1/2 \leq t < 0$, then $E_t = E(-D)$ for $-1 \leq t < -1/2$.

Assume given a logarithmic Higgs field φ that is strictly parabolic for the filtration, so

$\varphi : E \to E_{-1/2} \otimes \Omega^1_X(\log D)$.

In this case, the direct image $F.$ on Y is going to have trivial parabolic structure so we just want to calculate F_0.

The L^2 Dolbeault complex is defined in the usual way of [DPS16] outside of the origin in X. It has a unique extension to a two-term complex of locally free sheaves, the *locally free extension of the Dolbeault complex* expressed as

$$Dol(E., X/S)_{\mathrm{lf}} := \left[E_0 \to E_{-1/2} \otimes \omega_{X/S}(D)\right].$$

Use the notations for the blow-up $\widetilde{X} \xrightarrow{\beta} X$ established above, with exceptional divisors A and B.

Let $\widetilde{E}$ be the pullback of $E = E_0$ to $\widetilde{X}$. This isn't quite the same as the pullback parabolic bundle. We note that $\widetilde{E}$ is constant on A and B, equal to the vector space E_P with its quotient Q_P. The parabolic structure along A is just the pullback one, with

$$E_{A,0} = \widetilde{E}$$

and

$$E_{A,-1/2} = \ker(E_{A,0} \to Q_{P,A})$$

where $Q_{P,A}$ means the vector space Q_P considered as a trivial bundle on A.

Along B we get extra sections in $E_{B,0}$, namely at a general point of B,

$$E_{B,0} = \ker\left(\widetilde{E}(B) \to Q_{P,B}(B)\right).$$

The parabolic structure is trivial along B, in other words there are no non-integral levels. This is because the levels 1/2 on both divisors A and $\widetilde{D}$ combine together to give a piece without parabolic structure (it may be seen in greater detail by looking at $E.$ as a direct sum of two parabolic line bundles).

Let E' denote the level 0 piece of the parabolic bundle on $\widetilde{X}$. It then has a weight filtration, with $W_0E' = E'$ and $W_{-2}E' = \ker(E' \to Q'_{\widetilde{D}})$ where Q' means the quotient adjusted to be a quotient of E'.

Applying the general theory we have

$$\widetilde{Dol} = \left[E' \to W_{-2}E' \otimes \Omega_{\widetilde{X}/S}(\widetilde{D})\right].$$

In our situation where B has multiplicity two, we have

$$\Omega_{\widetilde{X}/S} = \eta^*\Omega_{X/S}(A + B)$$

since it should have degree -1 on A and degree 0 on B, and should agree with the relative dualizing sheaf except[7] along B. Note that $A^2 = -2$ while $B^2 = -1$.

Using the birational transformation $\beta : \widetilde{X} \to X$, we would like to calculate the complex

$$R\beta_* \widetilde{Dol} = R\beta_* \left[E' \to W_{-2}E' \otimes \Omega_{\widetilde{X}/S}(\widetilde{D}) \right]$$

on X. It will be the same as the previous complex away from the origin, and it will turn out also that the components are locally free at the origin.

Let $\beta^A : X^A \to X$ be the first blow up, and denote by

$$\beta^B : \widetilde{X} \to X^A$$

the second blowing up. Start by looking at $R\beta^B_* \widetilde{Dol}$ as a complex on X^A.

The first claim is that along points of A that don't meet the origin (which may be identified in X^A and $\widetilde{X}$), this complex is the same as the pullback of the locally free extension $Dol(E., X/S)_{\mathrm{lf}}$.

In degree 0 we have E' on the one hand, and the pullback of E on the other hand. These are the same along points of A.

[7]For comparison, the calculation of the relative dualizing sheaf is as follows: We have

$$\omega_{\widetilde{X}/S} = \eta^* \omega_{X/S}(A + 2B).$$

To see this, note that we divided by ω_S on both sides so we can just do the calculation for the ω_X etc.

Suppose we blow up a point in a surface with coordinates x, y, yielding new coordinates u, v with $x = uv$ and $y = v$ for example. Then

$$dx \wedge dy = (udv + vdu) \wedge dv = vdu \wedge dv.$$

This gives

$$\omega_{X^A/S} = \eta^{A,*} \omega_{X/S}(A_{X^A})$$

and

$$\omega_{\widetilde{X}/S} = \eta^{B,*} \omega_{X^A/S}(B).$$

However, note that

$$\eta^{B,*} \mathcal{O}_{X^A}(A_{X^A}) = \mathcal{O}_{\widetilde{X}}(A + B).$$

Thus we get

$$\omega_{\widetilde{X}/S} = \eta^* \omega_{X/S}(A + 2B).$$

Thus,

$$\omega_{\widetilde{X}/S} = \Omega_{\widetilde{X}/S}(B).$$

In degree 1, noting that our points of A under consideration don't touch $\widetilde{D}$, we have on the one hand $E' \otimes \Omega_{X^A/S}$ and on the other hand, the pullback of $E_{-1/2} \otimes \Omega_{X/S}(D)$. These are again the same, so that shows the first claim.

The remaining question is what happens over B. In order to calculate the terms of the complex, let's suppose that E has rank 1. The rank 2 case with trivial Higgs field is a direct sum of two parabolic line bundles. This will serve to calculate the two terms of the complex. Then the Higgs field gives a map between these two and we get the desired higher direct image. A spectral sequence argument shows the desired quasiisomorphism in the presence of the Higgs field.

If the line bundle has trivial parabolic structure: then E' is the same as $\widetilde{E}$. The degree 0 piece is $\widetilde{E}$ and the degree 1 piece is

$$\widetilde{E}(-\widetilde{D}) \otimes \Omega_{\widetilde{X}/S}(\widetilde{D}) = \widetilde{E} \otimes \Omega_{\widetilde{X}/S}$$

$$= \eta^*(E \otimes \Omega_{X/S})(A + B).$$

The higher direct image of $\widetilde{E}$ down to X^A is E^A (the pullback of E to there), and then the higher direct image down to X is E resp. $E \otimes \Omega_{X/S}$. For these, note that on $\mathbb{P}^1$, the H^1 of a trivial bundle vanishes; and furthermore, along the blowing-down of an exceptional $\mathbb{P}^1$, the higher direct image of a bundle that is trivial along the exceptional divisor is the same as the usual pushforward.

Now suppose we have a level $-1/2$ parabolic structure. This means $E_t = E$ for $-1/2 \leq t \leq 0$ and $E_t = E(-D)$ for $-1 \leq t < -1/2$. Here $W_0 E = E$ and $W_{-2} E = E$.

From the parabolic structure we get

$$E' = \widetilde{E}(B).$$

Thus, the degree 0 piece is $\widetilde{E}(B)$ and the degree 1 piece is $\widetilde{E}(B) \otimes \Omega_{\widetilde{X}/S}$ (here as above the $-\widetilde{D}$ and $\widetilde{D}$ cancel).

For the degree 0 piece,

$$R\eta_* \widetilde{E}(B) = E.$$

This is because the bundle $\widetilde{E}(B)$ restricted to the exceptional $B \cong \mathbb{P}^1$ is of the form $\mathcal{O}_{\mathbb{P}^1}(-1)$ that has vanishing H^1 so the higher direct image is zero, and the direct image is the space of sections of E by Hartogs' theorem.

Also note that

$$\eta^* \mathcal{O}_X(D) = \eta^{B,*} \mathcal{O}_{X^A}(D_{X^A} + A_{X^A}) = \mathcal{O}_{\widetilde{X}}(\widetilde{D} + A + 2B).$$

Thus, for the degree 1 piece we get

$$W_{-2}E' \otimes \Omega_{\widetilde{X}/S}(\widetilde{D}) = \eta^*(E \otimes \Omega_{X/S})(A + 2B + \widetilde{D}).$$

$$= \eta^*(E \otimes \omega_{X/S}(D)).$$

We first take the $R\eta^B_*$ from $\widetilde{X}$ down to X^A. This yields

$$\eta^{A,*}(E \otimes \omega_{X/S})(D))$$

and then taking the direct image down to X gives $E \otimes \omega_{X/S}(D)$.

In conclusion, back to the case when E has rank two so it is a direct sum of line bundles for the two cases discussed above, if we define the locally free extension Dolbeault complex $Dol(E., X/S)_{\rm lf}$ as at the start of this subsection, then the higher direct image of $Dol(E., X/S)_{\rm lf}$ from X to S calculates F_0. For the line bundle L with parabolic level 0, $E_0 = L$ is the bundle and $E_{-1/2} = L(-D)$ so $E_{-1/2} \otimes \omega_{X/S}(D) = L \otimes \omega_{X/S}$. For the line bundle L' with parabolic level $-1/2$ we have $E_0 = L'$ and $E_{-1/2} = L'$ so $E_{-1/2} \otimes \omega_{X/S}(D) = L' \otimes \omega_{X/S}(D)$.

12.4.2 Alternative method

For comparison, we also do the calculations by going to a double cover $\alpha : Z \to X$ ramified over D, with involution $\sigma : Z \to Z$. Let $R \subset Z$ be the upper ramification divisor mapping isomorphically to D.

The parabolic pullback of E to Z is a bundle that we denote E_Z, given by

$$E_Z = \ker\left(\alpha^*(E)(R) \to Q(R)\right).$$

This has a Higgs field without poles. It projects to a relative Higgs field

$$\varphi_Z : E_Z \to E_Z \otimes \omega_{Z/Y},$$

where we recall that $\omega_{Z/Y} = \Omega^1_Z(\log C)/f_Z^*\Omega^1_Y(\log 0)$.

The upstairs Dolbeault complex is

$$Dol_Z = \left[E_Z \to E_Z \otimes \omega_{Z/Y}\right].$$

This has an action of σ covering the action on Z. We can take the direct image down to X, and consider the σ-invariant part

$$(\eta_* Dol_Z)^+.$$

This is a direct summand in the complex of locally free sheaves $\eta_* Dol_Z$ so it is itself a complex of locally free sheaves.

The first term in the complex is just

$$(\eta_* E_Z)^+ = E.$$

We also note that

$$\omega_{Z/Y} = \eta^* \omega_{X/Y}(R).$$

Thus, the second term in the complex is

$$(\eta_*((\eta^{par,*}(E \otimes \omega_{X/Y}))(R)).$$

We have

$$(\eta^{par,*}(E) \otimes \eta^* \omega_{X/Y}))(R) = \eta^* \omega_{X/Y}) \otimes \ker(\eta^*(E)(2R) \to Q(2R))$$

and when we take the invariant part of the direct image back to X we get

$$\omega_{X/Y} \otimes \ker(E(D) \to Q(D)) = E_{-1/2} \otimes \omega_{X/Y}(D).$$

This is the second sheaf in the Dolbeault complex, and it is indeed the reflexive extension of the sheaf we would get away from the ramification point of D by taking $W_{-2}E \otimes \Omega^1_{X/Y}(\log D)$.

So, we conclude in this calculation that the Dolbeault complex on X/Y is just the reflexive i.e. locally free extension of the one we would get by the usual formula away from the ramification point.

12.4.3 The nilpotent case

We now consider the case of a Higgs bundle with nontrivial nilpotent residue and no parabolic structure. We still want to decompose into line bundles. For this, introduce the notion of weight-filtered bundle, a bundle with weight filtrations (inspired by mixed Hodge weight filtrations) on the parabolic graded pieces. In this case the parabolic level filtration is trivial so the parabolic graded piece is just the restriction of the bundle to D. The weight filtration determines the pieces of the Dolbeault complex, and we can look at a direct sum decomposition into line bundles.

In our case the "mixed Hodge" weights will be 1 and -1, since the residue of the Higgs field is a nonzero nilpotent 2×2 matrix so its monodromy weight filtration has those weights.

For both rank 1 pieces the bundle E'_0 is the same as $\widetilde{E} = \eta^* E$ since there was no parabolic structure.

Let $C \subset X$ denote the fiber over the origin of Y, and let $\widetilde{C}$ be its strict transform in $\widetilde{X}$.

Putting parabolic level $-1/2$ along the fiber over Y yields

$$E'_{-1/2} = \widetilde{E}(-A - B - \widetilde{C})$$

Note that E_{-1} is $\widetilde{E}(-A - 2B - \widetilde{C})$, that is just minus the fiber.

Recall that

$$\mathcal{O}_{\widetilde{X}}(-\widetilde{D}) = \eta^*(\mathcal{O}_X(-D))(A + 2B).$$

Also similarly

$$\mathcal{O}_{\widetilde{X}}(-\widetilde{C}) = \eta^*(\mathcal{O}_X(-C))(A + 2B).$$

Consider a line bundle E with mixed weight 1. Then $W_0 E = E(-D)$ and $W_{-2} E = E(-D)$ (away from the origin).

In degree 0 our bundle is

$$W_0 E'_0 = \eta^*(E(-D))(A + 2B) = \eta^{B,*}((\eta^{A,*} E(-D))(A_{X^A}))(B)$$

since $\eta^{B,*}(\mathcal{O}_{X^A}(A_{X^A}) = \mathcal{O}_{\widetilde{X}}(A + B)$. So the direct image down to X^A is locally free and then the direct image down to X is locally free, and we get

$$R\eta_* W_0 E'_0 = E(-D).$$

For level $-1/2$ we have

$$W_0 E'_{-1/2} = \eta^*(E(-D))(A + 2B - A - B - \widetilde{C}) = \eta^*(E(-D))(B - \widetilde{C}) = \eta^*(E(-D - C))(A + 3B).$$

In this case, there is an $R^1\eta_*$ term. Because of that we'll do a modified version of the calculation later.

Look now at the degree 1 piece. We have

$$W_{-2} E'_0 = \eta^*(E(-D))(A + 2B).$$

Also recall

$$\Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^* \Omega_{X/Y}(A + B + \widetilde{D}) = \eta^*(\Omega_{X/Y}(D))(-B).$$

Putting these together, the degree 1 piece is

$$W_{-2} E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*(E \otimes \Omega_{X/Y}))(A + B).$$

When we push down to X^A then to X the result is locally free, equal to

$$R\eta_* W_{-2}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}).$$

For the parabolic level $-1/2$ piece we have

$$W_{-2}E'_{-1/2} = \eta^*(E(-D))(A+2B)(-A-B-\widetilde{C}) = \eta^*(E(-D))(B-\widetilde{C}),$$

so

$$W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D})$$
$$= W_{-2}E'_{-1/2} \otimes \eta^*(\Omega_{X/Y}(D))(-A-2B)(A+B) = \eta^*(E \otimes \Omega_{X/Y})(-\widetilde{C})$$
$$= \eta^*(E \otimes \Omega_{X/Y}(-C))(A+2B).$$

Done differently we note that

$$W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \widetilde{E}(-A-B-\widetilde{C}-\widetilde{D}) \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D})$$
$$= \widetilde{E}(-A-B-\widetilde{C}) \otimes \Omega_{\widetilde{X}/Y} = \widetilde{E}(-A-B-\widetilde{C}) \otimes \eta^*(\Omega_{X/Y})(A+B)$$
$$= \eta^*(E \otimes \Omega_{X/Y}(-C))(A+2B).$$

The higher direct image down to X is locally free as seen by doing in two stages, and

$$R\eta_* W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}(-C).$$

Let's look at the case of mixed weight -1. In this case $W_{-2}E' = E'(-\widetilde{D})$ and $W_0E = E$.

The degree zero piece in parabolic level zero yields just

$$\eta^*(E)$$

and the direct image down to X is just E.

