ON THE TOPOLOGY AND INDEX OF MINIMAL SURFACES II

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ABSTRACT. For an immersed minimal surface in $\mathbb{R}^3$, we show that there exists a lower bound on its Morse index that depends on the genus and number of ends, counting multiplicity. This improves, in several ways, an estimate we previously obtained bounding the genus and number of ends by the index.

Our new estimate resolves several conjectures made by J. Choe and D. Hoffman concerning the classification of low-index minimal surfaces: we show that there is no complete two-sided immersed minimal surface in $\mathbb{R}^3$ of index two, complete embedded minimal surface with index three, or complete one-sided minimal immersion with index one.

1. INTRODUCTION

Let $X : \Sigma \to \mathbb{R}^3$ be a complete minimal immersion. We assume throughout Σ has finite Morse index. By work of Fischer-Colbrie and Gulliver–Lawson [FC85, GL86, Gul86] in the two-sided case, and Ross [Ros92] and Ros [Ros06] in the one-sided case, this is equivalent to Σ having finite total curvature. Thus, such a Σ is conformally equivalent to a closed Riemann surface $\overline{\Sigma}$ of finite genus g with finitely many punctures $p_1, p_2, \ldots, p_r$, which correspond to the ends of Σ. We refer the reader to the introduction of [CM16] for a thorough overview of existing results concerning the index of complete minimal surfaces in $\mathbb{R}^3$.

In our previous work [CM16], we established the following lower bound for the index of two-sided surfaces:

$$(1.1) \qquad \text{index}(\Sigma) \geq \frac{2}{3}(g+r) - 1.$$

We used this to prove the non-existence of embedded minimal surfaces of index two. However, (1.1) fails to account for the multiplicity of the end and thus is not very useful for the study of immersed minimal surfaces.[1]

In this work, we substantially improve our estimate (1.1) both in terms of the numerical constants and also so as to account for the multiplicity of the ends. As a consequence, we obtain several new classification results for complete minimal surfaces of low index. Our main results are as follows.

Theorem 1.1. *Let $X : \Sigma \to \mathbb{R}^3$ be a two-sided complete immersed minimal surface. Suppose Σ has genus g and r ends $E_1, E_2, \ldots, E_r$, with multiplicities respectively $d_1, d_2, \ldots, d_r$. Then*

$$\text{index}(\Sigma) \geq \frac{1}{3}\left(2g + 2\sum_{j=1}^{r}(d_j + 1) - 5\right).$$

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[1]For example, applying [MR91, Corollary 15] to the kth-order Enneper's surface $X_k : \mathbb{C} \to \mathbb{R}$ (the minimal surface with Weierstrass data $g(z) = z^k, dh = z^k dz$ in the notation of [HK97]), we find that index $X_k = 2k - 1 \to \infty$ as $k \to \infty$. On the other hand, the right hand side of (1.1) is $-\frac{1}{3}$ for all k. Note that Theorem 1.1 yields the lower bound $\frac{1}{3}(2k + 1)$.

This is stronger than our bound (1.1) from [CM16] even in the case when there are no multiplicities (i.e., the ends are embedded). Indeed, we have:

Corollary 1.2. *Let $X : \Sigma \to \mathbb{R}^3$ be a two-sided complete immersed minimal surface of genus g and with r embedded ends. Then,*

$$\text{index}(\Sigma) \geq \frac{1}{3} \left(2g + 4r - 5\right).$$

Our method also extends to the one-sided case, in which case we find

Theorem 1.3. *Let Σ be a one-sided complete immersed minimal surface in $\mathbb{R}^3$ of finite total curvature. Suppose that the ends of Σ, $E_1, E_2, \ldots, E_r$, have multiplicities $d_1, \ldots, d_r$ and that the genus of the two-sided double cover $\hat{\Sigma}$ is g. Then,*

$$\text{index}(\Sigma) \geq \frac{1}{3} \left(g + 2 \sum_{j=1}^{r}(d_j + 1) - 4\right).$$

Our proof of Theorems 1.1 and 1.3 proceed as in [CM16] but with a more delicate choice of weighted space (see Section 3). Our new weighted spaces allow us to consider harmonic 1-forms in the stability inequality with even slower decay towards the ends of the minimal surface than those considered previously. We note that the construction of the index in this space (in the sense of Fischer-Colbrie [FC85]) in Section 4 is somewhat subtle. Indeed, the role of the parameter α in the cut-off function φ_R^α defined in Section 2.5 plays a crucial role in this section. This might be useful elsewhere.

Example 1.4. For example, on the catenoid $\Sigma = \mathbb{C}\backslash\{0\}$, the weighted spaces used in [CM16] allowed us to plug in (the real and imaginary components of) $\frac{dz}{z}$. Here, our new weight allows the additional forms dz and $\frac{dz}{z^2}$. This yields the *sharp* lower bound[2] of $\text{index}(\Sigma) \geq 1$.

1.1. **Applications.** As noted above, our improved bounds yield new classification results for low-index minimal surfaces. In [CM16], we proved there are no complete embedded minimal surfaces in $\mathbb{R}^3$ of index two. Here, can improve this in two ways: in Theorem 1.5 we extend this to rule out immersed index two surfaces and in Theorem 1.7 we extend this to rule out embedded index three surfaces. We also show that there are no index one one-sided minimal immersions in Theorem 1.8.

The following was conjectured by Choe [Cho90, p. 210].

Theorem 1.5. *There is no complete two-sided minimal immersion into $\mathbb{R}^3$ with index two.*

Proof. Consider $X : \Sigma \to \mathbb{R}^3$ of index two. By [Cho90, Theorem 7] or [Nay90, Corollary 3.3], we can assume that Σ has genus $g \geq 1$. Applying Theorem 1.1, we find that

$$2g + 2 \sum_{j=1}^{r}(d_j + 1) \leq 11.$$

This implies that $g \leq 2$. If $g = 2$ then

$$\sum_{j=1}^{r}(d_j + 1) \leq 3,$$

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[2]We have 6 linearly independent harmonic 1-forms to start. We subtract 3 to remove the "linear" forms $*dx^1, *dx^2, *dx^3$. Finally, we must divide by 3 when using the method developed by Ros [Ros06] to estimate the index.

implying that $r = 1$ and $d_1 \in \{1, 2\}$. If $d_1 = 1$, then $X(\Sigma)$ is a flat plane, a contradiction. Moreover, if $d_1 = 2$, we obtain a contradiction from the monotonicity formula (cf. [HK97, Proposition 3.1]).

Thus, $g = 1$. In this case, Theorem 1.5 yields

$$\sum_{j=1}^{r}(d_j + 1) \leq 4.$$

Combined with the Jorge–Meeks formula (2.1), we find

$$\int_{\Sigma} \kappa \geq -8\pi,$$

where κ is the Gauss curvature of Σ. By work of López [Lóp92], because $g = 1$, the only possibility is that $X : \Sigma \to \mathbb{R}^3$ is the Chen–Gackstatter surface (cf. [HK97, §2.2]). However, the Chen–Gackstatter surface has index 3 by work of Montiel–Ros [MR91, Corollary 15]. This concludes the proof. $\square$

Remark 1.6. We note that a similar argument can be used to give a new proof of the classification of complete immersed minimal surfaces of index one by López–Ros [LR89] (they must be either Enneper's surface or the catenoid).

The following was posed to us by Hoffman (see [CM16, p. 402]).

Theorem 1.7. *There is no complete embedded minimal surface in $\mathbb{R}^3$ with index three.*

Proof. Consider an embedded minimal surface $\Sigma \subset \mathbb{R}^3$ with index$(\Sigma) = 3$. Using Theorem 1.1, we find

$$g + 2r \leq 7.$$

Because $r \geq 3$ and $g \geq 1$, we find that $g = 1$ and $r = 3$. Thus by Costa's characterization of embedded minimal tori with 3-ends [Cos89, Cos91], Σ must be a member of the deformation family of the Costa surface, as constructed by Hoffman–Meeks (see [Cos91] and [HK97, §4]).

We show in Section 7 that the three ended tori in the in deformation family have index at least 4.[3] This is a contradiction. $\square$

We also confirm the following conjecture of Choe [Cho90, p. 210]:

Theorem 1.8. *There is no complete one-sided minimal immersion in $\mathbb{R}^3$ with index one.*

Proof. Consider $X : \Sigma \to \mathbb{R}^3$ a complete one-sided minimal immersion with index one. Theorem 1.3 implies that

$$g + 2\sum_{j=1}^{r}(d_j + 1) \leq 7.$$

Because $\sum_{j=1}^{r}(d_j + 1) \geq 4$ (cf. [HK97, Proposition 3.1]), this is a contradiction. $\square$

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[3]Nayatani has shown that the index of the Costa surface itself is exactly 5 [Nay93] (see also [Nay92, Mor09]). However, it seems to be unknown how the index behaves along the deformation family [Nay17, Mor17]. Choe's vision number argument [Cho90, Corollary 5] applies in this case to prove that it is at least 3, but this does not suffice for us here.

