Math 103: Limits of Finite Sums and the Definite Integral

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Outline

The general form of area estimates

If we want to estimate the area under the curve $y = f(x)$ on the interval $[a, b]$, we divide the interval up into n subintervals of length $\Delta x = \frac{b-a}{n}$.

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Then we pick a point c_k in the k-th subinterval and estimate the hight of the rectangle as $f(c_k)$.
Then an estimate of the area is given by the following sum

$$f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + ... + f(c_n) \cdot \Delta x$$

A quick review of summation notation

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Rules for finite sums, c is a constant.

1. $\sum_{k=1}^{n} a_k + b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. $\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k$
3. $\sum_{k=1}^{n} c = c \cdot n$

Useful Formulas

$$1 + 2 + 3 + ... + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2$$

n	L_n	U_n
10	.285	.385
20	.308	.358
30	.316	.350
50	.323	.343
100	.328	.338
1000	.333	.334

Definition

The **area** A of a region S that lies under the graph of a continuous function f is the limit of the sum of areas of the approximating rectangles:

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Where c_i is any value between x_{i-1} and x_i. A collection of such points are called **sample points**.

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$$\int_a^b f(x)dx = lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

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provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is , the definite integral $\int_a^b f(x)dx$ exists.

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Theorem

If f is integrable on [a, b], then

$$\int_a^b f(x)dx = lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ *and* $c_i = a + i\Delta x$.