Consider the parabolic level $-1/2$ piece. Upstairs on $\widetilde{X}$ this is

$$\eta^*(E)(-A-B-\widetilde{C}) = \eta^*(E(-C))(B).$$

This has direct image $E(-C)$ down on X.

Look now at the degree 1 piece. We have as before

$$W_{-2}E'_0 = \eta^*(E(-D))(A+2B).$$

Also recall

$$\Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*\Omega_{X/Y}(A + B + \widetilde{D}) = \eta^*(\Omega_{X/Y}(D))(-B).$$

Putting these together, the degree 1 piece is

$$W_{-2}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*(E \otimes \Omega_{X/Y}))(A + B).$$

As before when we push down to X^A then to X the result is locally free, equal to

$$R\eta_* W_{-2}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}).$$

For the parabolic level $-1/2$ piece we have

$$W_{-2}E'_{-1/2} = \eta^*(E(-D))(A + 2B)(-A - B - \widetilde{C}) = \eta^*(E(-D))(B - \widetilde{C}),$$

so

$$\begin{aligned} W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) &= \eta^*(E \otimes \Omega_{X/Y})(-\widetilde{C}) \\ &= \eta^*(E \otimes \Omega_{X/Y}(-C))(A + 2B). \end{aligned}$$

Done differently we note that

$$W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \widetilde{E}(-A - B - \widetilde{C} - \widetilde{D}) \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D})$$
$$= \widetilde{E}(-A - B - \widetilde{C}) \otimes \Omega_{\widetilde{X}/Y} = \widetilde{E}(-A - B - \widetilde{C}) \otimes \eta^*(\Omega_{X/Y})(A + B)$$
$$= \eta^*(E \otimes \Omega_{X/Y}(-C))(A + 2B).$$

The higher direct image down to X is locally free as seen by doing in two stages, and

$$R\eta_* W_{-2}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}(-C).$$

12.4.4 Modified version

Since we had an $R^1\eta_*$ term, let's do an alternative version of the calculation. We note that, in the case where the mixed weighted bundles come from a φ, we could replace W_0E and $W_{-2}E$ by W_1E and $W_{-1}E$. in degrees 0 and 1 respectively. This observation originally due to Zucker [Zuc79] for L^2 Dolbeault complexes of VHS was explained for the present setting in [DPS16]. So let's look at those.

Suppose E has mixed weight 1. Then (away from the origin) $W_1E = E$ and $W_{-1}E = E(-D)$. We did this calculation above (for the mixed weight -1 case) and for the degree 0 part we got:

$$W_1E'_0 = \eta^*(E) \quad R\eta_* W_1E'_0 = E$$

and

$$W_1E'_{-1/2} = \eta^*(E)(-A-B-\widetilde{C}) = \eta^*(E(-C))(B)$$

giving $R\eta_*W_1E'_{-1/2} = E(-C)$.

For the degree 1 part we got:

$$W_{-1}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*(E \otimes \Omega_{X/Y}))(A+B)$$

so

$$R\eta_*W_{-1}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}.$$

At parabolic level $-1/2$ we have

$$W_{-1}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*(E \otimes \Omega_{X/Y}(-C))(A+2B),$$

so

$$R\eta_*W_{-1}E'_{-1/2} \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = E \otimes \Omega_{X/Y}(-C).$$

Turn now to the case of mixed weight -1. In this case,

$$W_1E = E, \quad W_{-1}E = E.$$

For the degree 0 piece the calculation is the same as above:

$$R\eta_*W_1E'_0 = E \quad R\eta_*W_1E'_{-1/2} = E(-C).$$

For degree 1 we have

$$W_{-1}E'_0 \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D}) = \eta^*(E) \otimes \Omega_{\widetilde{X}/Y}(\widetilde{D})$$

$$= \eta^*(E \otimes \Omega_{X/Y}(D))(A+B-A-2B) = \eta^*(E \otimes \Omega_{X/Y}(D))(-B).$$

The higher direct image down to X^A is

$$\eta^{A,*}(E \otimes \Omega_{X/Y}(D))(-p_A)$$

where p_A is the point that is blown up the second time. This fits in an exact sequence

$$0 \to \eta^{A,*}(E \otimes \Omega_{X/Y}(D))(-p_A) \to \eta^{A,*}(E \otimes \Omega_{X/Y}(D)) \to \mathbb{C}_{p_A} \to 0.$$

Taking the direct image down to X we get a long exact sequence that shows

$$R^1\eta^A_*\eta^{A,*}(E \otimes \Omega_{X/Y}(D))(-p_A)$$

as the cokernel of the map

$$R^0\eta^A_*\eta^{A,*}(E\otimes\Omega_{X/Y}(D))\to\mathbb{C}_p$$

induced by the previous map. (with p being the origin). We have

$$R^0\eta^A_*\eta^{A,*}(E\otimes\Omega_{X/Y}(D))=E\otimes\Omega_{X/Y}(D)$$

and a local section not vanishing at the origin corresponds to a section of $\eta^{A,*}(E\otimes\Omega_{X/Y}(D))$ that does not vanish at p_A. Therefore, this map is surjective and the cokernel is 0. We get

$$R^1\eta^A_*\eta^{A,*}(E\otimes\Omega_{X/Y}(D))(-p_A)=0$$

and

$$R^0\eta^A_*\eta^{A,*}(E\otimes\Omega_{X/Y}(D))(-p_A)=E\otimes\Omega_{X/Y}(D)(-p).$$

We conclude that in the mixed weight -1 case, for parabolic level 0 and in degree 1,

$$R\eta_*W_{-1}E'_0\otimes\Omega_{\widetilde{X}/Y}(\widetilde{D})=E\otimes\Omega_{X/Y}(D)(-p).$$

Look at the case of parabolic level $-1/2$. Now,

$$W_{-1}E'_{-1/2}\otimes\Omega_{\widetilde{X}/Y}(\widetilde{D})=\eta^*(E)\otimes\Omega_{\widetilde{X}/Y}(\widetilde{D})(-A-B-\widetilde{C})$$

$$=\eta^*(E\otimes\Omega_{X/Y}(D))(A+B-A-2B-A-B-\widetilde{C})=\eta^*(E\otimes\Omega_{X/Y}(D-C)).$$

Then

$$R\eta_*W_{-1}E'_{-1/2}\otimes\Omega_{\widetilde{X}/Y}(\widetilde{D})=(E\otimes\Omega_{X/Y}(D-C).$$

12.4.5 Conclusion for the Dolbeault complexes

Let's now put this back into the situation of our rank 2 bundle E with no parabolic structure but a Higgs field whose residue is nilpotent. We'll denote by W_iE the weight filtered bundles given their reflexive extensions as bundles across the origin.

We get, for the modified Dolbeault complex Dol' using W_1 and W_{-1}:

$$Dol'_0=\left[W_1E\longrightarrow W_{-1}E\otimes\Omega_{X/Y}(D)\longrightarrow W_{-1}(E_D\otimes\ldots)_p\right]$$

and

$$Dol'_{-1/2}=\left[W_1E(-C)\longrightarrow W_{-1}E\otimes\Omega_{X/Y}(D-C)\right].$$

We could also write the parabolic level 0 piece as follows. Introduce notation for the kernel

$$0 \to W_{-1}^{-p}E \to W_{-1}E \to W_{-1}(E_D)_p \to 0$$

then

$$Dol'_0 = \left[W_1E \longrightarrow W_{-1}^{-p}E \otimes \Omega_{X/Y}(D)\right].$$

We note that, since the divisor D becomes vertical at the point p, the nonzero residue map of the Higgs field at that point becomes zero when projected into the relative differentials, so φ does induce a map

$$W_1E \xrightarrow{\varphi} W_{-1}^{-p}E \otimes \Omega_{X/Y}(D)$$

to be used for the above complex.

12.5 Smooth spectral variety

We now specialize the previous calculations further, to the local situation where our rank 2 Higgs bundle[8] is the direct image of a line bundle $\mathcal{L}$ on its spectral variety $p : \Sigma \to X$ where p has degree 2 with simple ramification over the divisor D. Let $R \subset \Sigma$ be the reduced preimage of D, so it is a smooth divisor in Σ mapping isomorphically to D.

We have

$$p^*\Omega^1_X(\log D) = \Omega^1_\Sigma(\log R).$$

thus we have a map

$$\Omega^1_\Sigma \to p^*\Omega^1_X(\log D).$$

We suppose that the Higgs field on $E = p_*\mathcal{L}$ is given by multiplication by a spectral 1-form viewed as a section $\alpha \in H^0(\Sigma, \Omega^1_\Sigma)$.

Choose local coordinates x, y on X, such that the map $X \to S$ is given by $(x, y) \mapsto y$. Choose local coordinates (x, z) on Σ such that the map $\Sigma \to X$ is given by

$$(x, z) \mapsto (x, y) = (x, x^2 - z^2).$$

The divisor R is given by $z = 0$ and D is given by $y = x^2$.

Write

$$\alpha = a(x, z)dx + b(x, z)dz.$$

[8] Recall that we are working in a local neighborhood—in the applications to Hecke transformations the spectral variety will have higher degree, decomposing into a disjoint union of local pieces that are either etale or of the present rank 2 form.

The projection $\alpha_{\Sigma/S}$ of α to a section of $\omega_{\Sigma/S}$ is given as follows. The relative differentials are the reflexive closure of

$$\mathcal{O}_\Sigma\langle dx, dz\rangle/\langle d(x+z)(x-z)\rangle$$
$$= \mathcal{O}_\Sigma\langle du/u, dv/v\rangle/\langle du/u + dv/v\rangle$$

where $u = x+z$ and $v = x-z$. Thus we may think of du/u as a generator of $\omega_{\Sigma/S}$ subject to the relation $dv/v = -du/u$.

We have
$$\alpha = \frac{a+b}{2}du + \frac{a-b}{2}dv$$
and
$$\alpha_{\Sigma/S} = \left[u\frac{a+b}{2} + v\frac{a-b}{2}\right]\frac{du}{u}.$$

We assume that a and b take generic nonzero values at the origin.

Then the curve G defined by $(\alpha_{\Sigma/S} = 0)$ is smooth at the origin.

Also, under this genericity hypothesis the residue of the resulting Higgs field is nontrivially nilpotent along D.

There are two cases. In the first case, we have a nontrivial parabolic level $-1/2$, in which case the parabolic structure is given by the quotient

$$E_D = p_*\mathcal{L}|_{2R} \to p_*\mathcal{L}_R$$

over D. We have $E_{-1/2} = p_*\mathcal{L}(-R)$.

In the second case, we have no parabolic structure but the weight filtration is given by $W_{-1} = W_0$ being the kernel of the above map, and W_1/W_0 is the quotient $\mathcal{L}_R$. We have $W_{-1}E = p_*\mathcal{L}(-R)$.

12.5.1 The parabolic case

We now plug in the previous calculations. Consider first the case with parabolic level $-1/2$. Then we saw in Subsections 12.4.1 and 12.4.2 that

$$Dol(E, X/S)_0 = \left[E_0 \to E_{-1/2} \otimes \omega_{X/S}(D)\right].$$

We have $E_0 = p_*\mathcal{L}$ and $E_{-1/2} = p_*\mathcal{L}(-R)$, whereas

$$p^*\omega_{X/S}(D) \cong \omega_{\Sigma/S}(R).$$

We may therefore write

$$Dol(E, X/S)_0 = p_* \left[\mathcal{L} \xrightarrow{\alpha_{\Sigma/S}} \mathcal{L} \otimes \omega_{\Sigma/S} \right].$$

In turn, this complex is quasiisomorphic to the second bundle restricted over the curve G of zeros of the differential here:

$$Dol(E, X/S)_0 \overset{q.i.}{\sim} \mathcal{L} \otimes \omega_{\Sigma/S}|_G[-1].$$

Taking the direct image down to S we obtain G as spectral variety (it needs to be checked that it is the spectral variety of the Higgs field over S) with spectral line bundle being the restriction of $\mathcal{L} \otimes \omega_{\Sigma/S}$ to G.
Caution: This is a local calculation near the ramification point of the horizontal divisor, and on branches of the curve G that pass through the ramification point. The formula might be different at other points where G passes through points of Σ on other branches lying over D.

12.5.2 The nilpotent case

Next consider the case where there is no parabolic structure but the residue of the connection is a nontrivial nilpotent transformation. The weight filtration is given by the quotient. In this case, there are two pieces of the Dolbeault complex (modified as in Subsections 12.4.4 and 12.4.5) to consider.

First, we had

$$Dol(X/S, E)_0 = \left[W_1 E \to W_{-1} E \otimes \Omega^1_{X/S}(D)(-p) \right]$$

where the $(-p)$ indicates that we take the kernel of the map to the fiber of the quotient sheaf over p.

We have $W_1 E = p_* \mathcal{L}$ and $W_{-1} E = p_* \mathcal{L}(-R)$. Thus

$$W_{-1} E \otimes \Omega^1_{X/S}(D)(-p) = p_*(\mathcal{L} \otimes \omega_{\Sigma/S}(-q))$$

where here

$$0 \to \mathcal{L} \otimes \omega_{\Sigma/S}(-q) \to \mathcal{L} \otimes \omega_{\Sigma/S} \to \mathbb{C}_q \to 0$$

is the ideal sheaf kernel of the evaluation at the point $q \in \Sigma$ lying over the origin.

We get

$$Dol(X/S, E)_0 = p_* \left[\mathcal{L} \xrightarrow{\alpha_{\Sigma/S}} \mathcal{L} \otimes \omega_{\Sigma/S}(-q) \right].$$

This is quasi-isomorphic to the second piece restricted over the curve G:

$$Dol(X/S,E)_0 \overset{q.i.}{\sim} (\mathcal{L}\otimes\omega_{\Sigma/S})|_G(-q)[-1]$$

We obtain the following statement:

So, with the above notations in the situation where the original Higgs bundle had trivial parabolic structure and nilpotent residue, the line bundle on G yielding the level 0 part of the parabolic bundle is

$$F_0 = g_*(\mathcal{L}\otimes\omega_{\Sigma/S}|_G(-q))$$

where $g: G\to S$ is the covering.

We recall that the level $-1/2$ Dolbeault complex was the same, but without the $-p$, then twisted by $-C$ where $C = f^{-1}(0)\subset X$ was the fiber. When we restrict to G the fiber becomes $g^{-1}(0) = 2q$ so putting these together gives

$$Dol(X/S,E)_{-1/2} \overset{q.i.}{\sim} (\mathcal{L}\otimes\omega_{\Sigma/S}|_G(-2q)[-1])$$

and hence

$$F_{-1/2} = g_*(\mathcal{L}\otimes\omega_{\Sigma/S}|_G(-2q)).$$

Caution: As before, this is only a local calculation near the ramification point of the horizontal divisor, and on branches of the curve G that pass through the ramification point. Again, the formula might be different at other points where G passes through points of Σ on other branches lying over D.

12.5.3 Rephrasing

Let's rephrase the above calculations in terms of the pullback to $G\subset\Sigma$ of the relative differentials $\omega_{X/S}$. This is useful because, at other points, this is the relevant term.

We have, as was used above,

$$\omega_{\Sigma/S}(R) = p^*(\omega_{X/S}(D)) = (p^*\omega_{X/S})(2R).$$

This gives

$$\omega_{\Sigma/S} = (p^*\omega_{X/S})(R).$$

On the curve G, the divisor R is the same as q. We conclude the following statement.