Remark 1.9. Ambrozio–Buzano–Carlotto–Sharp have observed [ABCS18b] that our previous work [CM16] (along with López–Ros [LR89] and our work[4] with Ketover [CKM17] on the degeneration of bounded index minimal surfaces) leads to compactness results for minimal surfaces with low index but comparably large genus in (non-bumpy) Riemannian 3-manifolds with positive scalar curvature. Our Theorems 1.1 and 1.7 can be used to sharpen some of their results.

We also note that the same authors have observed that our work [CM16] is also relevant to the study of boundary degeneration of index one free boundary minimal surfaces [ABCS18a]. The analysis of the index of free boundary surfaces in a half-space has some features in common with our arguments in Section 7 concerning the index of the deformation family.

Finally, we explain how our main results imply a qualitative version of the fact that finite index is equivalent to finite total curvature. Theorems 1.1 and 1.3, combined with the Jorge–Meeks formula yield the following result (the upper bound is due[5] to Ejiri–Micallef [EM08]).

Theorem 1.10. *If $X : \Sigma \to \mathbb{R}^3$ is a non-planar two-sided minimal immersion, then*

$$\frac{1}{3} + \frac{1}{6\pi} \int_{\Sigma} (-\kappa) \leq \mathrm{index}(\Sigma) \leq -3 + \frac{3}{2\pi} \int_{\Sigma} (-\kappa).$$

For $X : \Sigma \to \mathbb{R}^3$ a non-planar one-sided minimal immersion, we have

$$\frac{1}{3} + \frac{1}{6\pi} \int_{\Sigma} (-\kappa) \leq \mathrm{index}(\Sigma) \leq -6 + \frac{3}{\pi} \int_{\Sigma} (-\kappa).$$

That such a bound should hold was originally conjectured by Fischer-Colbrie [FC85] and more explicitly by Grigoryan–Netrusov–Yau [GY03, GNY04]. It can be seen as a quantitative version of the fact that finite total curvature is equivalent to finite index [FC85, GL86, Gul86]. We emphasize that by combining [Nay93, Theorem 4] with [Ros06, Theorem 17] one can give a different proof of the lower bound in the case of two-sided surfaces, albeit with a much worse constant.

Proof. If $X : \Sigma \to \mathbb{R}^3$ is a two-sided minimal immersion, then Theorem 1.1 yields

$$\mathrm{index}(\Sigma) \geq \frac{1}{3}\left(2g + 2\sum_{j=1}^{r}(d_j + 1) - 5\right)$$

$$= \frac{2}{3}\left(g - 1 + \frac{1}{2}\sum_{j=1}^{r}(d_j + 1)\right) + \frac{1}{3}\sum_{j=1}^{r}(d_j + 1) - 1$$

$$= -\frac{1}{6\pi}\int_{\Sigma}\kappa + \frac{1}{3}\sum_{j=1}^{r}(d_j + 1) - 1.$$

The final equality follows from the Jorge–Meeks formula (2.1). Since $\sum_{j=1}^{r}(d_j + 1) \geq 4$ for a non-flat minimal surface (cf. [HK97, Proposition 3.1]), we find

$$\mathrm{index}(\Sigma) \geq \frac{1}{3} - \frac{1}{6\pi}\int_{\Sigma}\kappa,$$

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[4]See also the subsequent work of Buzano–Sharp [BS18].

[5]The same bound with a worse constant was originally proven by Tysk [Tys87]. See also [Nay93, GY03].

as claimed.

On the other hand, [EM08, Theorem 3.2] yields

$$\text{index}(\Sigma) \leq -\frac{1}{\pi} \int_\Sigma \kappa + 2g - 3$$

$$= -\frac{3}{2\pi} \int_\Sigma \kappa - \sum_{j=1}^{r} (d_j + 1) - 1$$

$$\leq -\frac{3}{2\pi} \int_\Sigma \kappa - 3$$

In the second step, we have used the Jorge–Meeks formula (2.1).

The argument in the one-sided case uses a similar argument based on Theorem 1.3 for the lower bound. For the upper bound, we combine [EM08, Theorem 3.2] with the fact that the index of the two-sided double cover is at least the index of the one-sided surface (and the total curvature is doubled). $\square$

1.3. **Organization.** In Section 2 we give the relevant definitions and background, as well as recalling several useful computations. In Section 3 we descrbe our new weighted space and study the harmonic forms lying in this space. In Section 4, we construct weighted eigenfunctions realizing the index. In Section 5 we prove the main theorem, Theorem 1.1 and in Section 6 we prove the non-orientable version, Theorem 1.3. Finally, in Section 7 we study the index of the deformation family containing the Costa surface.

2. FINITE MORSE INDEX IMMERSIONS

Throughout this note, we shall always assume that $X : \Sigma \to \mathbb{R}^3$ is a complete immersed minimal surface in $\mathbb{R}^3$ of finite index. Apart from Sections 2.2 and 6, we shall always assume that Σ is two-sided. We will denote by $|\cdot|$ the Euclidean distance and by $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ the extrinsic ball of radius R. Moreover, C will denote a positive constant, allowed to change from line to line.

2.1. **Two-sided immersions.** For $X : \Sigma \to \mathbb{R}^3$ two-sided minimal immersion in $\mathbb{R}^3$, we consider the stability operator defined by $L := -\Delta + 2\kappa$, where Δ is the intrinsic Laplacian associated to $X^*(g_{\mathbb{R}^3})$, and the associated quadratic form

$$Q(\phi, \phi) := \int_\Sigma |\nabla \phi|^2 + 2\kappa \phi^2.$$

Here, κ is the Gauss curvature of Σ. We note that Σ is of finite index if and only if it has finite total curvature [FC85] and, hence, it is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$, punctured at finitely many points $p_1, \ldots, p_r$ [Oss64]. Moreover, the Gauss map extends across the punctures as a meromorphic map and, in particular, such a Σ is properly immersed and we may also compute the index by taking the limit of extrinsic balls:

$$\text{index}(\Sigma) = \lim_{R \to \infty} \text{index}(\Sigma \cap B_R).$$

where $\mathrm{index}(\Sigma \cap B_R)$ is the number of negative eigenvalues of L in $\Sigma \cap B_R(0)$ with Dirichlet boundary condition. We refer to [CM16] and the references therein for details.

For a each $j = 1, 2, \ldots, p_r$, we consider D_j a small punctured disk neighborhood of p_j in $\overline{\Sigma}$. We refer to $E_j = X(D_j)$ as an end of Σ. Following [JM83], for a fixed $R > 0$, we consider $\Gamma_{R,j} = \{q \in \mathbb{S}^2 \mid Rq \in E_j\}$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, and observe that $\Gamma_{R,j}$ converges smoothly as $R \to \infty$ to a great circle of $\mathbb{S}^2$ with finite multiplicity, which we refer to as the multiplicity of E_j and denote by d_j. Moreover, Jorge-Meeks [JM83] showed that:

$$(2.1) \qquad -\frac{1}{4\pi} \int_\Sigma \kappa = g - 1 + \frac{1}{2}\left(r + \sum_{j=1}^r d_j \right).$$

2.2. One-sided immersions.
For an one-sided immersions $X : \Sigma \to \mathbb{R}^3$, the Morse index and total curvature can be defined using the two-sheeted orientable covering:

$$\pi : \widehat{\Sigma} \to \Sigma.$$

For instance, if $\tau : \widehat{\Sigma} \to \widehat{\Sigma}$ denotes the change of sheets involution and N is the unit normal vector of $\widehat{\Sigma}$, then $\tau \circ N = -N$ and an infinitesimal compactly supported deformation of Σ correspond to a function ϕ with compact support in $\widehat{\Sigma}$ satisfying $\phi \circ \tau = -\phi$. Thus, if Q is the quadratic form associated with the second variation of area for $\widehat{\Sigma}$, the index of Σ may be defined as the number of negative eigenvalues of Q with respect to deformations that are anti-invariant with respect to τ.

The total curvature of Σ, on the other hand, is equal to the degree of its "Gauss map" $n : \Sigma \to \mathbb{RP}^2$, which in turn can be defined via the following diagram:

$$\begin{array}{ccc} \widehat{\Sigma} & \xrightarrow{N} & \mathbb{S}^2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \Sigma & \xrightarrow{n} & \mathbb{RP}^2 \end{array}$$

where $p : \mathbb{S}^2 \to \mathbb{RP}^2$ is the standard covering of $\mathbb{RP}^2$. It follows from Ros [Ros06, Theorem 17] that Σ has finite Morse index if and only if it has finite total curvature. The total curvature of Σ can then be estimated by applying the Jorge-Meeks formula to $\widehat{\Sigma}$. Indeed, if $\widehat{\Sigma}$ has genus g and s ends, then s must be even, say $s = 2r$ and the ends of $\widehat{\Sigma}$ must be of the form $E_1, E_2, \ldots, E_r, \tau(E_1), \tau(E_2), \ldots, \tau(E_r)$, and so:

$$\deg(n) = \deg N$$

$$= g - 1 + \frac{1}{2}\left(s + \sum_{j=1}^s d_j \right).$$

Finally, since E_j and $\tau(E_j)$ must have the same multiplicity, the above reduces to:

$$(2.2) \qquad -\frac{1}{2\pi} \int_\Sigma \kappa = \deg(n) = g - 1 + \left(r + \sum_{j=1}^r d_j \right).$$

2.3. **Distance function at an end E_j.** We finish this part of the paper with Sections 2.3 through 2.5. They carry important calculations that will be used later, but could be skipped at first reading. We assume two-sidedness again until Section 6 (where we will apply these computations to the two-sided double cover).

The following calculation will be important for proving integrability. Write $X : D_j\backslash\{0\} \to \mathbb{R}^3$ for a conformal parametrization of one of its end E_j with multiplicity d_j. Assume that the Gauss map satisfies $g(0) = 0$. If z is the local coordinate in $D_j\backslash\{0\}$, we follow the proof of [CM16, Lemma 10] (cf. [HK97, Proposition 2.1]) and write the Weierstrass representation:

$$X(z) = \text{Re} \int (\phi_1, \phi_2, \phi_3)$$

where

$$\phi_1(z) = \left(\frac{A}{z^{d_j+1}} + \frac{B_1(z)}{z^{d_j}} \right) dz$$

$$\phi_2(z) = \left(\frac{iA}{z^{d_j+1}} + \frac{B_2(z)}{z^{d_j}} \right) dz$$

$$\phi_3(z) = \left(\frac{B_3(z)}{z^{d_j}} \right) dz,$$

for $B_1(z), B_2(z), B_3(z)$ holomorphic on D_j and $A \in \mathbb{C} \setminus \{0\}$. Integrating this, we see that

$$X(z) = \text{Re} \left(-\frac{A}{d_j} z^{-d_j} + O(z^{-d_j+1}), -\frac{iA}{d_j} z^{-d_j} + O(z^{-d_j+1}), O(z^{-d_j+1}) \right),$$

so

(2.3) $$|X(z)| \simeq |z|^{-d_j}.$$

Continuing on, because $g(z) = O(z)$, the unit normal can be written as

$$N = \left(\frac{2\,\text{Re}(g)}{|g|^2 + 1}, \frac{2\,\text{Im}(g)}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right) = (0, 0, -1) + O(z).$$

This allows us to estimate $N \cdot \nabla_{\mathbb{R}^3}|x|$ as $z \to 0$,

$$N \cdot \nabla_{\mathbb{R}^3}|x| = N \cdot \frac{X(z)}{|X(z)|} = O(z).$$

Since $|X(z)| \simeq |z|^{-d_j}$, we find that

(2.4) $$|N \cdot \nabla_{\mathbb{R}^3}|x|| \le C|z| \le C|x|^{-1/d_j}.$$

2.4. **The Gaussian curvature at an end E_j.** Assuming as in the previous section that $X : D_j \setminus \{0\} \to \mathbb{R}^3$ is conformally parametrizing the end E_j. Write d_j for the multiplicity of E_j and assume that $g(0) = 0$. Write

$$B_3(0) = bz^n + O(z^{n+1})$$

for some $n \ge 0$ and $b \ne 0$.

Recall that the Weierstrass representation gives (see [HK97, Theorem 2.1])

$$(\phi_1(z), \phi_2(z), \phi_3(z)) = \left(\frac{1}{2}(g^{-1} - g)dh, \frac{i}{2}(g^{-1} + g)dh, dh \right),$$

for ϕ_i considered above. Note that

$$g = \frac{\phi_3}{\phi_1 - i\phi_2} = \frac{B_3(z)z}{2A + z(B_1 - iB_2)} = \frac{b}{2A}z^{n+1} + O(z^{n+2})$$

and

$$dh = bz^{-d_j+n} + O(z^{-d_j+n+1}).$$

Now, recall that the Gaussian curvature can be written in terms of g and dh (see [HK97, (2.10)]) as

$$\kappa = \frac{-16}{(|g| + |g|^{-1})^4} \left| \frac{dg/g}{dh} \right|^2.$$

We compute

$$\left| \frac{dg/g}{dh} \right|^2 = \left(\frac{n+1}{b} \right)^2 |z|^{2(d_j-n-1)}(1 + O(z))$$

Thus, there is $c > 0$ depending on E_j so that

$$\kappa(X(z)) = -c|z|^{2(d_j+n+1)}(1 + O(z)).$$

It is useful to write this using $|X(z)| \simeq |z|^{-d_j}$ as

$$(2.5) \qquad\qquad \kappa(x) = O(|x|^{-2-\frac{2(n+1)}{d_j}}).$$

for $x \in E_j$ with $|x| \to \infty$. All that will matter in the sequel is the super-quadratic decay.

2.5. **A one-parameter family of cut-off functions.** Let $\xi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\chi(s) = 0$ for $s \leq 0$ and $\chi(s) = 1$ for $s \geq 1$. In particular, there exists a constant $C > 0$ such that $|\xi'|, |\xi''| \leq C$. For any $\alpha > 1$ and $R > 1$, we define $\varphi_R^\alpha : \Sigma \to \mathbb{R}$ given by:

$$\varphi_R^\alpha(x) = \xi \circ \psi_R^\alpha(x),$$

where $\psi_R^\alpha(x) = 1 + \frac{1}{\alpha-1} - \frac{\log |x|}{(\alpha-1)\log R}$. Note that ψ_R^α is a modification of the standard log-cutoff function (which corresponds to taking $\alpha = 2$). We will use $\alpha = 2$ in most places throughout, but will take α large in the proof of Proposition 4.3.

Note that φ_R^α satisfies $\varphi_R^\alpha(x) = 1$ for $|x| \leq R$ and $\varphi_R^\alpha(x) = 0$ for $|x| \geq R^\alpha$. The goal of the remainder of this section is to estimate $|\nabla \varphi_R^\alpha|$ and $|\Delta \varphi_R^\alpha|$ where ∇, Δ are the intrinsic gradient and Laplacian associated to $X^*(g_{\mathbb{R}^3})$. These are clearly supported in the compact region $X^{-1}(B_{R^\alpha}(0) \setminus B_R(0))$. First, we note:

$$|\nabla \varphi_R^\alpha| = |\xi'||\nabla \psi_R^\alpha|$$

$$\leq C \left| \frac{1}{|x|(\alpha-1)\log R} \nabla |x| \right|$$

$$(2.6) \qquad\qquad \leq \frac{C}{|x|(\alpha-1)\log R}.$$

Next, since Σ is minimal, we recall that $|x|\Delta|x| = 2 - |\nabla|x||^2$. Hence:

$$\Delta \log |x| = \frac{1}{|x|}\Delta|x| - \frac{1}{|x|^2}|\nabla|x||^2$$

$$= \frac{2}{|x|^2}(1 - |\nabla|x||^2)$$

Since the gradient of $|x|$ in Σ is just the tangent projection of the gradient of $|x|$ in $\mathbb{R}^3$, we have that $1 - |\nabla|x||^2 = (N \cdot \nabla_{\mathbb{R}^3}|x|)^2$. Using this:

$$\Delta\psi_R^\alpha(x) = -2\frac{(N \cdot \nabla_{\mathbb{R}^3}|x|)^2}{(\alpha - 1)|x|^2 \log R},$$

and thus, by (2.4), if $k = \min\{d_1, d_2, \ldots, d_r\}$, we get the estimate:

$$|\Delta\varphi_R^\alpha(x)| \leq \frac{C}{(\alpha - 1)|x|^{2+2/k} \log R}.$$

Therefore, we finally obtain (taking $C > 0$ larger if necessary):

$$(2.7) \qquad |\Delta\varphi_R^\alpha| \leq |\xi''||\nabla\varphi_R^\alpha|^2 + |\xi'||\Delta\varphi_R^\alpha| \leq \frac{C}{|x|^2(\alpha - 1)^2 \log^2 R} + \frac{C}{(\alpha - 1)|x|^{2+2/k} \log R}.$$

3. A NEW WEIGHTED SPACE AND MORE HARMONIC 1-FORMS

We define the weighted space $L_*^2(\Sigma)$ to be the completion of smooth compactly supported functions with respect to the norm

$$\|f\|_{L_*^2(\Sigma)}^2 := \int_\Sigma f^2(1 + |x|^2)^{-1}(\log(2 + |x|))^{-2},$$

where $|x|$ is the Euclidean distance. This norm clearly comes from an inner product, making $L_*^2(\Sigma)$ into a Hilbert space.