Proposition 12.7. *Suppose given a Higgs bundle with smooth spectral variety Σ having simple ramification over a horizontal divisor D_H such that D_H s smooth but has a simple ramification over the base S. We have considered two ways of making a parabolic Higgs bundle with a spectral line bundle $\mathcal{L}$. These result in a higher direct image Higgs bundle F. whose local expression near this point is as follows.*

In the case of parabolic levels $0, -1/2$ along D_H for the Higgs bundle on X, we get

$$F_0 = g_*(\mathcal{L} \otimes j^*\omega_{X/S}(q))$$

where $j : G \to X$ denotes the composed map. Recall here that $\mathcal{L}$ being the spectral line bundle means that its direct image to X is the level 0 piece of the parabolic structure.

In the case where the parabolic levels along D_H are trivial but the residue of the Higgs field along D_H is nilpotent, we get

$$F_0 = g_*(\mathcal{L} \otimes j^*\omega_{X/S})$$

and

$$F_{-1/2} = g_*(\mathcal{L} \otimes j^*\omega_{X/S}(-q)).$$

Remark 12.8. The result of the proposition is local near a singular point. The statement refers to the way of filling in the structure of the spectral line bundle on the spectral covering G over S, noting that over a general point of S we have a canonical identification between the spectral line bundle on G, and the restriction of $\mathcal{L}$ to G tensored with $\omega_{X/S}$.

12.6 Globalization

Let's now look at a global situation. The notations and hypotheses of Subsection 3.11 are heretofore in effect. Thus, $f : X \to S$ is a map from a smooth projective surface to a smooth projective curve, and $D \subset X$ is a simple normal crossings divisor, $T \subset S$ is a divisor consisting of a finite set of points $t_1, \ldots, t_k$, and let $K \subset X$ be a closed subset containing the "other" points of type 3.11.1(f).

Write $D = D_H + D'_V$ such that the irreducible components of D_H dominate S and the irreducible components of D'_V map into T. We let D_V denote the full inverse image of T.

Assume that $f(K)$ is a finite subset of S.

Suppose that for any point $x \in X$, one of the following holds:

1. $x \in X - D$ and f is either smooth at x (type 3.11.1(a))or has a simple normal crossing (type 3.11.1(b));

2. $x \in D_H$, f is smooth at x and D_H is étale over S at x (type 3.11.1(c));

3. $x \in D_H$, D_H is smooth at x, f is smooth at x, $f(x) = t_i \in T$, and $f|_{D_H}$ has a simple ramification point at x (type 3.11.1(d));

4. or $x \in K$ (type 3.11.1(f)).

For the moment, this supposes that there aren't any points of type 3.11.1(e); those will be treated in the next subsection.

Notice that $D'_V \subset K$.

Suppose (E, φ) is a parabolic logarithmic Higgs bundle over (X, D_H). We assume that over D_H, either the parabolic structure is trivial and φ has nilpotent residue, or it has levels $0, -1/2$ and φ is strictly parabolic.

Let $p : \Sigma \to X$ be the spectral covering of (E, φ) with spectral line bundle $\mathcal{L}$ over Σ so that $E_0 = p_*(\mathcal{L})$, and spectral 1-form α inducing the Higgs field φ.

We assume that away from the subset K, Σ is smooth and the covering p has at most simple ramification; let $R \subset \Sigma$ be the reduced inverse image of D_H, assume R is smooth and p has simple ramification along R away from points of K.

Let $\alpha_{\Sigma/S}$ be the relative spectral form viewed as a section of $p^*\omega_{X/S}(R)$. Let G be the curve of zeros of this form, defined away from K.

We have the following hypothesis: that the closure in X of the image of G does not meet K. In particular G is proper. Let $g : G \to S$ be the map, factoring as $g = f \circ j$ through the morphism $j : G \to X$. Let $\mathcal{L}|_G$ be the restriction of $\mathcal{L}$ to G.

Let $Q := R \cap G$ be the trace of the reduced divisor R onto the curve G.

We are also assuming that the restriction of the spectral 1-form α to the vertical direction in Σ over a point of type 3.11.1(d) is nonzero.

Lemma 12.9. *The hypothesis that α is nonzero in the vertical direction at implies that G is transverse to R at points of type 3.11.1(d), so Q is reduced at such points.*

Proof. Let's calculate in coordinates x, t on X, such that t gives the map $X \to S$. Assume the horizontal divisor is $x^2 - t = 0$ and the covering Σ is $w^2 = x^2 - t$. Thus, x and w give a system of coordinates on Σ. Write $\alpha = adx + bdw$ with $a = a(x, w)$ and $b = b(x, w)$ holomorphic functions of x, w. The equation $t = x^2 - w^2$ tells us that dividing out by dt is equivalent to setting $xdx = wdw$. The form dx provides a frame for the sheaf of relative differentials $\Omega^1_{X/S}$ and in terms of this frame,

$$\alpha^{\mathrm{rel}} = (a + bx/w)dx.$$

Thus, the equation for the upper critical locus $\alpha^{\mathrm{rel}} = 0$ becomes

$$aw + bx = 0.$$

The hypothesis $b(0,0) \neq 0$ implies that the linear term of this equation at the origin is nonzero. The ramification divisor R is given by $w = 0$, that is to say it is the x-axis in this coordinate system, and $b(0,0) \neq 0$ tells us that the above equation has a simple zero along the x-axis, so the zero set G is transverse to R. □

The following statement gives Theorem 3.17 in the case when there aren't any points of type 3.11.1(e).

Theorem 12.10. *With all the above hypotheses, in the case of parabolic levels* $0, -1/2$ *we have*

$$F_0 = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \otimes \mathcal{O}_G(Q) \right).$$

In the nilpotent case,

$$F_0 = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \right)$$

and

$$F_{-1/2} = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \otimes \mathcal{O}_G(-Q) \right).$$

Proof. In the nilpotent case, G does not intersect R at points distinct from ramification points, because we know that the map $gr^W_1 \to gr^W_{-1}$ is an isomorphism at any point of D_H etale over the base. We notice that, at points of G mapping to points of D_H but in sheets of Σ that are étale, the formula is as stated since the term of the Dolbeault complex has $W_{-1}E = E(-D_H)$ and $\Omega^1_{X/S}(\log D_H) = \omega_{X/S}(D_H)$ locally on Σ at those points.

Also, the divisor consisting of ramification points q is the same as $Q = R \cap G$ in this case, in view of the non-intersection of G with other points of R. Thus, for the nilpotent case our previous calculations give the required results.

For the parabolic case, there might be points where G intersects R at points where D_H is étale over S. In this case, the component of $E_{-1/2}$ coming from the local neighborhood in Σ is $p_*(\mathcal{L}(-R))$ and

$$p^*\Omega^1_{X/S}(\log D_H) = (p^*\omega_{X/S})(2R)$$

so the required bundle in the degree 1 term of the Dolbeault complex is

$$p_*(\mathcal{L} \otimes p^*\omega_{X/S}(R)).$$

When we restrict to G, we get the line bundle

$$\mathcal{L}|_G \otimes j^*\omega_{X/S} \otimes \mathcal{O}_G(Q)$$

at these points. At the ramification points of D_H, we notice that Q is the same as the divisor of ramification points q considered above. Therefore, near the ramification points the required bundle is also

$$\mathcal{L}|_G \otimes j^*\omega_{X/S} \otimes \mathcal{O}_G(Q).$$

This yields the stated formula. □

12.7 Type 3.11.1(e) points

The calculation needs to be extended to cover the points of type 3.11.1(e) in the classification of Subsection 3.11. Suppose $x \in X$ is a point of type 3.11.1(e). Recall this means the horizontal divisor D has a node at x, with both branches etale over S, and the spectral variety $\Sigma \to X$ decomposes into a collection of ordinary double points over x.

Choose a neighborhood $x \in U \subset X$, and let ${}^U\Sigma \subset \Sigma$ be a neighborhood of one of the double points over x.

Consider the blow-up map $b : {}^U\widetilde{X} \to U \subset X$. Let Z be the normalization of ${}^U\widetilde{\Sigma} := {}^U\Sigma \times_U {}^U\widetilde{X}$. Notice that Z is the blow-up of ${}^U\Sigma$ at the double point.

Let D^+ and D^- be the two branches of the strict transform of D, and let F be the strict transform of the fiber. Let $B \subset \widetilde{X}$ be the exceptional divisor and $C \subset Z$ its inverse image in Z. Thus $B \cong \mathbb{P}^1$ and $C \cong \mathbb{P}^1$ with the map $C \to B$ a double cover ramified over the points $d^+ := D^+ \cap B$ and $d^- := D^- \cap B$.

Let D^+_Z and D^-_Z denote the reduced inverse images of these divisors in Z.

Let $\mathcal{L}$ be the spectral line bundle on Σ, with $\mathcal{L}_Z$ its pull-back to Z. Let $E = \varpi_*(\mathcal{L})$ be the Higgs bundle on $U \subset X$, with $\widetilde{E} := b^*(E)$ its inverse image on ${}^U\widetilde{X}$. Since we're working with neighborhoods both downstairs and upstairs, E has rank 2 here.

The assumption on the nilpotent residue of the Higgs field φ means that there is a decomposition $E = E_1 \oplus E_{-1}$ into a direct sum of two line bundles, such that the image of the residue of φ is E_{-1} over both branches of the divisor D. The pullback decomposes as $\widetilde{E} = \widetilde{E}_1 \oplus \widetilde{E}_{-1}$.

Let $p : Z \to {}^U\widetilde{X}$ denote the map. The bundle $\widetilde{E}$ is obtained from $p_*(\mathcal{L}_Z(-C))$ by glueing together the line bundles on the two branches of Z over general points of B. Since $\mathcal{L}_Z$ is pulled back from a line bundle on Σ, the restriction $\mathcal{L}_Z|_B$ is a trivial line bundle that we'll denote $\mathcal{L}_B$. The glueing is done using this trivialization, so we have an exact sequence

$$0 \to \widetilde{E} \to p_*(\mathcal{L}_Z) \to (-) \to 0$$

where the cokernel is also the cokernel in the sequence

$$0 \to \mathcal{L}_B \to p_*(\mathcal{L}_Z) \to (-) \to 0$$

over B. The exact sequence of the elementary transformation is

$$0 \to p_*(\mathcal{L}_Z(-C)) \to \widetilde{E} \to \mathcal{L}_B \to 0.$$

Notice now that $\mathcal{L}_B \cong \widetilde{E}_{1,B}$ as quotient of $\widetilde{E}|_B$, as it is the trivial subbundle whose values over the two ramification points correspond to the unramified part.

On the subsheaf given by the weight filtration we get

$$0 \to p_*(\mathcal{L}_Z(-C - D_Z^+ - D_Z^-)) \to W_{-1}\widetilde{E} \to \widetilde{E}_{1,B}(-d^+ - d^-) \to 0.$$

The sheaf of relative logarithmic differentials is

$$\omega_{\widetilde{X}/S}(\log) = b^*\omega_{X/S}(D^+ + D^- + B).$$

Pulling back to Z gives

$$p^*\omega_{\widetilde{X}/S}(\log) = (bp)^*\omega_{X/S} \otimes \mathcal{O}_Z(2D_Z^+ + 2D_Z^- + C).$$

Also, $\omega_{\widetilde{X}/S}(\log)|_B \cong \mathcal{O}_B(1)$. We have

$$\begin{aligned} p_*(\mathcal{L}_Z(-C - D_Z^+ - D_Z^-)) \otimes \omega_{\widetilde{X}/S}(\log) &= p_*(\mathcal{L}_Z(-C - D_Z^+ - D_Z^- + 2D_Z^+ + 2D_Z^- + C)) \otimes b^*\omega_{X/S} \\ &= p_*(\mathcal{L}_Z(D_Z^+ + D_Z^-)) \otimes b^*\omega_{X/S}. \end{aligned}$$

Thus we have an exact sequence

$$0 \to p_*(\mathcal{L}_Z(D_Z^+ + D_Z^-)) \otimes b^*\omega_{X/S} \to W_{-1}\widetilde{E} \otimes \omega_{\widetilde{X}/S}(\log) \to \widetilde{E}_{1,B}(-d^+ - d^-) \otimes \mathcal{O}_B(1) \to 0.$$

Lemma 12.11. *Suppose $G \subset \Sigma$ is a curve that decomposes, near the double point, into a disjoint union of two smooth branches whose tangent vectors at the double point are distinct. Then the map on spaces of sections on the neighborhood*

$$\Gamma(U, p_*(\mathcal{L}_Z(D_Z^+ + D_Z^-)) \otimes b^*\omega_{X/S}) \to (G \cap {}^U\Sigma, \mathcal{L}|_G \otimes \omega_{X/S}|_G)$$

is an isomorphism.

Proof. The curve G is isomorphic to its strict transform inside Z. The two branches intersect C in different points. The line bundle $\mathcal{L}_Z(D_Z^+ + D_Z^-)$ is $\mathcal{O}_C(2)$ on C, so its sections (which extend to local sections in a neighborhood) span the space of sections over G. $\square$

Theorem 12.12. *In the presence of points of type 3.11.1(e), let F' be the parabolic bundle defined by*

$$F'_0 = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S}\right)$$

and

$$F'_{-1/2} = g_* \left(\mathcal{L}|_G \otimes j^* \omega_{X/S} \otimes \mathcal{O}_G(-Q)\right),$$

and let F be the parabolic Higgs bundle corresponding to the higher direct image local system. Then there is an injective morphism $F' \hookrightarrow F$.

Proof. The expression of the higher direct image Higgs bundle comes from a direct image of a sheaf supported on the relative critical locus, so it is local over the relative critical locus. Near a point of type 3.11.1(e), it comes from blowing up once $b : \widetilde{X} \to X$. There is a sheaf $\mathcal{G}$ supported on $\widetilde{X}$ whose direct image down to S is the local piece of F, and $\mathcal{G}$ is supported on the image of G union the exceptional divisor B. The expression is local on $\widetilde{\Sigma}$ in turn, and the sheaf $\mathcal{G}$ is a quotient of $W_{-1}\widetilde{E} \otimes \omega_{\widetilde{X}/S}(\log)$ (where here E is the piece of the full Higgs bundle, corresponding to the local piece of Σ). Thus, any sections of $W_{-1}\widetilde{E} \otimes \omega_{\widetilde{X}/S}(\log)$ over a neighborhood of B will generate a subsheaf of F.

The exact sequences above give a map from sections of $p_*(\mathcal{L}_Z(D_Z^+ + D_Z^-)) \otimes b^*\omega_{X/S}$ to sections of $W_{-1}\widetilde{E} \otimes \omega_{\widetilde{X}/S}(\log)$, and Lemma 12.11 says that these restrict on G to sections of $\mathcal{L}|_G \otimes \omega_{X/S}|_G)$. Those are the sections that appear in the definitions of F'_0 and $F'_{-1/2}$ (note that there is no difference in the two parabolic level spaces locally at a point of type 3.11.1(e)).

Let F' be the subsheaf of F generated by such sections near the points of type 3.11.1(e), and equal to F as calculated in Theorem 12.10 elsewhere. This is the stated subsheaf. $\square$

Remark 12.13. We'll apply this statement by calculating the degree of the parabolic sheaf F' given by Theorem 12.12. If it has parabolic degree 0, then since we know that F also has parabolic degree 0, the map $F' \hookrightarrow F$ is an isomorphism, and this will yield the same calculation as in Theorem 12.10 for the case when there are points of type 3.11.1(e).

13 Drinfeld's construction

Drinfeld's original construction of Hecke eigensheaves was done in [Dri83] and later consolidated and extended by Laumon [Lau95], Gaitsgory [Gai97, Gai15] and others.

In order to make a comparison with our above constructions, we'll give here a preliminary approach to the interpretation of Drinfeld's construction in the setting of Higgs bundles.