3.1. Harmonic 1-forms.
We will show next that the extra logarithmic weight allows for more harmonic 1-forms. Write $\mathscr{H}^1(\Sigma)$ for the space of all harmonic 1-forms on Σ.

Proposition 3.1. *If $X : D\backslash\{0\} \to \Sigma$ be a conformal parametrization of an end E_j of Σ. If z is the local coordinate in $D\backslash\{0\}$, the 1-forms $\frac{dz}{z}, \frac{dz}{z}^2, \ldots, \frac{dz}{z^{d_j+1}}$ are all integrable in $L_*^2(E_j)$, where d_j is the multiplicity of E_j.*

Proof. Let $\omega = \frac{dz}{z^l}$, where $1 \leq l \leq d_j+1$. By (2.3), we have that $|X(z)| \simeq |z|^{-d_j}$. Additionally, because the squared norm of a 1-form times the area element is a pointwise conformally invariant quantity, we can estimate:

$$\left\|\frac{dz}{z^l}\right\|_{L_*^2(E_j)}^2 \leq \int_{D_j\backslash\{0\}} \frac{1}{|z|^{2l}}(1 + |x|^2)^{-1}(\log(2 + |x|))^{-2}|dz|^2$$

$$\leq C\int_{D_j\backslash\{0\}} \frac{1}{|z|^{2l}}\frac{|z|^{2d_j}}{1 + |z|^{2d_j}}\frac{1}{\log^2(2 + |z|^{-2d_j})}|dz|^2$$

$$\leq C\int_0^1 r^{2(d_j-l)+1}(\log r)^{-2}\,dr$$

For any $l \in \{1, \ldots, d_j + 1\}$, the above integral is finite.[6] $\square$

Proposition 3.2. *If Σ has genus g and r ends, with multiplicities $d_1, d_2, \ldots, d_r$, the dimension of $\mathscr{H}^1(\Sigma) \cap L_*^2(\Sigma)$, i.e., harmonic forms in $L_*^2(\Sigma)$, is $2g + 2\sum_{j=1}^r(d_j + 1) - 2$.*

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[6]Note that $\left(\frac{1}{\log r}\right)' = -r^{-1}(\log r)^{-2}$.

Proof. By Osserman [Oss64], Σ is conformal to a Riemann surface with genus g and r punctures, say $p_1, p_2 \ldots, p_r$, corresponding respectively to the ends $E_1, E_2, \ldots E_r$. By Riemann-Roch,[7] the dimension of holomorphic 1-forms on $\overline{\Sigma}$ which have poles of order at most $d_j + 1$ at the point p_j, for $j = 1, \ldots, r$ is exactly

$$g + \sum_{j=1}^{r}(d_j + 1) - 1.$$

By taking real and imaginary parts, we find at least $2g + 2\sum_{j=1}^{r}(d_j + 1) - 2$ harmonic forms in $L_*^2(\Sigma)$.

Conversely, for any harmonic form $\omega \in L_*^2(\Sigma)$, the form $\widetilde{\omega} := \omega + i * \omega$ is holomorphic on Σ (cf. [FK80, I.3.11]). Using (2.3) and the definition of $L_*^2(\Sigma)$, it is easy to check that $\widetilde{\omega}$ must extend meromorphically to $\overline{\Sigma}$, with a pole of order $d_j + 1$ at each point p_j. This proves the assertion. $\qquad\square$

4. WEIGHTED EIGENFUNCTIONS

Proposition 4.1. *Suppose $f \in L_*^2(\Sigma)$ is a smooth function on Σ satisfying the weighted eigenvalue equation:*

$$\Delta f - 2\kappa f + \lambda(1 + |x|^2)^{-1}(\log(2 + |x|))^{-2} f = 0.$$

Then $|\nabla f| \in L^2(\Sigma)$.

Proof. Let $\varphi = \varphi_R^\alpha$ with $\alpha = 2$. We will show integrability of $|\nabla f|$ by proving that $\int_\Sigma \varphi^2 |\nabla f|^2$ stays bounded as $R \to \infty$. First we compute:

$$\int_\Sigma \varphi^2 |\nabla f|^2 = -\int_\Sigma \varphi^2 f \Delta f - \frac{1}{2}\int_\Sigma \nabla \varphi^2 \cdot \nabla f^2$$

$$= -\int_\Sigma 2\kappa\varphi^2 f^2 + \lambda \int_\Sigma \varphi^2 f^2 (1 + |x|^2)^{-1}(\log(2 + |x|))^{-2} - \frac{1}{2}\int_\Sigma \nabla \varphi^2 \cdot \nabla f^2$$

Since κ decays super-quadratically by (2.5), and $f \in L_*^2$ we only need to check the integrability of $\int_\Sigma \nabla \phi^2 \cdot \nabla f^2$ as $R \to \infty$. Then:

$$-\int_\Sigma \nabla \varphi^2 \cdot \nabla f^2 = \int_\Sigma f^2 \Delta \varphi^2$$

(4.1)
$$= 2\int_\Sigma f^2 \varphi \Delta \varphi + 2\int_\Sigma f^2 |\nabla \varphi|^2$$

We analyze each term separately. First note that, by (2.7):

$$\left|\int_\Sigma f^2 \varphi \Delta \varphi\right| \le \int_\Sigma f^2 |\varphi||\Delta \varphi|$$

(4.2)
$$\le \underbrace{\int_\Sigma f^2 \frac{C|\varphi|}{|x|^2 \log^2 R}}_{(\mathrm{I})} + \underbrace{\int_\Sigma f^2 \frac{C|\varphi|}{|x|^{2+2/k} \log R}}_{(\mathrm{II})}.$$

where $k \ge 1$ is the highest multiplicity of an end of Σ.

---

[7]Specifically, this follows by considering the divisor $\mathfrak{U} = \prod_{j=1}^{r} p_j^{-d_j - 1}$ in e.g., [FK80, §III.4].

Because $\varphi(x)$ is only nonzero for $|x| \in [R, R^2]$, we have:

$$(\mathrm{I}) = \int_\Sigma f^2 \frac{C|\varphi|}{|x|^2 \log^2 R} \leq C \int_\Sigma f^2 \frac{|\varphi| \log^2 |x|}{|x|^2 \log^2 |x| \log^2 R}$$

$$\leq C \int_\Sigma f^2 \frac{|\varphi| \log^2 R^2}{|x|^2 \log^2 |x| \log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{1}{|x|^2 \log^2 |x|}.$$

This tends to zero as $R \to \infty$ since $f \in L_*^2$.

Moreover, because $|\varphi| \leq 1$, we get:

$$(\mathrm{II}) = \int_\Sigma f^2 \frac{C|\varphi|}{|x|^{2+2/k} \log R} \leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{1}{|x|^2 \log^2 |x|} \frac{\log |x|}{|x|^{2/k}},$$

which also goes to zero as $R \to \infty$, since $f \in L_*^2$ and $|x|^{2/k} \gg \log |x|$ as $x \to \infty$.

Finally, for the second term in (4.1):

$$\int_\Sigma f^2 |\nabla \varphi|^2 \leq \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{C}{|x|^2 \log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{\log^2 |x|}{|x|^2 \log^2 |x| \log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{\log^2 R^2}{|x|^2 \log^2 |x| \log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} f^2 \frac{1}{|x|^2 \log^2 |x|}.$$

This tends to zero as $R \to \infty$, like before. This completes the proof. $\qquad\square$

Lemma 4.2. *For ω a 1-form in $L_*^2(\Sigma) \cap \mathscr{H}^1(\Sigma)$, we have that*

$$\int_\Sigma |\nabla \omega|^2 < \infty.$$

Proof. Using the Bochner formula as in Lemma 13 of [CM16], the proof follows similarly to Proposition 4.1 above. $\qquad\square$

We now prove that the weighted analogue of Fischer–Colbrie's result constructing the index on a complete minimal surface [FC85, Proposition 2] holds for $L_*^2(\Sigma)$.