A main ingredient of Drinfeld's construction is the following general definition: given a local system Λ on C, we obtain a local system $\Lambda^{(\otimes m)}$ on the symmetric power $Sym^m(C)$ with singularities along the big diagonal.

In the case $\mathrm{rk}(\Lambda) = 2$, Drinfeld starts with $\Lambda^{(\otimes m)}$ as input and uses Radon transform to construct a local system on an open subset of a projective space bundle over Bun. The main theorem of [Dri83] says that it descends from the projective space bundle down to a perverse sheaf on the moduli stack **Bun**. Going to the coarse moduli spaces of stable bundles we get a local system on an open subset of the moduli space of stable bundles. We would like to calculate the Higgs sheaf associated to the Radon transform and use that to show that the Higgs sheaves we constructed above are the same as the ones corresponding to Drinfeld's perverse sheaf.

Fix a line bundle M on C of sufficiently high degree m. Denote by

$$P := |M| = \mathbb{P}H^0(C, M)$$

the linear system, i.e. the space of divisors $x = x_1 + \ldots + x_m$ on C such that $\mathcal{O}_C(x_1 + \ldots + x_m) \cong M$. It is the fiber over the point M of the map on the right here:

$$P \overset{i}{\hookrightarrow} Sym^m(C) \to Pic^m(C)$$

where we took the opportunity to give the name i to the inclusion.

Serre duality says that

$$H^0(C, M)^* \cong H^1(C, M^\vee \otimes \omega_C).$$

Set

$$Q := \mathbb{P}H^1(C, M^\vee \otimes \omega_C),$$

so Q is naturally the dual projective space to P. In other words, Q is identified with the space of hyperplanes in P and vice-versa.

In a certain sense dual to i is a map $\mathbf{L} : Q \to \mathbf{Bun}$ to the moduli stack, or with the same notation a rational map to the moduli space of stable bundles. This map associates to

$\xi \in H^1(C, M^\vee \otimes \omega_C)$, considered as an extension class i.e. an element of $Ext^1(M, \omega_C)$, the bundle in the middle of the corresponding extension

$$0 \to \omega_C \to E \to M \to 0.$$

It is a point in the moduli stack of rank 2 bundles having determinant $M \otimes \omega_C$. The appearance of ω_C on the left of the extensions we look at is just there to accommodate the twist in Serre duality.

Let $X_{M\otimes\omega_C}$ denote the coarse moduli space of semistable rank 2 bundles up to S-equivalence, having determinant $M \otimes \omega_C$. Thus $\mathbf{L}$ may be interpreted (losing some information) as a rational map $\mathbf{L} : Q \dashrightarrow X_{M\otimes\omega_C}$.

The duality between P and Q is reflected in the incidence correspondence

$$I \subset P \times Q, \quad I := \{(x, \xi) \text{ s.t. } x \in \xi \text{ i.e. } \langle \xi, x\rangle = 0\}.$$

Let $p : I \to P$ and $q : I \to Q$ be the projections. Thus, p induces an isomorphism between $q^{-1}(\xi)$ and the hyperplane in P associated to ξ, whereas q induces an isomorphism between $p^{-1}(x)$ and the hyperplane in Q associated to x.

Let $i : P \hookrightarrow Sym^m(C)$ be the inclusion. Drinfeld's basic *Radon transform* means constructing the perverse sheaf

$$\mathbf{Rad} := Rq_* \left(p^* i^* \Lambda^{(\otimes m)}\right)$$

on Q.

Let $\Delta_P^{(m)} \subset P$ be the intersection of the big diagonal in $Sym^m(C)$ with P. This is the singular set of $i^*\Lambda^{(\otimes m)}$. Thus, the singular set of $p^*i^*\Lambda^{(\otimes m)}$ on I is

$$D_H := p^{-1}(\Delta^{(m)}).$$

Let $U \subset Q$ be the open set over which D_H is a relative normal crossings divisor, that is to say where all strata of D are etale under the projection q. Then $\mathbf{Rad}_U := \mathbf{Rad}|_U$ is a local system over U.

The first main part of the geometric Langlands correspondence for rank 2 bundles, according to Drinfeld and Laumon, may be formulated as follows:

Theorem 13.1 (Drinfeld-Laumon [Dri83, Lau95])**.** *There is a Zariski open subset $U' \subset U$ such that the map $\mathbf{L}' : U' \to X_{M\otimes\omega_C}$ is well-defined, and maps into the complement of the wobbly locus* $\mathrm{Wob}_{M\otimes\omega_C} \subset X_{M\otimes\omega_C}$*. The restriction of $\mathbf{Rad}_U$ to U' denoted $\mathbf{Rad}_{U'}$ is a local system on U', constant on the fibers of $\mathbf{L}'$. It is therefore isomorphic to the pullback of a local* *system* $\mathbf{E}$ *on* $X_{M\otimes\omega_C} - \mathrm{Wob}_{M\otimes\omega_C}$*. That local system is a Hecke eigensheaf (in an appropriate sense taking into account the determinant* $M\otimes\omega_C$*).*

Another proof was given by Gaitsgory in his thesis [Gai97].

13.1 A spectral variety

The perverse sheaf **Rad** corresponds to a $\mathscr{D}$-module, and by Sabbah's and Mochizuki's theory [Moc07a, Moc07b, Sab05], it has a structure of purely imaginary pure twistor $\mathscr{D}$-module. Over the open subset $U'\subset Q$ the fiber at $\lambda=0$ in the twistor line, is a Higgs bundle. Let

$$\Sigma_{\mathbf{Rad},U'}\subset T^*U'$$

be the spectral variety of this Higgs bundle. We would like to express this in terms of the fiber of the Hitchin fibration.

Consider the moduli space $X_{M\otimes\omega_C}$ of semistable rank 2 bundles up to S-equivalence, with determinant $M\otimes\omega_C$, with the smooth open subset $X^\circ_{M\otimes\omega_C}$ of stable bundles. Let $M_{H,M\otimes\omega_C}$ denote the moduli space of semistable Higgs bundles with determinant $M\otimes\omega_C$, and let $h: M_{H,M\otimes\omega_C}\to\mathbb{A}^N$ be its Hitchin fibration.

A general point $b\in\mathbb{A}^N$ corresponds to a spectral curve $\widetilde{C}\xrightarrow{\pi} C$ provided with a tautological differential $\alpha\in H^0(\widetilde{C},\omega_{\widetilde{C}})$.

Let (E,φ) be the Higgs bundle associated to Λ, with spectral covering $\widetilde{C}$, spectral 1-form α and spectral line bundle U. Denote by $(E^{(m)},\varphi^{(m)})$ the Higgs bundle associated to $\Lambda^{(\otimes m)}$, that we view as having a parabolic structure in codimension ≤ 1.

The spectral covering of $(E^{(m)},\varphi^{(m)})$ is described as is $Sym^m(\widetilde{C})$, see Lemma 13.4 below. If m is big enough, this is a projective space bundle over $Pic^0(\widetilde{C})$.

We assume that the point $b\in\mathbb{A}^N$ corresponds to (E,φ) and hence to Λ, in that the spectral covering of (E,φ) is $\widetilde{C}$ and spectral 1-form of (E,φ) is α.

Let $\mathcal{P}_{M\otimes\omega_C}$ denote the Prym variety of line bundles V on $\widetilde{C}$ such that $\pi_*(V)$ has determinant $M\otimes\omega_C$. Equivalently it means that the norm of the divisor defining V down to C is the divisor of $M\otimes\omega_C^{\otimes 2}$, in particular the line bundles V have degree $m+4g-4$.

The tautological one-form $\alpha_{\mathcal{P}}$ on $\mathcal{P}_{M\otimes\omega_C}$ leads to a rational map $\mathcal{P}_{M\otimes\omega_C}\dashrightarrow T^*X^\circ_{M\otimes\omega_C}$, and this may be pulled back using the dominant rational map $Q\dashrightarrow X^\circ_{M\otimes\omega_C}$ to get a map

$$\mathcal{P}_{M\otimes\omega_C}\times_{X_{M\otimes\omega_C}}Q\dashrightarrow T^*Q.$$

Theorem 13.2. *The restriction $\Sigma_{\mathbf{Rad},U'}$ over $U' \subset Q$ is isomorphic, as a variety mapping to T^*U', with the pullback $\mathcal{P}_{M\otimes\omega_C} \times_{X_{M\otimes\omega_C}} U'$.*

Corollary 13.3. *In the case when C is a curve of genus $g = 2$, the local system constructed by Drinfeld-Laumon-Gaitsgory on $X_{M\otimes\omega_C} - \mathrm{Wob}_{M\otimes\omega_C}$ is the same as the local system we have constructed in the previous chapters of this paper.*

The next subsections are devoted to the proofs. Some parts work for rank 2 local systems and bundles on a curve C of arbitrary genus g. For the calculations of parabolic structures along the wobbly divisor we'll restrict to the case of curves of genus $g = 2$ where we understand well the geometry.

13.2 The incidence correspondence

We have fixed a line line bundle M of degree m on C, and defined

$$P := \mathbb{P}H^0(M), \quad Q := \mathbb{P}H^1(M^* \otimes \omega_C).$$

These projective spaces are dual by Serre duality. We assume $m \gg 0$, so they both have dimension $m + 1 - g$. Let

$$(p, q) : I \hookrightarrow P \times Q$$

be the incidence correspondence.

There is a rational map $\mathbf{L} : Q \dashrightarrow Bun_{M\otimes\omega_C}$ from Q to the moduli space $Bun_{M\otimes\omega_C}$ of rank 2 vector bundles $\mathcal{L}$ with determinant $M \otimes \omega_C$, sending a point represented by a nonzero class $\xi \in H^1(M^* \otimes \omega_C) = Ext^1(M, \omega_C)$ to the isomorphism class of the bundle in the extension ξ

$$0 \to \omega_C \to \mathcal{L} \to M \to 0.$$

On the other hand there is a map $i : P \to Sym^m(C)$ sending a point represented by a nonzero section $f \in H^0(M)$ to the divisor $z = z_1 + \ldots + z_m$ of zeros of f. The image is the set of points of $Sym^m(C)$ mapping to $[M] \in Pic^m(C)$.

Let (E, φ) be a rank 2 Higgs bundle corresponding to a local system Λ. We recall that there is a perverse sheaf denoted $\Lambda^{(m)}$ on $Sym^m(C)$ obtained by descending $\Lambda^{\boxtimes m}$ on C^m via the action of the symmetric group.

This corresponds to a parabolic Higgs bundle that we'll denote by $(E^{(m)}, \varphi^{(m)})$. Since the divisor in $Sym^m(C)$ has non-normal crossings singularities, we don't exactly know what a parabolic structure means, so we'll instead say that we work with the pure twistor D-module and look at the Higgs fiber. This has a parabolic structure given by the V-filtration in codimension ≤ 1, and we get a parabolic Higgs bundle on an open subset $Sym^m(C)^\circ$ complement of a set of codimension 2.

Let $\widetilde{C} \xrightarrow{\pi} C$ be the spectral covering of (E, φ). We assume that $\widetilde{C}$ is smooth, and since the degree is 2 it automatically has simple ramification. Let $\alpha \in H^0(\widetilde{C}, \pi^*\omega_C)$ be the tautological 1-form, and let U be the spectral line bundle. Thus $E \cong \pi_*(U)$.

We have a covering

$$Sym^m(\widetilde{C}) \to Sym^m(C)$$

and a tautological 1-form $\alpha^{(m)}$ given as the descent from $\widetilde{C}^m$ of the sum of the pullbacks of α from the components. The covering has a line bundle $U^{(m)}$ descended from the line bundle $U^{\boxtimes m}$ on $\widetilde{C}^m$; it is characterized by the condition that $U^{\boxtimes m}$ is the pullback of $U^{(m)}$.

Let $\boldsymbol{E} \subset Sym^m(\widetilde{C})$ be the divisor consisting of points $\tilde{t}$ such that $t_i = t_j$ but $\tilde{t}_i \neq \tilde{t}_j$, where t_i are the images in C of $\tilde{t}_i$.

Lemma 13.4. *The spectral data in codimension ≤ 1 for the parabolic Higgs bundle $(E^{(m)}, \varphi^{(m)})$ is given by the covering $Sym^m(\widetilde{C})$ with its tautological 1-form $\alpha^{(m)}$ and spectral line bundle $U^{(m)}$. The parabolic structure is the standard one with weights $0, 1/2$ using the divisor $\boldsymbol{E}$.*

The inverse image of P in $Sym^m(\widetilde{C})$ is the subvariety $Sym^m(\widetilde{C})_M$ consisting of divisors whose associated line bundle has trace down to C equal to M. If m is big enough, it is smooth, being a projective space bundle over a translate of the Prym variety that is the kernel of $Pic^m(\widetilde{C}) \to Pic^m(C)$.

Proof. The exterior tensor product Higgs bundle $E^{\boxtimes m}$ over C^m has spectral variety $\widetilde{C}^m$. On there, the spectral one-form for the tensor product is the sum of the pullbacks of α to each of the components, and the spectral line bundle on $\widetilde{C}^m$ is $U^{\boxtimes m}$. These descend to the given spectral data outside of codimension 2. Notice here that we might also need to remove a codimension 2 subset bigger than just the singular locus of the parabolic divisor, in case the image of $Sym^m(\widetilde{C})$ in the logarithmic cotangent bundle has singularities at the branch points of $\widetilde{C}/C$ along the multidiagonal (we didn't calculate if this happens or not).

In order to understand the line bundle, we note the following remark. Let $g : C^m \to Sym^m(C)$ be the projection. The level 0 piece of the parabolic structure, on the parabolic bundle over $Sym^m(C)$ obtained by descending $E^{\boxtimes m}$ from C^m down to $Sym^m(C)$, is equal to the subsheaf of sections of $g_*(E\boxtimes m)$ invariant by the symmetric group action. This can be seen using the metric interpretation of parabolic structures.

Consider now the commutative (although not cartesian) diagram

$$\begin{array}{ccc} \widetilde{C}^m & \to & C^m \\ \downarrow & & \downarrow \\ Sym^m(\widetilde{C}) & \to & Sym^m(C) \end{array}.$$

The sheaf $E^{\boxtimes m}$ is the direct image of $U^{\boxtimes m}$ from $\widetilde{C}^m$. The permutation group action preserves the diagram. We can take the direct image of $U^{\boxtimes m}$ by the left vertical and then bottom arrows, and the invariant sections therein are the direct image of the invariant sections on $Sym^m(\widetilde{C})$. This says that our sheaf of invariant sections is the direct image of $U^{(m)}$ from $Sym^m(\widetilde{C})$ to $Sym^m(C)$, which means in turn that $U^{(m)}$ is the spectral line bundle. One might have expected that there could be a correction term by some multiple of the ramification divisor but this argument shows that that isn't the case.

The parabolic structure is the standard one coming from reflections in the monodromy of the local system; this happens along the divisor $\boldsymbol{E}$ in the spectral variety.

The last part comes from the standard properties of symmetric powers of curves, noting that P is the projective space of the linear system $|M|$. □

Let $F_B := Rq_*(p^* E_B^{(m)})$ be the perverse sheaf on Q obtained by Radon transform. Drinfeld's theorem 13.1 says that this is constant on the fibers of the map $\mathbf{L}$, so it descends to a perverse sheaf on $Bun_{M\otimes\omega_C}$.

Theorem 13.5 (Deligne cf [Lau87]). *The rank of this sheaf at a general point is* 2^{3g-3}.