Proposition 4.3. *Suppose that Σ has index $\mathrm{index}(\Sigma) = k < \infty$ in the usual L^2-sense. Then there exists a k-dimensional subspace W of $L_*^2(\Sigma)$ with an L_*^2-orthornormal basis of L_*^2-eigenfunctions for the stability operator $f_1, \ldots, f_k$. Letting the associated eigenvalues be $\lambda_1, \ldots, \lambda_k$, each $\lambda_i < 0$ and moreover $Q(\phi, \phi) \geq 0$ for $\phi \in C_0^\infty(\Sigma) \cap W^\perp$, where $W^\perp \subset L_*^2(\Sigma)$ is the $L_*^2(\Sigma)$ orthogonal complement of W.*

Proof. Choose R_0 sufficiently large so that $k = \mathrm{index}(\Sigma) = \mathrm{index}(\Sigma \cap B_\rho)$ for $\rho > R_0$. Let $\{f_{1,\rho}, \ldots, f_{k,\rho}\}$ and $\{\lambda_{1,\rho}, \ldots, \lambda_{k,\rho}\}$ denote the $L_*^2(\Sigma \cap B_\rho)$-weighted Dirichlet eigenfunctions and eigenvalues, respectively, of an $L_*^2(\Sigma \cap B_\rho)$-orthonormal basis constructed by minimizing the Rayleigh quotient

$$Q(\phi, \phi) / \|\phi\|_{L_*^2(\Sigma \cap B_\rho)}^2$$

(note that there are exactly k such eigenfunctions since the L^2 and L_*^2 norms are equivalent in B_ρ). It is not hard to check that this implies that

$$\Delta f_{i,\rho} - 2\kappa f_{i,\rho} + \lambda_{i,\rho}(1 + |x|^2)^{-1}(\log(2 + |x|))^{-2} f_{i,\rho} = 0$$

in $\Sigma \cap B_\rho$.

Our goal is to prove that $f_{i,\rho}$ converges to the desired eigenfunctions as for an appropriate sequence of ρ going to infinity. To that end, note that $\Sigma \setminus B_R$ must be stable for any $R > R_0$. Thus, if η is any smooth function vanishing in B_R, the stability inequality implies that for any $\phi \in C_c^\infty(\Sigma)$,

$$(4.3) \qquad -\int_\Sigma 2\kappa(\eta\phi)^2 \le \int_\Sigma |\nabla(\eta\phi)|^2 = \int_\Sigma \eta^2 |\nabla\phi|^2 + 2\eta\phi \langle \nabla\eta, \nabla\phi \rangle + \phi^2 |\nabla\eta|^2.$$

In particular, arguing this as in [FC85, Proposition 2] and [CM16, Proposition 8], choosing $\eta = 1 - \varphi_R^\alpha$, with $\alpha = 2$,

$$\int_\Sigma (1 - \eta^2)(|\nabla\phi|^2 + 2\kappa\phi^2) \le Q(\phi, \phi) + \int_\Sigma 2\eta\phi \langle \nabla\eta, \nabla\phi \rangle + \phi^2 |\nabla\eta|^2$$

$$\le Q(\phi, \phi) + \int_{\Sigma \cap B_{R^2}} \eta^2 |\nabla\phi|^2 + 2\phi^2 |\nabla\eta|^2$$

$$\le 2Q(\phi, \phi) + \int_{\Sigma \cap B_{R^2}} 2\phi^2 |\nabla\eta|^2 - 2\kappa\phi^2.$$

Thus, we find

$$(4.4) \qquad \int_{\Sigma \cap B_R} |\nabla\phi|^2 \le 2Q(\phi, \phi) + \underbrace{\left(\frac{C}{R^2 \log^2 R} + \sup_{\Sigma \cap B_{R^2}} 4|\kappa| \right)}_{:=2C_R} \int_{\Sigma \cap B_{R^2}} \phi^2.$$

Because $\max\{\lambda_{1,\rho}, \ldots, \lambda_{k,\rho}\}$ is decreasing with ρ, there is $\epsilon_0 > 0$ so that for $\rho \ge R > R_0$, we have $\lambda_{i,\rho} < -\epsilon_0$. On the other hand, the inequality we have just proven shows that $\lambda_{i,\rho} \ge -C_R(1 + R^2) \log^2(2 + R^2)$.

Moreover, (4.4) yields

$$\int_{\Sigma \cap B_R} f_{i,\rho}^2 + |\nabla f_{i,\rho}|^2 \le (1 + 2C_R) \int_{\Sigma \cap B_{R^2}} f_{i,\rho}^2 \le (1 + 2C_R)(1 + R^2) \log^2(2 + R^2).$$

From this, as in [FC85, CM16], for every i fixed, a diagonal argument (together with the fact that $W^{1,2}(\Sigma \cap B_R)$ compactly embeds into $L_*^2(\Sigma \cap B_R)$) implies the existence of a sequence $\rho_j \to \infty$ and a function $f_i \in L_*^2(\Sigma)$ such that

$$f_{i,\rho_j} \to f_i \quad \text{in } L_*^2(K)$$

for any compact domain K of Σ. Because $\lambda_{i,\rho}$ is uniformly bounded from below, it is clear that f_i is an L_*^2-eigenfunction with eigenvalue $\lambda_i = \lim_{j \to \infty} \lambda_{i,\rho_j} < 0$.

To finish our proof we must show that no mass escapes to infinity, i.e., that $\|f_i\|_{L_*^2(\Sigma)} = 1$. To this end, we return to (4.3) and plug in $\phi = f_{i,\rho}$ (extending ϕ to be zero outside of B_ρ) and $\eta = 1 - \varphi_R^\alpha$, for some $\alpha > 1$ and $R > R_0$ to be chosen below. We compute:

$$-\int_\Sigma 2\kappa(\eta f_{i,\rho})^2 \le \int_\Sigma \eta^2 |\nabla f_{i,\rho}|^2 + 2\eta f_{i,\rho} \langle \nabla\eta, \nabla f_{i,\rho} \rangle + f_{i,\rho}^2 |\nabla\eta|^2$$

$$= \int_\Sigma \eta^2 |\nabla f_{i,\rho}|^2 + \frac{1}{2} \left\langle \nabla \eta^2, \nabla f_{i,\rho}^2 \right\rangle + f_{i,\rho}^2 |\nabla \eta|^2$$

$$= \int_\Sigma \eta^2 |\nabla f_{i,\rho}|^2 - \eta^2 \left(f_{i,\rho} \Delta f_{i,\rho} + |\nabla f_{i,\rho}|^2 \right) + f_{i,\rho}^2 |\nabla \eta|^2$$

$$= \int_\Sigma f_{i,\rho}^2 |\nabla \eta|^2 - \eta^2 f_{i,\rho} \Delta f_{i,\rho}$$

$$= \int_\Sigma f_{i,\rho}^2 |\nabla \eta|^2 - 2\kappa(\eta f_{i,\rho})^2 + \lambda_{i,\rho}(1 + |x|^2)^{-1} (\log(2 + \log |x|))^{-2} (\eta f_{i,\rho})^2.$$

Hence,

$$\epsilon_0 \|\eta f_{i,\rho}\|_{L_*^2(\Sigma)}^2 \leq (-\lambda_{i,\rho}) \|\eta f_{i,\rho}\|_{L_*^2(\Sigma)}^2$$

$$\leq \int_\Sigma f_{i,\rho}^2 |\nabla \eta|^2$$

$$\leq \frac{1}{(\alpha - 1)^2 R^2 \log^2 R} \|f_{i,\rho}\|_{L^2(\Sigma \setminus B_R)}^2$$

$$\leq \frac{(1 + R^2) \log^2(2 + R)}{(\alpha - 1)^2 R^2 \log^2 R} \|f_{i,\rho}\|_{L_*^2(\Sigma \setminus B_R)}^2$$

$$\leq \frac{(1 + R^2) \log^2(2 + R)}{(\alpha - 1)^2 R^2 \log^2 R}.$$

This implies that there is some $c > 0$ (independent of ρ, R, α) so that

$$(4.5) \qquad \int_{\Sigma \setminus B_{R^\alpha}} (1 + |x|^2)^{-1} [\log(2 + |x|)]^{-2} f_{i,\rho}^2 \leq c \frac{(1 + R^2) \log^2(2 + R)}{(\alpha - 1)^2 R^2 \log^2 R},$$

for any $R \in (R_0, \rho^{1/\alpha}]$.

First, we choose any $R > R_0$ and consider it fixed below. Given any $0 < \delta < 1$, note that we may find $\alpha > 1$ and integer $j_0 \in \mathbb{Z}_+$, both depending on δ, such that for any $j > j_0$:

(a) $\rho_j^{1/\alpha} \geq R$, so that we may use equation (4.5), and

(b) $(\alpha - 1)^2 \geq c \dfrac{(1 + R^2) \log^2(2 + R)}{\delta R^2 \log^2 R}.$

Thus, we have a compact set $K = \overline{\Sigma \cap B_{R^\alpha}}$ such that for every $j > j_0$:

$$\|f_{i,\rho_j}\|_{L_*^2(\overline{\Sigma \setminus K})} < \delta.$$

This proves that f_{i,ρ_j} does not loose any mass at infinity. The proof may now be completed as in [FC85, p. 126]. $\qquad \square$

5. Proof of Theorem 1.1

We first recall the notation of [CM16]. For ω harmonic 1-form, we write:

$$X_\omega := (\langle \omega, dx_1 \rangle, \langle \omega, dx_2 \rangle, \langle \omega, dx_3 \rangle).$$

By [Ros06, Lemma 1], we have

$$\Delta X_\omega - 2\kappa X_\omega = 2 \langle \nabla \omega, h \rangle N,$$

where this equation is to be interpreted in the component by component sense. Here, h is the second fundamental form. Moreover, if $\langle \nabla \omega, h \rangle \equiv 0$, then $\omega \in \mathrm{span}\{*dx^1, *dx^2, *dx^3\}$ by [Ros06, Lemma 1].

As in [CM16], for vector fields X and Y along Σ with components (X_1, X_2, X_3) and (Y_1, Y_2, Y_3), we denote by $Q(X, Y)$ the sum $\sum_{i=1}^{3} Q(X_i, Y_i)$. For example, for a vector field X along Σ, we have that

$$Q(X, X) = -\int_{\Sigma} \langle \Delta X - 2\kappa X, X \rangle,$$

where the integrand is the Euclidean inner product of the following vector fields along X

$$(\Delta X_1 - 2\kappa X_1, \Delta X_2 - 2\kappa X_2, \Delta X_3 - 2\kappa X_3) \qquad \text{and} \qquad (X_1, X_2, X_3).$$

By Proposition 4.3, there are L^2_*-eigenfunctions $f_1, \ldots, f_k \in L^2_*(\Sigma)$ which span $W \subset L^2_*(\Sigma)$ and so that for $\phi \in C_0^\infty(\Sigma) \cap W^\perp$, we have $Q(\phi, \phi) \geq 0$. By Proposition 3.2, $V := L^2_*(\Sigma) \cap \mathcal{H}^1(\Sigma)$ has dimension

$$2g + 2\sum_{j=1}^{r}(d_j + 1) - 2$$

Suppose that $\omega \in V$ satisfies $X_\omega \in W^\perp$ (where the orthogonal complement is taken with respect to the L^2_*-inner product). We will prove that $\omega \in \mathrm{span}\{*dx_1, *dx_2, *dx_3\}$. This finishes the proof (cf. [CM16, §5]).

Suppose that $\omega \in V$ satisfies $X_\omega \in W^\perp$. Following [Ros06, CM16], we will show that $Q(X_\omega, Y) = 0$ for any compactly supported smooth vector field Y with $Y \in W^\perp$. Choose R sufficiently large so that B_R contains the support of Y. We set

$$X_t := \varphi_R(X_\omega + tY + f_1\vec{c}_1 + \cdots + f_k\vec{c}_k),$$

where φ_R is the test function constructed in Section 2.5 with $\alpha = 2$. Here, the vectors $\vec{c}_j \in \mathbb{R}^3$ depend on X_ω, φ_R and are chosen so that $X_t \in W^\perp$. In particular, we are requiring that

$$\int_{\Sigma} \varphi_R(X_\omega + f_1\vec{c}_1 + \cdots + f_k\vec{c}_k)f_j(1 + |x|^2)^{-1}(\log(2 + |x|))^{-2} = 0,$$

where we have used the fact that $Y \in W^\perp$ and $\varphi_R Y = Y$. Because $X_\omega, f_j \in L^2_*(\Sigma)$ and the $f_1, \ldots, f_k$ form an L^2_*-orthonormal basis for W, the dominated convergence theorem guarantees that the $\vec{c}_j$ tend to 0 when sending $R \to \infty$.

As in [CM16], the stability inequality implies that:

$$Q(X_\omega, Y)^2 \leq Q(Y, Y)$$

(5.1)
$$\times \left(\underbrace{Q(\varphi_R X_\omega, \varphi_R X_\omega)}_{(I)} + 2\sum_{i=1}^{k} \underbrace{Q(\varphi_R X_\omega, \varphi_R f_i \vec{c}_i)}_{(II)} + \sum_{i,j=1}^{k} \underbrace{Q(\varphi_R f_i \vec{c}_i, \varphi_R f_j \vec{c}_j)}_{(III)} \right)$$

We claim that the term in parenthesis tends to zero as $R \to \infty$. To show this, we consider each term in (5.1) separately. We have

$$(I) = Q(\varphi_R X_\omega, \varphi_R X_\omega) = \int_{\Sigma}(|\nabla(\varphi_R X_\omega)|^2 + 2\kappa(\varphi_R)^2|X_\omega|^2)$$

$$= -\int_{\Sigma} \langle \Delta(\varphi_R X_\omega) - 2\kappa\varphi_R X_\omega, \varphi_R X_\omega \rangle$$

$$= -\int_\Sigma (\varphi_R)^2 \langle \Delta X_\omega - 2\kappa X_\omega, X_\omega \rangle$$

$$- \int_\Sigma (\varphi_R \Delta \varphi_R |X_\omega|^2 + 2 \langle \nabla \varphi_R X_\omega, \varphi_R \nabla X_\omega \rangle)$$

$$= -\int_\Sigma \left(\varphi_R \Delta \varphi_R |X_\omega|^2 + \frac{1}{2} \langle \nabla (\varphi_R)^2, \nabla |X_\omega|^2 \rangle \right)$$

$$= \int_\Sigma |\nabla \varphi_R|^2 |X_\omega|^2.$$

Since $|\nabla \varphi_R| \leq \frac{C}{|x| \log R}$, we have:

$$\int_\Sigma |\nabla \varphi_R|^2 |X_\omega|^2 \leq \int_{\Sigma \cap (B_{R^2} \setminus B_R)} |X_\omega|^2 \frac{C}{|x|^2 \log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} |X_\omega|^2 \frac{1}{|x|^2 \log^2 |x|} \frac{\log^2 |x|}{\log^2 R}$$

Note that $\max_{r \in [R,R^2]} \frac{\log^2 r}{\log^2 R} = 4$ so taking $C > 0$ larger, we find:

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} |X_\omega|^2 \frac{1}{|x|^2 \log^2(|x|)} \frac{\log^2 |x|}{\log^2 R}$$

$$\leq C \int_{\Sigma \cap (B_{R^2} \setminus B_R)} |X_\omega|^2 \frac{1}{|x|^2 \log^2 |x|}$$

$$\leq C \int_{\Sigma \setminus B_R} |X_\omega|^2 \frac{1}{|x|^2 \log^2 |x|}$$

Since $|X_\omega| \in L^2_*(\Sigma)$, this converges to 0 as $R \to \infty$.

For the second term, we compute:

$$\text{(II)} = Q(\varphi_R X_\omega, \varphi_R f_i \vec{c}_i)$$

$$= -\int_\Sigma \varphi_R \langle X_\omega, \vec{c}_i \rangle (\Delta(\varphi_R f_i) - 2\kappa \varphi_R f_i)$$

$$= -\int_\Sigma (\varphi_R)^2 \langle X_\omega, \vec{c}_i \rangle (\Delta f_i - 2\kappa f_i) - \int_\Sigma \varphi_R \langle X_\omega, \vec{c}_i \rangle (\Delta \varphi_R f_i + 2 \langle \nabla \varphi_R, \nabla f_i \rangle)$$

$$(5.2) \qquad = \lambda_i \int_\Sigma (\varphi_R)^2 \langle X_\omega, \vec{c}_i \rangle f_i (1 + |x|^2)^{-1} (\log(2 + |x|))^{-2} - \int_\Sigma \varphi_R \Delta \varphi_R \langle X_\omega, \vec{c}_i \rangle f_i$$

$$- 2 \int_\Sigma \varphi_R \langle X_\omega, \vec{c}_i \rangle \langle \nabla \varphi_R, \nabla f_i \rangle.$$

The first term in (5.2) tends to zero as $R \to \infty$ by the dominated convergence and choice of $X_\omega \in W^\perp$. The second term in (5.2) tends to zero as follows. By (2.7):

$$\left| \int_\Sigma \varphi_R \Delta \varphi_R \langle X_\omega, \vec{c}_i \rangle f_i \right| \leq |\vec{c}_i| \int_\Sigma \frac{C |\varphi_R|}{|x|^2 \log^2 R} |X_\omega| |f_i|$$

$$+ |\vec{c}_i| \int_\Sigma \frac{C |\varphi_R|}{|x|^{2+2/k} \log R} |X_\omega| |f_i|.$$

Since $X_\omega, f_i \in L^2_*(\Sigma)$, by arguing as in (4.2), the above integrals are bounded as $R \to \infty$ and thus the right-hand side converges to zero since $|\vec{c}_i| \to 0$.