Laumon states in [Lau87, Remarque 5.5.2] that this was communicated by Deligne.

We would like to approximate the Dolbeault calculation of a Higgs bundle F on Q corresponding to the perverse sheaf F_B. We know that this exists by applying Mochizuki's theory, and we'll try to calculate it over a Zariski open subset.

Let

$$\begin{array}{ccccc} \widetilde{I} & \to & \widetilde{P} & \to & Sym^m(\widetilde{C}) \\ \downarrow & & \downarrow & & \downarrow \\ I & \to & P & \to & Sym^m(C) \end{array}$$

be the cartesian diagram of pullbacks of the spectral variety of $E^{(m)}$. Thus, pulling back the statement from Lemma 13.4, $\widetilde{P}$ is the spectral variety of $E^{(m)}|_P$ and $\widetilde{I}$ is the spectral variety of $p^*E^{(m)}$.

Lemma 13.6. *The spectral variety* $\widetilde{P}$ *is irreducible.*

Proof. From Lemma 13.4, $\widetilde{P}$ is the subvariety $Sym^m(\widetilde{C})_M \subset Sym^m(\widetilde{C})$ consisting of divisors whose associated line bundle has trace down to C equal to M. It is a projective bundle a translate of the kernel of $Pic^m(\widetilde{C}) \to Pic^m(C)$. We need to show that this kernel is irreducible.

The map between real tori viewed in terms of the exponential exact sequence as

$$\frac{H^1(\widetilde{C},\mathcal{O})}{H^1(\widetilde{C},\mathbb{Z})} \to \frac{H^1(C,\mathcal{O})}{H^1(C,\mathbb{Z})}$$

may be written, using Poincaré duality, in the form

$$\frac{H_1(\widetilde{C},\mathbb{R})}{H_1(\widetilde{C},\mathbb{Z})} \to \frac{H_1(C,\mathbb{R})}{H_1(C,\mathbb{Z})}.$$

The map $H_1(\widetilde{C},\mathbb{Z}) \to H_1(C,\mathbb{Z})$ is surjective, since $\widetilde{C}/C$ is a double cover and the ramification set is nonempty. An element of the kernel is a point in $H_1(\widetilde{C},\mathbb{R})$ that maps to $H_1(C,\mathbb{Z})$, and by the surjectivity it can be modified by an element of $H_1(\widetilde{C},\mathbb{Z})$ so that it maps to zero in $H_1(C,\mathbb{R})$; thus we have a point in the kernel of $H_1(\widetilde{C},\mathbb{R}) \to H_1(C,\mathbb{R})$ and that covers the connected component of the identity in the kernel we are looking at. This shows that our kernel is connected. □

Let $\tilde{q} : \widetilde{I} \dashrightarrow Q$ be the composition of the map q with the covering. The holomorphic L^2 Dolbeault complex of $p^*E^{(m)}$ relative to the map q has a cokernel sheaf in top degree

$$Dol_{L^2}(I \xrightarrow{q} Q, p^*E^{(m)}) \to \mathcal{G}.$$

We can write what this is, away from the parabolic divisors. The spectral line bundle on $\widetilde{I}$ is

$$U_{\widetilde{I}} := \tilde{p}^*(U^{(m)}|_{\widetilde{P}}.$$

There is a tautological 1-form denoted $\alpha_{\widetilde{I}}$.

The relative dimension of I/Q is one less than the dimension of P or Q since I is a family of hyperplanes; it is $m-g$. Consider the sequence

$$U_{\widetilde{I}} \otimes \Omega^{m-g-1}_{\widetilde{I}/Q} \xrightarrow{\wedge\alpha_{\widetilde{I}}} U_{\widetilde{I}} \otimes \Omega^{m-g}_{\widetilde{I}/Q} \to \widetilde{\mathcal{G}} \to 0$$

where $\widetilde{\mathcal{G}}$ is defined to be the cokernel. Let $\Gamma \subset \widetilde{I}$ be the support of $\widetilde{\mathcal{G}}$.

This is the relative critical locus, and one may alternatively say that Γ is the subset of points on which the projection $\alpha^{\mathrm{rel}}_{\widetilde{I}}$ of the tautological 1-form into a section of $\Omega^1_{\widetilde{I}/Q}$ vanishes.

Lemma 13.7. *Over an appropriate open subset, $\mathcal{G}$ is the direct image from $\widetilde{I}$ down to I of $\widetilde{\mathcal{G}}$.*

Proof. The relative Dolbeault complex $Dol_{L^2}(I \xrightarrow{q} Q, p^*E^{(m)})$ is the pushforward of the complex $[\ldots \to U_{\widetilde{I}} \otimes \Omega^{m-g-1}_{\widetilde{I}/Q} \to U_{\widetilde{I}} \otimes \Omega^{m-g}_{\widetilde{I}/Q}]$ so over the open subset where $\widetilde{I}/I$ is finite and flat, the cokernel $\mathcal{G}$ of the pushforward is the pushforward of the cokernel. $\square$

The main result we need is the following, whose proof is deferred until after the statements of some lemmas—the lemmas in turn being proven later too.

Theorem 13.8. *There is a unique irreducible component Γ^{main} that surjects onto P. It contains a dense Zariski open subset that maps isomorphically to an open subset of $\widetilde{P}$. This irreducible component Γ^{main} maps to Q by a generically finite map of degree 2^{3g-3}.*

We are going to define a rational map $v : \widetilde{P} \dashrightarrow Q$. Suppose $\tilde{z} \in \widetilde{P}$. Use this to define a line bundle

$$V(\tilde{z}) := \pi^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{z}_1 + \ldots, \tilde{z}_m)$$

on $\widetilde{C}$, and set $\mathcal{L}_{\tilde{z}} := \pi_*(V(\tilde{z}))$.

One calculates $\det(\mathcal{L}_{\tilde{z}}) = M \otimes \omega_C$. This uses the condition that $\tilde{z} \in \widetilde{P}$ saying in particular that the image $z \in Sym^m(C)$ is in the linear system $|M| = P$, i.e. $\mathcal{O}_C(z_1 + \ldots + z_m) \cong M$.

Therefore $V(\tilde{z}) \in \mathcal{P}_{M \otimes \omega_C}$. The moduli point of $\mathcal{L}_{\tilde{z}}$ in $X_{M \otimes \omega_C}$ is the image of $V(\tilde{z})$ under the natural projection from the Prym. To define $v(\tilde{z})$ we need to specify an expression of $\mathcal{L}_{\tilde{z}}$ in an extension.

The fact that $\widetilde{C}$ is a spectral curve implies that there is a natural isomorphism

$$\pi_*(\pi^*(\omega_C)) \cong \omega_C \oplus \mathcal{O}_C, \tag{58}$$

in particular we get a morphism $\omega_C \to \mathcal{L}_{\tilde{z}}$. For general $\tilde{z}$ (the condition being that there aren't opposite pairs of points), this is a subbundle. Because of the determinant calculation we obtain an exact sequence

$$0 \to \omega_C \to \mathcal{L}_{\tilde{z}} \to M \to 0$$

and hence an extension $v(\tilde{z}) \in H^1(M^* \otimes \omega_C)$ (well-defined up to scalars). This defines our map $v : \widetilde{P} \dashrightarrow Q$.

Remark 13.9. It follows from the above discussion that the diagram of rational maps

$$\begin{array}{ccc} \widetilde{P} & \to & Q \\ \downarrow & & \downarrow \\ \mathcal{P}_{M \otimes \omega_C} & \to & X_{M \otimes \omega_C} \end{array}$$

is commutative, so we get a map $\widetilde{P} \dashrightarrow \mathcal{P}_{M\otimes\omega_C} \times_{X_{M\otimes\omega_C}} Q$.

Lemma 13.10. *If $\tilde{z} \in \widetilde{P}$ maps to $z \in P$ and to $v(\tilde{z})$ by the above construction then the point $(z, v(\tilde{z}))$ is in I. In particular, the point $(\tilde{z}, v(\tilde{z}))$ defines a point of $\tilde{I}$ so we get a (rationally defined) section $\sigma : \widetilde{P} \dashrightarrow \widetilde{I}$.*

Lemma 13.11. *The degree of $v : \widetilde{P} \dashrightarrow Q$ is 2^{3g-3} and indeed the fiber over a general point $\xi \in Q$ is naturally identified with the inverse image of $\mathcal{L}_\xi \in Bun_{M\otimes\omega_C}$ in the Prym variety that is the fiber of the Hitchin map corresponding to our given spectral curve $\widetilde{C}/C$.*

Lemma 13.12. *The section σ maps $\widetilde{P}$ into Γ.*

Proof of Theorem 13.8. We claim that over a general point of $\widetilde{P}$ there is exactly one point of Γ. Suppose $\tilde{z} \in \widetilde{P}$ is a general point mapping to $z \in P$. Consider the subspace $\widetilde{V} \subset T_{\tilde{z}}\widetilde{P}$ on which the tautologial 1-form $\alpha_{\widetilde{P}}$ vanishes. For $\tilde{z}$ general, this maps to a codimension one subspace $V \subset T_z P$. There will be a unique hyperplane of the projective space P containing z and such that the tangent space of the hyperplane at z contains V. This hyperplane represents a point of I, and its pair with $\tilde{z}$ defines a point of $\widetilde{I}$. Assuming that $\widetilde{P}/P$ is etale at $\tilde{z}$, the condition on the tangent spaces is equivalent to the vanishing of the tautological vertical one-form $\alpha^{\mathrm{rel}}_{\widetilde{I}}$ at the point. This shows the claim.

That implies that the section σ provided by Lemmas 13.10 and 13.12 is an isomorphism of $\widetilde{P}$ (which is irreducible by Lemma 13.6) onto the irreducible component $\Gamma^{\mathrm{main}} \subset \Gamma$. The general point of this component is therefore finite over Q since they have the same dimension. It has the required degree by Lemma 13.11. This gives the statement of the theorem. $\square$

For the proofs of the lemmas we begin with the following observation using $m \gg 0$.

Remark 13.13. Suppose $\xi \in Q$ and $\mathcal{L}_\xi$ is stable. Then $H^1(\mathcal{L}_\xi) = 0$, indeed an element of $H^0(\mathcal{L}_\xi^* \otimes \omega_C)$ would be a map $\mathcal{L}_\xi \to \omega_C$ contradicting stability. There is an exact sequence

$$0 \to H^0(\omega_C) \to H^0(\mathcal{L}_\xi) \to H^0(M) \to H^1(\omega_C) \to 0.$$

Thus the image of $H^0(\mathcal{L}_\xi)$ is a codimension 1 subspace of $H^0(M)$ corresponding to a hyperplane in P. This hyperplane is $I_\xi = p(q^{-1}\{\xi\}) = \xi^\perp$.

Proof of Lemma 13.10. Let $\xi = v(\tilde{z})$. We have $\mathcal{L}_\xi \cong \pi_*(V(\tilde{z}))$. Recall that $\pi^*(\omega_C) \subset V(\tilde{z})$, so the formula (58) gives a map

$$\omega_C \oplus \mathcal{O}_C = \pi_*(\pi^*(\omega_C)) \hookrightarrow \mathcal{L}_\xi.$$

This gives a map $\mathcal{O}_C \to \mathcal{L}_\xi$ not factoring through ω_C, so it projects to a nonzero section in $H^0(M)$ and hence gives a point of P. The cokernel of $\omega_C \oplus \mathcal{O}_C \to \mathcal{L}_\xi$ is the sheaf $\pi_*(V(\tilde{z})_{\tilde{z}}$ which is supported on z. For the proof it suffices to assume that $\tilde{z}$ is general, and in this case at least, the zero scheme of the section $\mathcal{O}_C \to M$ is z. The characterization of Remark 13.13 tells us that $z \in \xi^\perp$, so $(z, \xi) = (z, v(\tilde{z}))$ is in I. □

Proof of Lemma 13.11. In the space of Higgs bundles whose determinant is $M \otimes \omega_C$, let $\mathcal{P}$ denote the fiber of the Hitchin fibration over the point corresponding to the spectral curve $\widetilde{C}/C$. Let

$$\varpi : \mathcal{P} \dashrightarrow Bun_{M\otimes\omega_C}$$

be the rationally defined map of forgetting the Higgs field. The points of $\mathcal{P}$ are line bundles V on $\widetilde{C}$ such that the determinant of $\pi_*(V)$ is $M \otimes \omega_C$, and $\varpi(V) = \pi_*(V)$.

Suppose $\xi \in Q$ is a general point. As is well-known (cf eg Lemma 4.9), $\varpi^{-1}(\mathcal{L}_\xi)$ is a finite set with 2^{3g-3} elements.

On the other hand, $v^{-1}(\xi) \subset \widetilde{P}$ is the subset of points $\tilde{z} \in \widetilde{P}$ such that the resulting extension $v(\tilde{z})$, displaying as

$$0 \to \omega_C \to \mathcal{L}_{\tilde{z}} \to M \to 0,$$

is equal to the extension ξ.

We are going to establish an isomorphism between these two sets, from which it follows that $v^{-1}(\xi) \subset \widetilde{P}$ is a finite set with 2^{3g-3} elements.

Suppose given $\tilde{z} \in v^{-1}(\xi)$. Recall that $\mathcal{L}_{\tilde{z}} := \pi_*(V(\tilde{z}))$ and the condition $\tilde{z} \in v^{-1}(\xi)$ says that this bundle is isomorphic to $\mathcal{L}_\xi$. Therefore, $V(\tilde{z}) \in \varpi^{-1}(\mathcal{L}_\xi)$.

Suppose given $V \in \varpi^{-1}(\mathcal{L}_\xi)$. Then the map $\omega_C \to \mathcal{L}_\xi$ gives a nonzero map $\pi^*(\omega_C) \to V$, so there is a divisor $\tilde{z}$ on $\widetilde{C}$ such that $V \cong \pi^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{z})$, and the extension $v(\tilde{z})$ is equal to ξ. This gives a point $\tilde{z} \in v^{-1}(\xi)$.

Let's note that these two constructions are inverses. In the previous paragraph, recall that by definition $V(\tilde{z})$ is the line bundle $\pi^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C}}(\tilde{z})$ that is isomorphic to V, so the composition

$$V \mapsto \tilde{z} \mapsto V(\tilde{z})$$

is the identity. For the composition in the other direction, notice that if given $\tilde{z} \in v^{-1}(\mathcal{L}_\xi)$ then since $\mathcal{L}_\xi$ is stable (that's the case for a general ξ) the identification $\mathcal{L}_\xi \cong \pi_*(V(\tilde{z}))$ is unique and gives rise to a uniquely defined (up to a scalar) map $\pi^*(\omega_C) \to V(\tilde{z})$, which by definition is our given one. When we make the construction of the previous paragraph we get back to the given point $\tilde{z}$. This gives the required isomorphism. □

For the proof of Lemma 13.12 we are going to look at a general point $\tilde{z} \in \widetilde{P}$. In particular, the $\tilde{z}_i$ are distinct, not ramification points of $\widetilde{C}/C$, and there are no opposite pairs under the involution τ of $\widetilde{C}/C$.

Fix $\xi = v(\tilde{z}$. We would like to consider a path $\tilde{z}(t)$ with the given point as $\tilde{z}(0)$, and such that $z(t) = \pi(\tilde{z}(t)) \in P$ stay in the hyperplane $I_\xi \subset P$ corresponding to ξ. By Remark 13.13, we obtain general such paths of points $x(t)$ by considering a path $f(t) \in H^0(\mathcal{L}_\xi)$. If we suppose $f(0)$ is the canonical section corresponding to $\tilde{z}$, then as we deform $f(0)$ we'll get a deformation of the points $\tilde{z}(t)$. This projects to a general tangent vector of I_ξ based at the original point, and because of our genericity hypothesis $\widetilde{P} \to P$ is etale near our points, so we get in this way the required general tangent vector to the fiber $q_{\widetilde{I}}^{-1}(\xi)$.