For third term in (5.2), we use Hölder's inequality:

$$\left| \int_\Sigma \varphi_R \langle X_\omega, \vec{c}_i \rangle \langle \nabla\varphi_R, \nabla f_i \rangle \right| \le |\vec{c}_i| \left(\int_\Sigma |\nabla\varphi_R|^2 |X_\omega|^2 \right)^{\frac{1}{2}} \left(\int_\Sigma |\nabla f_i|^2 \right)^{\frac{1}{2}}.$$

Since $|\nabla f_i| \in L^2$ by Proposition 4.1, $\vec{c}_i \to 0$ as $R \to \infty$, and $\int_\Sigma |\nabla\varphi_R|^2 |X_\omega|^2$ also converges to zero, this also converges to zero as $R \to \infty$.

Finally, we have that the third term in (5.1) satisfies

$$\begin{aligned}
(\text{III}) &= Q(\varphi_R \vec{c}_i f_i, \varphi_R \vec{c}_j f_j) \\
&= -\frac{1}{2} \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R f_j (\Delta(\varphi_R f_i) - 2\kappa\varphi_R f_i) - \frac{1}{2} \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R f_i (\Delta(\varphi_R f_j) - 2\kappa\varphi_R f_j) \\
&= \frac{1}{2} (\lambda_i + \lambda_j) \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R^2 f_i f_j (1 + |x|^2)^{-1} (\log(2 + |x|))^{-2} - \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R \Delta\varphi_R f_i f_j \\
&\quad - \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R \langle \nabla\varphi_R, f_i \nabla f_j + f_j \nabla f_i \rangle \\
&= \frac{1}{2} (\lambda_i + \lambda_j) \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma \varphi_R^2 f_i f_j (1 + |x|^2)^{-1} (\log(2 + |x|))^{-2} + \langle \vec{c}_i, \vec{c}_j \rangle \int_\Sigma |\nabla\varphi_R|^2 f_i f_j.
\end{aligned}$$

This tends to zero as $R \to \infty$ because the $\vec{c}_i$ are tending to zero and, by arguing as above, so does $\int_\Sigma |\nabla\varphi_R|^2 |f_i f_j|$, since $f_i, f_j \in L^2_*(\Sigma)$.

The above computations show that $Q(X_\omega, Y) = 0$ for all compactly supported smooth vector fields $Y \in W^\perp$. Finally, fix an arbitrary smooth vector field $\tilde{Y} \in W^\perp$ and let

$$\tilde{Y}_R := \varphi_R(\tilde{Y} + f_1\vec{c}_1 + \cdots + f_k\vec{c}_k),$$

where the $\vec{c}_j$ are chosen so that $\tilde{Y}_R \in W^\perp$. Observe (as above) that $\vec{c}_j \to 0$ as $R \to \infty$ by the dominated convergence theorem. Hence, because $\tilde{Y}_R$ has compact support, we have that

$$\begin{aligned}
0 = Q(X_\omega, \tilde{Y}_R) \\
= -\int_\Sigma \left\langle \Delta X_\omega - 2\kappa X_\omega, \tilde{Y}_R \right\rangle \\
= -2 \int_\Sigma \langle \nabla\omega, h \rangle \left\langle N, \tilde{Y}_R \right\rangle.
\end{aligned}$$

Lemma 4.2 shows that $|\nabla\omega| \in L^2(\Sigma)$, and because the second fundamental form satisfies $|h| \le (1 + |x|^2)^{-\frac{1}{2}(1+\frac{1}{k})}$ (cf. (2.5)), we may use the dominated convergence theorem to see that

$$\int_\Sigma \langle \nabla\omega, h \rangle \left\langle N, \tilde{Y} \right\rangle = 0.$$

Similarly, we may show that for any vector $\vec{\alpha} \in \mathbb{R}^3$ and eigenfunction $f_i \in W$ from Proposition 4.3, then

$$\int_\Sigma \langle \nabla\omega, h \rangle \langle N, f_i\vec{\alpha} \rangle = 0.$$

Putting this together, we obtain $\langle \nabla\omega, h \rangle = 0$. Thus, $\omega \in \text{span}\{*dx_1, *dx_2, *dx_3\}$. This completes the proof.

6. Proof of Theorem 1.3

We will prove Theorem 1.3 by essentially redoing the proof of Theorem 1.1 for anti-invariant harmonic 1-forms in $L^2_*(\widehat{\Sigma})$, where $\widehat{\Sigma}$ is the two-sided double cover of Σ.

Note that the space $\mathscr{H}^1(\widehat{\Sigma})$ decomposes as

$$\mathscr{H}^1(\widehat{\Sigma}) = \mathscr{H}^1_+(\widehat{\Sigma}) \oplus \mathscr{H}^1_-(\widehat{\Sigma}),$$

where $\mathscr{H}^1_\pm(\widehat{\Sigma}) = \{\omega \in \mathscr{H}^1(\widehat{\Sigma}) \mid \tau^*\omega = \pm\omega\}$. Note that the Hodge star operator $*$ of $\widehat{\Sigma}$ gives an isomorphism between $\mathscr{H}^1_+(\widehat{\Sigma})$ and $\mathscr{H}^1_-(\widehat{\Sigma})$. In particular, the dimension of $\mathscr{H}^1_-(\widehat{\Sigma}) \cap L^2_*(\widehat{\Sigma})$ is equal to $g + 2\sum_{j=1}^r d_j + 1 - 1$.

The key point is that if $\omega \in \mathscr{H}^1_-(\widehat{\Sigma}) \cap L^2_*(\widehat{\Sigma})$ then $X_\omega \circ \tau = -X_\omega$, so its coordinates can be used as test functions for Q as in the proof of Theorem 1.1: suppose $\omega \in \mathscr{H}^1_-(\widehat{\Sigma}) \cap L^2_*(\widehat{\Sigma})$ and $Y : \widehat{\Sigma} \to \mathbb{R}^3$ is compactly supported with $Y \circ \tau = -Y$, we consider the cutoff function $\varphi_R : \widehat{\Sigma} \to R$, taking R big enough so that $\varphi_R \equiv 1$ on the support of Y. Then, since $\varphi_R \circ \tau = \varphi_R$, the vector

$$X_t = \varphi_R X + tY = \varphi_R(X + tY)$$

is anti-invariant. Given this, we can argue precisely as for Theorem 1.1.

7. On the deformation family of the Costa surface

Consider $\{\Sigma_t\}_{t \geq 1}$ the 1-parameter family of embedded genus one minimal surfaces with three ends, see Costa [Cos91]. Recall that each Σ_t is conformally equivalent to $\mathbb{C}/L(it) \setminus \{\pi(0), \pi(1/2), \pi(it/2)\}$, where $L(it) = \{m + int \mid m, n \in \mathbb{Z}\}$ and $\pi : \mathbb{C} \to \mathbb{C}/L(it)$ is the canonical projection. We way arrange that the following isometries of $\mathbb{R}^3$:

$$\tau_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \qquad \tau_2(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

descend to $\overline{\Sigma_t}$ as reflections in x and y coordinates through the center of $L(it)$, where we write $z = x + iy$ for the canonical coordinate of $\mathbb{C}$, see [Cos91, HK97].

By work of Nayatani [Nay92], we know that $\text{index}(\Sigma_1) = 5$ and, by Choe [Cho90], that $\text{index}(\Sigma_t) \geq 3$ for all t. Suppose that $\text{index}(\Sigma_t) = 3$ for certain parameter t. We will show that this leads to a contradiction.

7.1. Decomposing the 1-form argument with respect to the symmetries.
By Proposition 3.2, $\mathscr{H}^1(\Sigma_t) \cap L^2_*(\Sigma_t)$ has dimension 12. Thus, if we write $\mathscr{H}$ for the subspace of such 1-forms on Σ_t that are L^2_*-orthogonal to $*dx^1, *dx^2, *dx^3$, then $\dim \mathscr{H} = 9$.