Let α be the tautological form on $\widetilde{C}$, and let α_V be α viewed as a section of $\pi^*(\omega_C)$ and then in turn viewed as a section of $V := V(\tilde{z}) = \pi^*(\omega_C)(\tilde{v})$.

We have $\mathcal{L}_\xi = \pi_*(V)$ and α_V, viewed now as a section of $\mathcal{L}_\xi$, is the same as the section $f(0)$.

For any connection ∇ on V, holomorphic near the $\tilde{z}_i$ but possibly meromorphic elsewhere,

$$(\nabla \alpha_V)_{\tilde{z}_i} = T_{\tilde{z}_i}(\widetilde{C}) \stackrel{\cong}{\to} V_{\tilde{z}_i}.$$

Letting ϵ denote an infinitesimal value of t, we get

$$\epsilon^{-1} f(\epsilon) \in V_{\tilde{z}_i}.$$

The zero $\tilde{z}_i(\epsilon)$ of the section $f(\epsilon)$, infinitesimally near to $\tilde{z}_i = \tilde{z}_i(0)$, is the displacment of $\tilde{z}_i$ by ϵ times the tangent vector

$$\vec{w}_i := [(\nabla \alpha_V)_{\tilde{z}_i}]^{-1}(\epsilon^{-1} f(\epsilon)) \in T_{\tilde{z}_i}(\widetilde{C}).$$

If we now apply α considered as a 1-form to this vector, we get a number.

Lemma 13.14. *The number $\alpha(\vec{w}_i)$ is equal to the residue at $\tilde{z}_i$ of*

$$f'(0) := \frac{f(\epsilon) - f(0)}{\epsilon} \in H^0(\mathcal{L}_\xi)$$

where $f'(0)$ is viewed as a section of V and hence as a differential form on $\widetilde{C}$ with poles at the points of $\tilde{z}$.

Proof. This calculation may be done in local coordinates. Let x be the coordinate near the point $\tilde{z}_i$. The bundle $\pi^*(\omega_C)$ is the same as $\omega_{\widetilde{C}}$ near our point, so this bundle has frame (dx).

The bundle V locally is sections having a simple pole at the origin, so it has frame $x^{-1}dx$. The section $f(0)$ is $a(x)dx$ and $f'(0) = b(x)x^{-1}dx$. In other words,

$$f(\epsilon) = a(x)dx + \epsilon b(x)x^{-1}dx + o(\epsilon).$$

The zero of this section near the origin is given by

$$x(\tilde{z}_i(\epsilon)) = \epsilon b(0)/a(0) + o(\epsilon).$$

The derivative of this in ϵ is

$$\vec{w}_i = b(0)/a(0)\frac{\partial}{\partial x}$$

viewed as a tangent vector at the origin using our coordinate x. The tautological 1-form is the same as the section $f(0)$ but viewed as a 1-form instead of a section of V. In our notations it is still called $a(x)dx$. When we evaluate this on the tangent vector we get

$$(a(x)dx)(\vec{w}_i) = b(0).$$

This is exactly the residue of $f'(0)$ at the origin. □

Proof of Lemma 13.12. The tautological 1-form on $Sym^m(\widetilde{C})$ evaluated at a tangent vector that is composed of tangent vectors at distinct points of $\widetilde{C}$, is the sum of the values of α evaluated on those vectors. In the situation of the lemma, our family $f(t)$ yields curves of points $\tilde{z}_i(t)$ based at the $\tilde{z}_i$, whose first derivatives give tangent vectors $\vec{w}_i$ based at the $\tilde{z}_i$. The evaluation of the tautological form on this tangent vector to the fiber $q_{\widetilde{I}}^{-1}(\xi)$ is therefore the sum of the $\alpha(\vec{w}_i)$. By Lemma 13.14, this is the sum of the residues of the section $f'(0)$, viewed as a section of V and hence as a meromorphic section of $\pi^*(\omega_C)$, at the $\tilde{z}_i$. Since these comprise all the polar locus of the section, the residue theorem says that the sum of the residues is 0. We have now shown that the evaluation of the tautological form on a general tangent vector to the fiber of the map $q_{\widetilde{I}}^{-1}(\xi)$, is zero. This is exactly the condition for inclusion of our point $(\tilde{z}, \xi)$ in Γ, completing the proof of the lemma. □

13.3 Dolbeault part of a twistor D-module

In order to use the result of Theorem 13.8 in conjunction with Deligne's calculation 13.5, we need to delve into some general theory of mixed twistor $\mathscr{D}$-modules [Moc07a, Moc07b, Sab05].

Suppose $\mathcal{E}$ is a pure twistor D-module on a variety X. Then there is an open subset $U \subset X$ over which $\mathcal{E}$ is smooth. We get a vector bundle $\mathcal{E}_U$ on $\mathbb{A}^1 \times U$ with relative integrable λ-connection ∇. For each $\lambda \in \mathbb{A}^1$ this gives a vector bundle $\mathcal{E}_U^\lambda$ with λ-connection ∇^λ.

We'll call the fiber at $\lambda = 0$ the *Dolbeault part* of $\mathcal{E}$. This is a vector bundle $\mathcal{E}_{Dol,U}$ with Higgs field $\varphi := \nabla^0$. In particular, it has a spectral variety $\Sigma \hookrightarrow T^*U$ finite and dominant over U.

There will be a notion of extension to a parabolic bundle in codimension 1, that is to say over $X^{\leq 1} := X - D^{\mathrm{sing}}$ where $D := X - U$ is the complementary divisor. The parabolic structure should be given by the V-filtration construction. We don't discuss that here, as we are looking at generic constructions over U.

We can define the Dolbeault complex over U:

$$DOL(U, \mathcal{E}_{Dol,U}, \varphi) = \left[\mathcal{E}_{Dol,U} \xrightarrow{\wedge\varphi} \cdots \xrightarrow{\wedge\varphi} \mathcal{E}_{Dol,U} \otimes \Omega^n_U\right].$$

Let $\mathcal{G}(U, \mathcal{E}_{Dol,U}, \varphi)$ denote the cokernel of the last map.

Proposition 13.15. *Assume that $\mathcal{G}(U, \mathcal{E}_{Dol,U}, \varphi)$ has finite support, and let*

$$d := \dim H^0(\mathcal{G}(U, \mathcal{E}_{Dol,U}, \varphi))$$

be the total length. Let $H^n(\mathcal{E})$ be the cohomology of X with coefficients in the twistor D-module. This is a twistor D-module over a point, that is to say a vector bundle over $\mathbb{P}^1$. If $d \geq \mathrm{rk} H^n(\mathcal{E})$, then equality holds, and the Dolbeault fiber of $H^n(\mathcal{E})$ (i.e. the fiber over $\lambda = 0$) is naturally isomorphic to $H^0(\mathcal{G}(U, \mathcal{E}_{Dol,U}, \varphi))$.

Proof. We don't do this here. The idea would be to look at a fibration of X as a family of curves, and use that to calculate the cohomology. We can then apply the calculation of [DPS16], and be careful about the difference between the cohomology of a resolution and the cohomology of the twistor D-module using the decomposition theorem. □

We can now formulate a relative version. If $f : X \to Y$ is a map, then choose an open set $U_Y \subset Y$ and an open subset $U \in f^{-1}(U_Y)$ over which the map is smooth of relative dimension m and $\mathcal{E}$ (a twistor D-module on X) is smooth. We can form the relative Dolbeault complex

$$DOL(U/U_Y, \mathcal{E}_{Dol,U}, \varphi) = \left[\mathcal{E}_{Dol,U} \xrightarrow{\wedge\varphi_{X/Y}} \cdots \xrightarrow{\wedge\varphi_{X/Y}} \mathcal{E}_{Dol,U} \otimes \Omega^m_{U/U_Y}\right].$$

Let $\mathcal{G}(U/U_Y, \mathcal{E}_{Dol,U}, \varphi)$ be the cokernel.

Proposition 13.16. *Suppose $\mathcal{G}(U/U_Y, \mathcal{E}_{Dol,U}, \varphi)$ has support that is finite over U_Y, and let*

$$d := \mathrm{rk} f_*(\mathcal{G}(U/U_Y, \mathcal{E}_{Dol,U}, \varphi))$$

be the relative length. Let $\mathcal{F} := R^m f_*(\mathcal{E})$ be the higher direct image. This is a twistor D-module over Y. Assume that the open set U_Y has been chosen so that $\mathcal{F}$ is smooth on U_Y. If $d \geq \mathrm{rk}(\mathcal{F})$, then equality holds, and

$$\mathcal{F}_{Dol,U_Y} \cong f_*(\mathcal{G}(U/U_Y, \mathcal{E}_{Dol,U}, \varphi)).$$

Furthermore, the decomposition of $\mathcal{F}_{Dol,U_Y}$ over $y \in U_Y$ into a direct sum of pieces indexed by the points in the support of $\mathcal{G}(U/U_Y, \mathcal{E}_{Dol,U}, \varphi)$ lying over y, is the spectral decomposition of the Higgs bundle $(\mathcal{F}_{Dol,U_Y}, \phi)$.

The Higgs field on $\mathcal{F}_{Dol,U_Y}$ will be determined in the same way as in [DPS16].

We can now apply this to the Drinfeld situation.

Proof of Theorem 13.2. Comparing Deligne's calculation in Theorem 13.5 with Theorem 13.8, the dimension of the cohomology on each geneal fiber of the map q is equal to the number of points in the support of the upstairs cokernel sheaf $\widetilde{\mathcal{G}}$. It follows from Proposition 13.16 that the length of $\widetilde{\mathcal{G}}$ at each of these points is 1, and then that the spectral variety for the direct image Higgs bundle is the support of $\widetilde{\mathcal{G}}$, which is to say Γ^{main}.

To complete the proof of Theorem 13.2, we need to show that the map $\Gamma^{\mathrm{main}} \dashrightarrow T^*Q$ obtained by interpreting Γ^{main} as the critical locus, is the same as the map $\widetilde{P} \to T^*Q$ that is the pullback of the map from the Hitchin fiber to the cotangent bundle of the moduli space of bundles $X_{M\otimes\omega_C}$.

Both maps are given, over general points of $\widetilde{P}$, by a spectral 1-form. In the case of the map on the relative critical locus, this is the same as the spectral 1-form $\alpha_{\widetilde{I}}$ on $\widetilde{I}$ restricted to Γ^{main} that is birational to $\widetilde{P}$.

We have a commutative diagram

$$\begin{array}{ccc} \widetilde{P} & \to & Jac^{m+4g-4}(\widetilde{C}) \\ \downarrow & & \downarrow \\ Sym^m(\widetilde{C}) & \to & Jac^m(\widetilde{C}) \end{array}$$

where the right vertical arrow is the isomorphism given by tensoring with $\pi^*(\omega_C)$. The spectral 1-form α on $\widetilde{C}$ corresponds to a unique 1-form α_{Jac} whose pullback to $Sym^m(\widetilde{C})$ is the 1-form given by summing up the pullbacks of α on $\widetilde{C}^m$.

In the case of the map from the Hitchin fiber, the spectral 1-form is given by restricting the 1-form on $Jac^{m+4g-4}(\widetilde{C})$ coming from the spectral 1-form on $\widetilde{C}$, to the Prym variety. The top map of the above diagram factors through the projection $\widetilde{P} \to \mathcal{P}_{M\otimes\omega_C}$, so the top and right pullback of α_{Jac} is the 1-form corresponding to the Hitchin fiber.

The left and bottom pullback of α_{Jac} is the spectral form on $\widetilde{P}$ that was used to define the Higgs bundle over P. By definition the spectral form $\alpha_{\widetilde{I}}$ is the pullback of this form to $\widetilde{I}$. In turn, that pulls back to the spectral 1-form on Γ^{main}. In other words, the isomorphism

$$\Gamma^{\text{main}} \to \widetilde{P}$$

identifies the two forms. This implies that they provide the same map to T^*Q over a Zariski open subset, and therefore they give the same map whenever it is defined. This completes the verification that the two isomorphic spectral varieties sit in the same way inside T^*Q. $\square$

13.4 Uniqueness over the degree 1 space

In this subsection, we'll prove a uniqueness result for Higgs bundles over X_1 that have Y_1 as spectral variety. Let X denote X_1 (the intersection of two quadrics in $\mathbb{P}^5$) and let Y denote Y_1. Let $\text{Wob} := \text{Wob}_1$ be the wobbly locus.

Let X° be the complement of the singular locus of Wob, let $\text{Wob}^\circ = \text{Wob} \cap X^\circ$ and let Y° be the inverse image in Y. Over X° the tautological 1-form on Y° yields a map $Y^\circ \to T^*(X^\circ, \log \text{Wob})$.

Lemma 13.17. *Away from a subset of codimension* 2*, this map is an embedding.*

Proof. It is an embedding away from the wobbly locus, since Y is isomorphic there to a fiber of the Hitchin fibration, the total space of which is the cotangent bundle of X. In fact, that holds true over the complement $Y - E$ of the exceptional divisor, since the other points of Y, over the wobbly locus but not on E, map into $T^*(X)$. On the wobbly locus, the first question is whether the map separates tangent directions. The tangent directions along E map to tangent directions along Wob. The normal directions will map to nontrivial vectors in the logarithmic cotangent bundle, as soon as we know that the tautological 1-form on the Prym variety $\mathcal{P}$ is nonzero in a general normal direction to $\widehat{C}$. This is true for a general normal direction since α is a linear form on the abelian variety and there are two normal directions at each point. This shows that the map $Y^\circ \to T^*(X^\circ, \log \text{Wob})$ separates tangent vectors away from a codimension 2 subset.

To show that it is an embedding, recall that above every (general) point of Wob there are 2 points of E. We need to show that these don't get glued together by the map. Notice that no other pairs of points on E, one of which is general, can be glued together since the map from E to Wob factors through this double cover.

Two points of E mapping to the same point of Wob correspond to two points of $\widehat{C}$ of the form $\tilde{a}$ and $\tau\tilde{a}$ where τ is the involution of $\widetilde{C}$ over $\overline{C}$; together with corresponding tangent vectors at these points. We note that both points go to the zero-section of the residue map

$$T^*(X^\circ, \log \mathrm{Wob})|_{\mathrm{Wob}} \to \mathrm{Wob} \times \mathbb{C}.$$

The kernel of the residue map projects to a map to $T^*\mathrm{Wob}$. To distinguish the points we would like to show that their images in $T^*\mathrm{Wob}$ are different. The tautological form is going to send the points into the image of

$$T^*\overline{C} \times_{\overline{C}} \mathrm{Wob} \to T^*\mathrm{Wob}.$$

The condition that $\widetilde{C}/\overline{C}$ is a smooth spectral curve means that the two points $\tilde{a}$ and $\tau\tilde{a}$ map to distinct (opposite) points of $T^*\overline{C}$, showing that they map to distinct points in $T^*\mathrm{Wob}$. $\square$

Lemma 13.18. *Suppose $\mathcal{E}_\cdot$ is a vector bundle with a parabolic structure over* Wob° *together with a meromorphic Higgs field that is logarithmic along* Wob°*, such that the spectral variety of the Higgs field is the image of Y° in* $T^*(X^\circ, \log \mathrm{Wob})$*. Suppose that it corresponds to a flat bundle such that the cohomology of the restriction to a generic Hecke conic has dimension* 16*. Then the parabolic structure is obtained from a parabolic level $0 < \alpha \leq 1$ as follows: there is a line bundle $\mathcal{L}$ on Y such that $\mathcal{E}_a = \pi_*(\mathcal{L})$ for $0 \leq a < \alpha$ and $\mathcal{E}_a = \pi_*(\mathcal{L}(E))$ for $\alpha \leq a < 1$.*

Proof. If there is no parabolic structure along the exceptional divisor E, then each of the 2 points of E over a point of Wob will contribute a unipotent block of size 2 to the monodromy. Let's count the contributions of these to the cohomology: over the conic $\mathbb{P}^1$ we have a local system of rank 8, with 16 points (the intersections of the conic with Wob) on which there are two unipotent blocks of size 2. This gives a total contribution of -32 to the Euler characteristic $\chi = h^0 - h^1$. On the other hand, from the rank 8 local system over $\mathbb{P}^1$ the contribution of the Euler characteristic is $8 \cdot 2 = 16$. The sum is then -16. This means that $h^1 \geq 16$. The hypothesis that $h^1 = 16$ implies that there can't be any further parabolic structure. This is the case $\alpha = 1$ (that is equivalent to $\alpha = 0$ by an elementary transformation; for simplicity below we prefer calling this $\alpha = 1$).

Suppose now that there is a nontrivial parabolic structure defined along E. If it has a single parabolic level different from 0, then the calculation is the same as above giving $h^1 = 16$. If there were two parabolic levels at each point of E (or a single level distinct from 0 with multiplicity 2) then the h^1 would be too big compared to the hypothesis. Thus, there is only one parabolic level α different from 0 or 1, and one can see (looking at a transverse section and thinking about parabolic structures on Higgs bundles over a curve) that the only possibility to create the parabolic Higgs bundle is the one described at the end of the statement. □

Corollary 13.19. *The parabolic level in the previous lemma is either $\alpha = 1/2$ or $\alpha = 1$.*

Proof. The parabolic first Chern class on $\widetilde{X}$ pushes down to a class on X, which has to vanish for a flat bundle. The formula of [IS08] for this parabolic first Chern class on X simplifies to a simple average over the interval $[0, 1]$:

$$\mathrm{ch}_1^{\mathrm{par}}(\mathcal{E}.) = \int_{a=0}^{1} \mathrm{ch}_1(\mathcal{E}_a).$$

This should vanish. In view of the formulas for $\mathcal{E}_a$ in the lemma, we get

$$0 = \mathrm{ch}_1^{\mathrm{par}}(\mathcal{E}.) = \alpha \mathrm{ch}_1(\pi_*(\mathcal{L})) + (1-\alpha)\mathrm{ch}_1(\pi_*(\mathcal{L}(E))).$$

The GRR formula gives

$$0 = \pi_*(\mathrm{td}_1(Y/X) + \mathcal{L} + (1-\alpha)E).$$

Equivalently,

$$F^2 \cdot (\mathrm{td}_1(Y/X) + \mathcal{L} + (1-\alpha)E) = 0.$$

Recall from Proposition 4.7 that

$$\mathrm{td}(Y/X) = (1 - F + 5F^2/12)(1 - E/2 + (E^2 + EF)/9)$$

so $\mathrm{td}_1(Y/X) = -F - E/2$. If $\mathcal{L} = \mathcal{O}_Y(aF + bE)$ with $a, b \in \mathbb{Z}$ we get

$$0 = F^2((-F - E/2) + aF + bE + (1-\alpha)E) = (a-1)F^3 + (b + 1/2 - \alpha)F^2E.$$

Using the calculations of Proposition 4.8 this gives

$$32(a-1) + 64(b + 1/2 - \alpha) = 0.$$

It follows from the condition that $a, b \in \mathbb{Z}$ that $\alpha = 1/2$ or $\alpha = 1$. □

Remark 13.20. The case $\alpha = 1/2$ corresponds to the case we have been treating, in which we found our flat bundle. We claim that the numerical class of the line bundle $\mathcal{L}$ is uniquely determined by the condition of ch_2 being extremal.

Notice that there is a relation on the coefficients of E and F that is fixed by requiring $\mathrm{ch}_1 = 0$. There is another parameter, which we can view as being the coefficient of E.

The uniquenessof the value for which ch_2 is extremal may be seen by calculating the parabolic Chern class, up to some correction terms of the kind we have seen in Chapter 4. The correction terms are local at the non-normal crossings points of the wobbly divisor, and don't depend on the choice of $\mathcal{L}$. The resulting function of the coefficient of E in the divisor of $\mathcal{L}$ is a strictly concave quadratic function, whose maximum is at an integer value; in case the reader is interested, the calculation is reproduced below, but the interesting point is that this fact comes from the factor $1/2E$ in the relative Todd class combined with the parabolic level $\alpha = 1/2$. By the Bogomolov-Gieseker inequality, we can't choose an integral line bundle $\mathcal{L}$ such that $\mathrm{ch}_2 > 0$. It follows that the integral value for which $\mathrm{ch}_2 = 0$ is unique.

Here is an approach to the calculation referred to above. The condition $\mathrm{ch}_1 = 0$ tells us that $\mathcal{E}_\alpha$ may be written as $\pi_*\mathcal{O}((2-2b)F+bE)$ for $0 \leq \alpha \leq 1/2$ and $\pi_*\mathcal{O}((2-2b)F+(b+1)E)$ for $1/2 \leq \alpha \leq 1$. Call these two bundles $\mathcal{E}(b)$ and $\mathcal{E}'(b)$ respectively. At all steps below we'll allow ourselves to ignore any terms that are constant as functions of b. Set

$$c(b) := H \cdot \mathrm{ch}_2(\mathcal{E}(b)), \quad \Delta c(b) := H \cdot (\mathrm{ch}_2(\mathcal{E}'(b)) - \mathrm{ch}_2(\mathcal{E}(b))).$$

Using the convention that we ignore terms that are constant in b, the integral formula for the second Chern class becomes much easier in that it no longer depends on ch_1:

$$H \cdot \mathrm{ch}_2^{\mathrm{par}} = c(b) + \Delta c(b)/2.$$

From the GRR formula (and dropping terms not depending on b) we have:

$$c(b) = F\cdot(1-F+5F^2/12)(1-E/2+(E^2+EF)/9)(1+((2-2b)F+bE)+((2-2b)F+bE)^2/2) \text{ (degree 3 terms)}$$

$$F \cdot (((2-2b)F + bE)^2/2 + ((2-2b)F + bE)(-F - E/2))$$

$$= 2(1-b)^2F^3 + 2b(1-b)F^2E + b^2FE^2/2 - 2(1-b)F^3 - (1-b+b)F^2E - bFE^2/2$$

$$= (2b^2 - 2b)F^3 + (2b - 2b^2)F^2E + (b^2 - b)FE^2/2$$

$$= (b^2 - b)(2F^3 - F^2E + FE^2/2)$$

$$= (b^2 - b)(32 - 64 + 16) = 16(b - b^2).$$

The term $\Delta c(b)$ comes only from the quadratic term in the Chern characters of $\mathcal{E}(b)$ and $\mathcal{E}'(b)$, and these are indeed the same, so it is just

$$\Delta c(b) = c(b+1) - c(b) = -32b.$$

Now

$$\begin{aligned}\mathrm{ch}_2^{\mathrm{par}} &= c(b) + \Delta c(b)/2 + \text{ terms constant in } b \\ &= 16(b - b^2) - 16b = -16b^2.\end{aligned}$$

This has its extremum at $b = 0$ as claimed. This completes our parenthetical calculation.

Moving on, in order to prove Corollary 13.3 we need to rule out the possibility that $\alpha = 1$. This case corresponds to the situation that our bundle $\mathcal{E}$ has no parabolic structure along Wob°.

Corollary 13.21. *With trivial parabolic structure, for $\mathcal{L} = \mathcal{O}_Y(aF + bE)$ and $\mathcal{E} = \pi_*(\mathcal{L})$ we have*

$$H^2 \mathrm{ch}_1(\mathcal{E}) = 32(a-1) + 64(b - 1/2).$$

If this vanishes then $a = 2 - 2b$.

Proof. As in the proof of the previous corollary we get

$$H^2\mathrm{ch}_1(\mathcal{E}) = H^2\pi_*(\mathrm{td}_1(Y/X) + \mathcal{L}) = F^2((a-1)F + (b-1/2)E) = 32(a-1) + 64(b-1/2) = 32(a+2b-2).$$

This vanishes for $a = 2 - 2b$. □

Put $m := b - 1$ so $b = m + 1$ and $a = -2m$.

Lemma 13.22. *Let $\mathcal{L} = \mathcal{O}_Y(-2mF + (m+1)E)$ and $\mathcal{E} = \pi_*(\mathcal{L})$, then*

$$H \cdot \mathrm{ch}_2(\mathcal{E}) = -48m^2 - 48m - 8.$$

This value is ≤ -8 if $m \in \mathbb{Z}$.

Proof. We have that $\mathrm{ch}_2(\mathcal{E})$ is the degree 2 part of

$$\pi_* \left[(1 - F + 5F^2/12)(1 - E/2 + (E^2 + EF)/9)(1 - 2mF + (m+1)E + (-2mF + (m+1)E)^2/2)\right]$$

which is π_* of

$$5F^2/12+(E^2+EF)/9+(-2mF+(m+1)E)^2/2+EF/2-(F+E/2)(-2mF+(m+1)E).$$

We get

$$H\cdot \mathrm{ch}_2(\mathcal{E}) = F\cdot\big[5F^2/12+(E^2+EF)/9+(-2mF+(m+1)E)^2/2+EF/2-(F+E/2)(-2mF+(m+1$$

This expands to:

$$F^3(5/12+2m^2+2m)$$
$$+EF^2(1/9-2m(m+1)+1/2+m-(m+1))$$
$$+E^2F(1/9+(m+1)^2/2-(m+1)/2)$$

which, in view of Proposition 4.8, becomes

$$32(2m^2+2m+5/12)+64(-2m^2-2m+1/9-1/2)+32(m^2/2+m/2+1/9)$$

$$=-48m^2-48m+40/3+32/3-32=-48m^2-48m-8.$$

This is

$$-48m^2-48m-8=-48(m+1/2)^2+4.$$

The extremal value $m=-1/2$ is not allowed since m is supposed to be an integer. The extremal values for integers m are at $m=0$ and $m=-1$ and there the values are -8. $\square$

Let $\widetilde{X}\to X$ be a resolution of singularities of the cusps of Wob in codimension 2.

Lemma 13.23. *Suppose $\mathcal{E}$ is a vector bundle over a surface Z. Let $b:\widetilde{Z}\to Z$ be a birational map from another smooth surface obtained by blowing up some points. Suppose $\widetilde{\mathcal{E}}.$ is a parabolic bundle on $\widetilde{Z}$ that is isomorphic to $\mathcal{E}$ over an open subset, complement of a finite collection of points in Z, where b is an isomorphism. Suppose $\mathrm{ch}_1^{\mathrm{par}}(\widetilde{\mathcal{E}}.)=0$. Then*

$$\mathrm{ch}_2^{\mathrm{par}}(\widetilde{\mathcal{E}}.)\le 0$$

with equality only in the case where the parabolic structure is trivial and $\widetilde{\mathcal{E}}_0=b^*(\mathcal{E})$.

Proof. The corrections due to the parabolic structure are local. If any of the corrections were >0 then one could fill in such structures an arbitrary number of times to a stable vector bundle and contradict the Bogomolov-Gieseker inequality.

If any of the corrections is $= 0$ then one can fill that into an irreducible flat unitary bundle. The parabolic case of the Donaldson-Uhlenbeck-Yau theorem [SW01, Moc06, Moc09] would imply that the resulting bundle is flat, but a flat bundle can't have a nontrivial parabolic structure only on an exceptional divisor, so the parabolic structure would have to be trivial. □

Corollary 13.24. *If $\mathcal{E} = \pi_*(\mathcal{L})$ for $\mathcal{L}$ a line bundle on Y, then or any parabolic extension $\widetilde{\mathcal{E}}.$ of $\mathcal{E}|_{X^\circ}$ across the exceptional divisors in $\widetilde{X}$ such that the parabolic first Chern class is 0, then we have*

$$H.\mathrm{ch}_2^{\mathrm{par}}(\widetilde{\mathcal{E}}.) \leq -8.$$

In particular, no such Higgs bundle can correspond to a local system.

Proof of Corollary 13.3. This corollary rules out the possibility of a parabolic level $\alpha = 0$, so by Corollary 13.19 the level must be $\alpha = 1/2$. The parabolic Higgs bundle therefore has the same structure, by Lemma 13.18, as the parabolic Higgs bundle that we construct. This proves Corollary 13.3 for the degree 1 moduli space, up to the choice of line bundle of degree 0 on Y. The Drinfeld-Laumon construction gives a Hecke eigensheaf with the original rank 2 local system as eigenvalue. We have also shown the Hecke eigensheaf property. On the one hand, this fixes the choice of the line bundle of degree 0 on Y, and it also shows that our construction coincides with the Drinfeld-Laumon construction on the degree 0 moduli space. □

13.5 Tensor description in the degree 1 case

The description of the spectral variety given in Theorem 13.2 is used above in the case of $m \gg 0$. However, it turns out that in our special case of the moduli space X_1 of bundles of rank 2 and odd degree on a curve of genus 2, this construction almost leads directly to a description of the Hecke eigensheaf.

As pointed out to us by Hitchin, see [Hit22], Atiyah showed in 1955 [Ati55] that the moduli space of odd degree $PGL(2)$-bundles on C is a double covering of $\mathbb{P}^3 = \mathrm{Sym}^3(\mathbb{P}^1)$. See Proposition 13.28 and Theorem 13.30 below. Points of X_1 are thus in correspondence with unordered triples of points in the hyperelliptic $\mathbb{P}^1$, and up to a further covering, with unordered triples of points of C. This allows us to use the symmetric exterior tensor product $\Lambda^{(\boxtimes 3)}$. Because of the coverings involved, we will first look at the description over a general line.

Let Λ denote the eigenvalue rank 2 local system on C. This leads to a rank 8 local system $\Lambda^{(\boxtimes 3)}$ on the third symmetric power $Sym^3(C)$. If (E, φ) is the rank 2 Higgs bundle associated to Λ, then the Higgs bundle associated to $\Lambda^{(\boxtimes 3)}$ is $(E^{(3)}, \varphi^{(3)})$. By Lemma 13.4, its spectral variety is $Sym^3(\widetilde{C})$.

Suppose $\ell \subset X_1$ is a general line, consisting of bundles E fitting into an exact sequence

$$0 \to A \to E \to A^{-1}(p) \to 0$$

for a degree 0 line bundle A. Set $M := A^{\otimes -2}(3p)$ be the resulting degree 3 line bundle, and let $|M|$ be the linear system of sections of M. We have $|M| \cong \mathbb{P}^1 \hookrightarrow Sym^3(C)$. Points of $|M|$ are lines in $H^0(Hom(A, A^{-1}(p) \otimes \omega_C))$ and this space is Serre dual to $Ext^1(A^{-1}(p), A)$, so $|M|$ can be identified with the projectivized space of extensions, which is ℓ. Recall from Subsection 4.5 that the elements of $|M|$ are the fibers of the trigonal map $C \to \mathbb{P}^1 \cong \ell$ associated to the line ℓ.