Lemma 7.1. *Suppose that*

$$\mathscr{H} = \mathscr{H}^{++} \oplus \mathscr{H}^{+-} \oplus \mathscr{H}^{-+} \oplus \mathscr{H}^{--}.$$

is the decomposition of $\mathscr{H}$ into eigenspaces of τ_1, τ_2. Then:

$$\dim \mathscr{H}^{++} = 2$$
$$\dim \mathscr{H}^{+-} = 3$$
$$\dim \mathscr{H}^{-+} = 3$$
$$\dim \mathscr{H}^{--} = 1.$$

Proof. Write $\widetilde{\mathscr{H}} = \mathscr{H}^1(\Sigma_t) \cap L^2_*(\Sigma_t)$. Because $*dx^1, *dx^2, *dx^3$ are of type $+-, -+, --$ respectively, it suffices to show that the decomposition

$$\widetilde{\mathscr{H}} = \widetilde{\mathscr{H}}^{++} \oplus \widetilde{\mathscr{H}}^{+-} \oplus \widetilde{\mathscr{H}}^{-+} \oplus \widetilde{\mathscr{H}}^{--}$$

has

$$\dim \widetilde{\mathscr{H}}^{++} = 2$$
$$\dim \widetilde{\mathscr{H}}^{+-} = 4$$
$$\dim \widetilde{\mathscr{H}}^{-+} = 4$$
$$\dim \widetilde{\mathscr{H}}^{--} = 2.$$

Observe that the forms dx and dy on the torus $\overline{\Sigma_t}$ are of type $-+$ and $+-$, respectively. Moreover, these span $\mathscr{H}^1(\Sigma_t) \cap L^2(\Sigma_t)$.

Consider $\omega \in \widetilde{\mathscr{H}}^{++}$ (the other cases will follow similarly). Define the associated holomorphic 1-form $\alpha = \omega + i * \omega$. We will work near the end $\pi(0)$. We have that

$$\alpha = \alpha_1 \frac{dz}{z} + \alpha_2 \frac{dz}{z^2} + O(1)$$

near $z = 0$. Observe that

$$\frac{dz}{z} = \frac{xdx + ydy}{x^2 + y^2} + i\frac{xdy - ydx}{x^2 + y^2}$$
$$\frac{dz}{z^2} = \frac{(x^2 - y^2)dx + 2xydy}{(x^2 - y^2)^2 + 4x^2y^2} + i\frac{(x^2 - y^2)dy - 2xydx}{(x^2 - y^2)^2 + 4x^2y^2}$$

and that the real and imaginary components of these forms obey the following parity relations:

1-form	type
$\dfrac{xdx + ydy}{x^2 + y^2}$	$++$
$\dfrac{xdy - ydx}{x^2 + y^2}$	$--$
$\dfrac{(x^2 - y^2)dx + 2xydy}{(x^2 - y^2)^2 + 4x^2y^2}$	$-+$
$\dfrac{(x^2 - y^2)dy - 2xydx}{(x^2 - y^2)^2 + 4x^2y^2}$	$+-$

Because $\omega = \operatorname{Re}\alpha$ was assumed to be in $\widetilde{\mathscr{H}}^{++}$, we thus find that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 = 0$. A similar computation holds at the other ends $\pi(1/2)$ and $\pi(it/2)$. By [FK80, Proposition II.5.4], the sum of the residues of ω equals zero. Thus, because $\widetilde{\mathscr{H}}^{++} \cap L^2(\Sigma_t) = \{0\}$, we find that $\dim \widetilde{\mathscr{H}}^{++} \leq 2$. More specifically, to each $\omega \in \widetilde{\mathscr{H}}^{++}$ we have an injective map

$$\omega \mapsto (\operatorname{Res}_{\pi(0)} \alpha, \operatorname{Res}_{\pi(1/2)} \alpha, \operatorname{Res}_{\pi(it/2)} \alpha)$$

whose image is contained in the two-dimensional subspace $\{(u, v, w) : u + v + w = 0\} \subset \mathbb{R}^3$. This yields the asserted inequality. Arguing similarly, for the other parities, we find that the

assertion in the lemma holds in the weaker form where inequalities replace the equalities. However, since $\dim \widetilde{\mathscr{H}} = 12$, this implies the full assertion. $\qquad\square$

Next, we also decompose the space of smooth functions as on Σ_t as:

$$C^\infty(\Sigma_t) = C^{++} \oplus C^{+-} \oplus C^{-+} \oplus C^{--}. \tag{7.1}$$

Let W be the span of eigenfunctions in L^2_* with negative eigenvalue, corresponding to the Morse index of Σ_t. Recall the we are assuming W has dimension 3. We decompose W according to (7.1) as

$$W = W^{++} \oplus W^{+-} \oplus W^{-+} \oplus W^{--}$$

and write $\dim W^{\pm\pm} = w^{\pm\pm}$. We will show:

Lemma 7.2. *The following inequalities are true:*

$$2 = \dim \mathscr{H}^{++} \leq w^{-+} + w^{+-} + w^{++}$$
$$3 = \dim \mathscr{H}^{+-} \leq w^{--} + w^{++} + w^{+-}$$
$$3 = \dim \mathscr{H}^{-+} \leq w^{++} + w^{--} + w^{-+}$$
$$1 = \dim \mathscr{H}^{--} \leq w^{+-} + w^{-+} + w^{--}.$$

Proof. We will prove the first inequality and the rest will follow analogously. Suppose $w^{-+} + w^{+-} + w^{++} \leq 1$. For $\omega \in \mathscr{H}^{++}$, note that

$$\langle \omega, dx^1 \rangle \in C^{-+}, \langle \omega, dx^2 \rangle \in C^{+-}, \langle \omega, dx^3 \rangle \in C^{++}.$$

Thus, the requirement that $\langle \omega, dx^1 \rangle$ is orthogonal to W with respect to L^2_* amounts to at most w^{-+} linear equations (pairing an element of C^{-+} with e.g., C^{++} vanishes automatically, by a parity argument). Thus, requiring that X_ω is orthogonal to W takes at most $w^{-+} + w^{+-} + w^{++}$ equations.

By our assumption, we can thus find such a $\omega \in \mathscr{H}^{++} \setminus \{0\}$ that is orthogonal to W in L^2_*. Repeating the proof of Theorem 1.1, we conclude that $\omega \in \mathrm{span}\{*dx^1, *dx^2, *dx^3\}$, which is a contradiction since $\mathscr{H}$ is orthogonal to $\mathrm{span}\{*dx^1, *dx^2, *dx^3\}$. $\qquad\square$

If we assume that $\mathrm{index}(\Sigma_t) = 3$, then $\sum w^{\pm\pm} = 3$. Combined with Lemma 7.2, we find

$$w^{+-} = w^{-+} = 0, \qquad w^{++} = 2, \qquad w^{--} = 1. \tag{7.2}$$

We show below that this is a contradiction by comparing with Choe's vision number.[8]

7.2. Comparing with the vision number.

Write $\Pi_i = \{\tau_i(x) = x\}$ for the $\{x_i = 0\}$ planes of reflection symmetry in $\mathbb{R}^3$. Consider ϕ_ℓ the Jacobi field coming from the rotation around the x_3-axis. Write $H(\phi_\ell) = \{\phi_\ell = 0\}$. By the symmetries, we have that

$$\Pi_1 \cup \Pi_2 \subset H(\phi_\ell),$$

so the vision number $v(\phi_\ell)$ (i.e., the number of connected components of $\Sigma_t \setminus H(\phi_\ell)$, see Choe [Cho90]) satisfies $v(\phi_\ell) \geq 4$. See [HK97, p. 50] for a picture of one quadrant of the Costa surface.

On the other hand, since we have assumed that $\mathrm{index}(\Sigma_t) = 3$, we have by Choe [Cho90]:

$$3 = \mathrm{index}(\Sigma_t) \geq v(\phi_\ell) - 1 \geq 3,$$

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[8]Alternatively, one may find a contradiction by first showing that $w^{--} \leq w^{+-} \leq w^{++}$ by using the variational characterization of $\omega^{\pm\pm}$ in a quarter of Σ_t.

we find that[9] $v(\phi_\ell) = 4$. Write Σ' for one of the components of $\Sigma_t \setminus (H_1 \cup H_2)$. Because $v(\phi_\ell) = 4$, we must have that ϕ_ℓ does not change sign on the interior of each connected componnet of Σ' and vanishes at their boundary. Thus, Σ' is a stable minimal surface with fixed boundary. By the usual argument relating odd symmetry and Dirichlet boundary conditions, this implies that $w^{--} = 0$, contradicting the previous section.

Thus, we find that $\mathrm{index}(\Sigma_t) \geq 4$ for every t, completing the proof.

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[9]In fact, one can prove this holds, even without assuming that $\mathrm{index}(\Sigma_t) = 3$. See [HK97, Remark 7.3].

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