Theorem 13.25. *The Hecke eigensheaf on X_1 with eigenvalue Λ, according to Drinfeld's construction or our construction in of Section 4, restricted to ℓ becomes isomorphic to $\Lambda^{(\otimes 3)}|_\ell$.*

Proof. By Corollary 13.3, the Hecke eigensheaf on X_1 we constructed in Section 4 is the same as that of Drinfeld's construction. By construction, the spectral variety of the associated parabolic Higgs bundle $(\mathscr{F}_{1,\bullet}, \Phi_1)$ is the covering $Y_1 \to X_1$. The description in Lemma 4.15 of the covering Y_1 over a line yields the isomorphism

$$Y_1 \times_{X_1} \ell \cong Sym^3(\widetilde{C}) \times_{Sym^3(C)} \ell.$$

These spaces are isomorphic to the subvariety $Sym^3_M(\widetilde{C})$ of divisors on $\widetilde{C}$ whose norm to C is in the linear system $|M|$.

The parabolic structure has levels $0, 1/2$ and one verifies that the parabolic structure for $E^{(3)}$ is the standard one coming from the ramification points of the spectral covering, as is the case for $\mathscr{F}_{1,\bullet}|_\ell$.

The spectral line bundle for $E^{(3)}$ is pulled back from the line bundle $\boldsymbol{N}^{(3)}$ on $Sym^3(\widetilde{C})$ obtained using the spectral line bundle $\boldsymbol{N}$ on $\widetilde{C}$.

Let's check first that this has the right degree to get a degree 0 parabolic bundle of rank 8 (this will provide a check of Lemma 13.4). The spectral line bundle $\boldsymbol{N}$ has degree 2 on $\widetilde{C}$. The map $\widetilde{C}^3_M \to \widetilde{C}$ has degree 8. Indeed once we fix one of the points $\tilde{t}_1$ with image t_1 in C, the remaining divisor $t_2 + t_3$ in C is fixed, leading two choices of ordered triple of points; there are 4 lifts of each pair t_2, t_3 to $\tilde{t}_2, \tilde{t}_3$ so the fiber over $\tilde{t}_1$ has 8 points in all. The pullback $pr_i^*(L)$ therefore has degree 16 on $\widetilde{C}^3_M$, and the tensor product of three of these has degree 48.

The spectral line bundle $\boldsymbol{N}^{(3)}$ on $Sym^3_M(\widetilde{C})$ pulls back to this bundle of degree 48 via the $6:1$ map $\widetilde{C}^3_M \to Sym^3_M(\widetilde{C})$, so $\boldsymbol{N}^{(3)}$ has degree 8.

The map $Sym^3_M(\widetilde{C}) \to \ell$ ramifies at two types of points: there are the points of the divisor $Sym^3_M(\widetilde{C}) \cap \boldsymbol{E}_1$ that are of the form $\tilde{t}$ where $t_i = t_j$ but $\tilde{i} \neq \tilde{j}$, there are two such points over each point of $\ell \cap \mathrm{Wob}_1$. Then, there are ramification points $\tilde{t}$ such that one of the $\tilde{t}_i$ is a ramification point of $\widetilde{C}/C$. For each of the 4 ramification points of $\widetilde{C}/C$, the other two points in the trigonal fiber are specified and there are 4 ways of lifting these to pairs of points in $\widetilde{C}$, so we get 16 such ramification points. These constitute the movable ramification locus.

The parabolic degree of a bundle created using the standard parabolic structure at some simple ramification points, and not at others, is calculated by the same formula as the degree of a usual direct image, but not counting the ramification points that are used for the parabolic structure. Thus, we should count 16 ramification points instead of 32, and the required parabolic degree is one-half of this number, that is to say $16/2 = 8$. This is indeed the degree of the bundle $\boldsymbol{N}^{(3)}$ as is to be expected.

The pullback of the spectral line bundle from Y_1 under the map $Sym^3_M(\widetilde{C}) \to Y_1$ also has degree 8, since the spectral line bundle is chosen to create a parabolic structure over X_1 that has vanishing ch_1.

To identify the two spectral line bundles, we need to show that these two line bundles of degree 8 are the same, when the spectral line bundle on Y_1 is chosen as a function of $\boldsymbol{N}$ in the specified way that will be described next.

We recall from Subsection 7.4 that the spectral line bundle $\mathscr{L}_1$ on Y_1 for the Hecke eigensheaf is related to the spectral line bundle $\boldsymbol{N}$ on $\widetilde{C}$ for the original eigenvalue Higgs bundle in the following way.

First, one has from Lemma 7.5 the line bundle

$$\mathfrak{L} = \boldsymbol{t}^*_{\boldsymbol{N}(-\pi^*(\mathsf{p}))}\boldsymbol{\xi} \otimes \boldsymbol{\xi}^{-1}$$

on the Prym variety $\mathcal{P}$ of degree 0 line bundles on $\widetilde{C}$ with trivial norm down to C. This gives by translation the line bundle

$$\mathfrak{L}_1 = \boldsymbol{t}^*_{\mathcal{O}_{\widetilde{C}}(-\widetilde{\mathsf{p}}-\pi^*\mathsf{p})}\mathfrak{L}$$

on the Prym variety $\mathcal{P}_3$ of degree 3 line bundles with norm $\mathcal{O}_C(3\mathbf{p})$. Then

$$\mathscr{L}_1 = \varepsilon_1^* \mathfrak{L}_1 \otimes f_1^* \mathcal{O}_{X_1}(1)$$

is the spectral line bundle for the Hecke eigensheaf $\mathscr{F}_{1,\bullet}$.

The following lemma, whose proof will be given below, shows the comparison between the two spectral line bundles on $Sym^3_M(\widetilde{C})$.

Lemma 13.26. *The pullback of $\mathscr{L}_1$ by the map $Sym^3_M(\widetilde{C}) \to Y_1$ is isomorphic to $\boldsymbol{N}^{(3)}$.*

To finish the proof of the theorem, we leave it to the reader to check that the tautological 1-form for $E^{(3)}$ given in Lemma 13.4 is the same as the restriction of the tautological form on Y_1. Therefore, the parabolic Higgs bundles are isomorphic, giving the desired isomorphism of local systems. □

Proof of Lemma 13.26: Express the degree 0 line bundle as $A = \mathcal{O}_C(a + b - 2\mathbf{p})$. Thus $M = \mathcal{O}_C(2a + 2b - 2\mathbf{p})$.

A point of $Sym^3_M(\widetilde{C})$ is a divisor $\tilde{y} = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3$ on $\widetilde{C}$, such that the image divisor $y_1 + y_2 + y_3$ is in $|M|$. This yields a line bundle

$$U_3(\tilde{y}) := \pi^*(A)(\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3) = \mathcal{O}_{\widetilde{C}}(\tilde{a} + \tilde{a}' + \tilde{b} + \tilde{b}' + \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 - 2\mathbf{p}' - 2\mathbf{p}'')$$

on $\widetilde{C}$, whose norm to C gives the line bundle $A^{\otimes 2} \otimes \mathcal{O}_C(y_1 + y_2 + y_3 - 2\mathbf{p}) \cong \mathcal{O}_C(\mathbf{p})$, so $U_3(\tilde{t})$ is a point of $\mathcal{P}_3$. This describes the map $Sym^3_M(\widetilde{C}) \to \mathcal{P}_3$ that lifts in a unique way to a map to the blow-up Y_1.

The line bundle $\mathfrak{L}_1$ extends to a line bundle $\mathfrak{L}_{1,\mathrm{Jac}}$ on the Jacobian $Jac^3(\widetilde{C})$, as may be seen by its definition. The above description gives a map

$$Sym^3(\widetilde{C}) \to Jac^3(\widetilde{C}).$$

Furthermore, compose with the translation $Jac^3(C) \to Jac^0(C)$ that relates $\mathfrak{L}_1$ with $\mathfrak{L}$, to get a map

$$U_{\mathrm{Jac}} : Sym^3(\widetilde{C}) \to Jac^0(\widetilde{C})$$

defined by

$$U_{\mathrm{Jac}}(\tilde{y}) = \tilde{a} + \tilde{a}' + \tilde{b} + \tilde{b}' + \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 - 4\mathbf{p}' - 3\mathbf{p}''.$$

The pullback of $\mathfrak{L}$ by this map is the tensor product of the values of $\mathfrak{L}$ on each of the points translated back to $Jac^0(\widetilde{C})$. All the points except those of $\tilde{y}$ are constant, so tensoring with those values leads to just tensoring with constant lines.

Let $s : \widetilde{C}^3 \to Sym^3(\widetilde{C})$ be the projection, and let $\boldsymbol{N}_0 := j^*\mathfrak{L}$ denote the pullback of $\mathfrak{L}$ to $\widetilde{C}$ along the map $j : \widetilde{C} \to Jac^0(\widetilde{C})$ that sends $\tilde{t}$ to $\mathcal{O}_{\widetilde{C}}(\tilde{t} - \mathbf{p}')$.

We conclude that

$$s^*U^*_{\mathrm{Jac}}(\mathfrak{L}) \cong pr_1^*(\boldsymbol{N}_0) \otimes pr_2^*(\boldsymbol{N}_0) \otimes pr_3^*(\boldsymbol{N}_0).$$

This in turn implies that

$$U^*_{\mathrm{Jac}}(\mathfrak{L}) \cong (\boldsymbol{N}_0)^{(3)}.$$

Therefore, the restriction of $\mathfrak{L}_1$ to $Sym^3_M(\widetilde{C})$ is the same as the restriction of $(\boldsymbol{N}_0)^{(3)}$.

The restriction of the spectral line bundle $\mathscr{L}_1$ to $Sym^3_M(\widetilde{C})$ is thus $(\boldsymbol{N}_0)^{(3)} \otimes \mathcal{O}_\ell(1)$.

We have $\boldsymbol{N} = \boldsymbol{N}_0 \otimes \mathcal{O}_{\widetilde{C}}(\mathbf{p}' + \mathbf{p}'')$.

The line bundle $\mathcal{O}_{\widetilde{C}}(\mathbf{p}' + \mathbf{p}'')^{(3)}$ on $Sym^3(\widetilde{C})$ is given by a divisor whose pullback to $\widetilde{C}^3$ is the set of points $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$ such that one of the coordinates is either $\mathbf{p}'$ or $\mathbf{p}''$. The divisor in $Sym^3(\widetilde{C})$ is the set of sums $\tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3$ containing either $\mathbf{p}'$ or $\mathbf{p}''$. This condition is equivalent to the condition that the sum $t_1 + t_2 + t_3$ contains $\mathbf{p}$. Restrict now to $Sym^3_M(\widetilde{C})$. For A hence M general, there is a unique sum that may be written as $\mathbf{p} + t_2 + t_3$, so our divisor consists of all the lifts of these points to $\widetilde{C}$. That, in turn, is the fiber of $Sym^3_M(\widetilde{C}) \to \ell = |M|$ over the point $\mathbf{p} + t_2 + t_3 \in |M|$ that is also described as the image of $\mathbf{p}$ under the trigonal map. We have now shown that the restriction of $\mathcal{O}_{\widetilde{C}}(\mathbf{p}' + \mathbf{p}'')^{(3)}$ to $Sym^3_M(\widetilde{C})$ is isomorphic to the pullback of $\mathcal{O}_\ell(1)$.

Now, the restriction of $\mathscr{L}_1$ to $Sym^3_M(\widetilde{C})$ is

$$(\boldsymbol{N}_0)^{(3)} \otimes \mathcal{O}_\ell(1) = (\boldsymbol{N}_0)^{(3)} \otimes \mathcal{O}_{\widetilde{C}}(\mathbf{p}' + \mathbf{p}'')^{(3)} = \boldsymbol{N}^{(3)}.$$

This completes the proof of Lemma 13.26, tying up what was needed for the proof of Theorem 13.25. □

We now sketch how to go from here to a global description over X_1. The proofs are left to the reader.

Let $P \to X_1$ be the degree 4 map whose fiber over $x \in X_1$ is the set of four points corresponding to four lines through x. Thus $P \to \mathrm{Jac}(C)$ is a $\mathbb{P}^1$-bundle.

Lemma 13.27. *Given any pair of two lines ℓ_1, ℓ_2 passing through x, we obtain a point of C. If ℓ_3, ℓ_4 is the opposite pair of lines (so that altogether these are the four lines through x) then the corresponding point of C is the conjugate by the hyperelliptic involution $\boldsymbol{\iota}_C$.*

Given a point of P lying over $x \in X_1$ it is one of the lines, so there are three pairs of lines containing that one; this gives a point of $Sym^3(C)$. Choosing a different point over x results in changing two of these three by the hyperelliptic involution, and altogether the four lines yield three pairs of pairs of points, hence three pairs of conjugate points in C, hence three points of $\mathbb{P}^1$. We get a diagram

$$\begin{array}{ccc} P & \rightarrow & Sym^3(C) \\ \downarrow & & \downarrow \\ X_1 & \rightarrow & Sym^3(\mathbb{P}^1). \end{array}$$

The hyperelliptic involution $\boldsymbol{\iota}_C$ acts on $Sym^3(C)$ by acting on all three of these points, so we can factor and obtain a diagram:

$$\begin{array}{ccc} P & \rightarrow & Sym^3(C)/\boldsymbol{\iota}_C \\ \downarrow & & \downarrow \\ X_1 & \rightarrow & Sym^3(\mathbb{P}^1). \end{array}$$

Here the vertical maps have degree 4 and the horizontal maps have degree 32.

Proposition 13.28. *The bottom map is the map given by squaring coordinates in $\mathbb{P}^5$, from X_1 to $\mathbb{P}^3 = Sym^3(\mathbb{P}^1)$. There are 6 planes in $\mathbb{P}^3$ corresponding to the six Weierstrass points of C, and the map $X_1 \to \mathbb{P}^3$ has ramification of order 2 along these. In particular it maps to a covering $X' \to \mathbb{P}^3$ of degree 2 ramified on these 6 planes.*

?? changed a question mark to the following remark :

Remark 13.29. It would be good to look more closely at the various group actions of $(\mathbb{Z}/2)^n$ for $n = 4, 5, 6$ as well as the Heisenberg group. This isn't done in the current version.

Theorem 13.30. *The space X' is the moduli space of PGL_2 bundles of odd degree. Let $\Delta \subset \mathbb{P}^3$ be the discriminant. Then the pullback of Δ to X' and X_1 are the wobbly loci of those spaces respectively.*

We can form the local system $\Lambda^{(\boxtimes 3)}$ on $Sym^3(C)$. There are two ways of descending Λ to a local system $\Lambda_{\mathbb{P}^1}$ on $\mathbb{P}^1$ with order two monodromy at the 6 points. For each of these we get local systems $\Lambda_{\mathbb{P}^1}^{(\boxtimes 3)}$.

The rank 8 local system $\Lambda^{(\boxtimes 3)}$ has singularities on the big diagonal of $Sym^3(C)$.

The rank 8 local system $\Lambda_{\mathbb{P}^1}^{(\boxtimes 3)}$ has singularities on the discriminant and the six planes in $\mathbb{P}^3$, and its pullbacks to X' and X_1 have singularities (generically finite of order 2) on the wobbly loci.

Theorem 13.31. *The pullback of our Hecke eigenvector local system of X_1 to P is isomorphic to the pullback of $\Lambda^{(\boxtimes 3)}$ to P. The pullbacks of $\Lambda_{\mathbb{P}^1}^{(\boxtimes 3)}$ to X' or X_1 are isomorphic to our Hecke eigenvector local system.*

We note that the discriminant Δ has a cuspidal locus along the small diagonal; the pullback will give the cuspidal locus of the wobbly locus.

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