RICCI CURVATURE AND MINIMAL HYPERSURFACES WITH LARGE BETTI NUMBERS

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Abstract. In any dimension $n+1 \geq 4$ we construct a sequence of closed $(n+1)$-dimensional Riemannian manifolds with positive Ricci curvature admitting embedded two-sided minimal hypersurfaces such that the following hold:

(i) any such hypersurface has Morse index one;
(ii) the first Betti numbers of the hypsersurfaces are not uniformly bounded along the sequence.

1. Introduction

Recent years have seen significant advances in the existence theory of minimal hypersurfaces in higher-dimensional Riemannian manifolds, largely propelled by the influential work of F.C. Marques and A. Neves [MN14, MN16, MN17, MN21]. Their contributions refined and extended the min-max framework originally developed by F. Almgren and J. Pitts in the 1980s, and have inspired a new wave of results in the field. Notably, the works X. Zhou [Zho20], A. Song [Son23a], and others [CM20, IMN18, LMN18] have further built upon this foundation.

A key feature of this theory is its reliance on weak convergence in the varifold setting, which ensures the existence of a limiting minimal hypersurface but does not provide precise control over its topology – leaving open questions about the specific geometric structure of the resulting surface. However, a remarkable strength of these modern refinements is their ability to predict the Morse index of the minimal hypersurface, offering insight into its stability and the number of independent directions in which it can be deformed to reduce area.

This interplay between existence, index prediction, and topological ambiguity underscores the power of the theory and its ongoing evolution in geometric analysis. On several instances, estimates that show the Morse index controlling the topology of the hypersurface can be established, *e.g.* [ACS18, CKM17, CM23, MR20, MVP23, Son23b]. However, as the ambient geometry becomes more complex, the interplay between index and topology becomes subtler, and many fundamental questions remain open.

One guiding question in this direction is the following folklore conjecture: if Σ^n is a closed, two-sided, minimal hypersurface of Morse index 1 on a closed manifold (M^{n+1}, g) of positive Ricci curvature, then the first Betti number $b_1(\Sigma^n)$ must be bounded by a universal constant $c(n)$, see [Nev14, Section 8]. This is known to be the case when $n+1 = 3$ [Yau87]. The main result of this paper is the construction of counterexamples to this conjecture starting in dimension $n+1 = 4$:

Theorem A. *For $n+1 \geq 4$, there exists a sequence of smooth, closed, $(n+1)$-dimensional, Riemannian manifolds (M_k^{n+1}, g_k) with embedded, two-sided minimal hypersurfaces $\Sigma_k \subset M_k$ satisfying the following properties:*

i) $\mathrm{Ric}_{g_k} > 0$ *for any* $k \in \mathbb{N}$*;*
ii) $\mathrm{index}(\Sigma_k) = 1$ *for any* $k \in \mathbb{N}$*;*
iii) $b_1(\Sigma_k) \to \infty$ *as* $k \to \infty$*, where* b_1 *stands for the first Betti number.*

To describe our construction in detail, we recall that if X^n is a closed manifold, the *suspension* $\Sigma_0 X$ of X (alternatively also called the *spinning* of X) is a $(n+1)$-dimensional manifold obtained by a surgery on $X \times S^1$ as follows: let $D^n \subset X$ be an embedded disk and define (see [Dua24, GGR25, Rei24]):

$$\Sigma_0 X = \left((X \setminus D^n) \times S^1\right) \cup_{S^{n-1} \times S^1} \left(S^{n-1} \times D^2\right) .$$

An interesting feature of $\Sigma_0(\cdot)$ is that it preserves various topological properties, such as the fundamental group (see [GGR25, Lemma 5.2]).

Theorem A is a direct consequence of the following result:

Theorem B. *Let M^n, $n \geq 3$, be a closed manifold that admits a Riemannian metric of positive Ricci curvature. Then, for all $\ell \geq 0$, there exists a Riemannian metric of positive Ricci curvature on*

$$\Sigma_0 M \#_\ell (S^2 \times S^{n-1})$$

and a totally geodesic two-sided embedding

$$M \# (-M) \#_\ell (S^1 \times S^{n-1}) \subseteq \Sigma_0 M \#_\ell (S^2 \times S^{n-1})$$

of index one.

In particular, we have the following Corollary for dimension $n+1 = 4$:

Corollary C. *Let $N = S^3/\Gamma$ be a spherical space form, i.e. $\Gamma \subseteq O(4)$ is a finite subgroup that acts freely on S^3. Then, for any $\ell \geq 0$, the manifold*

$$N \# (-N) \#_\ell (S^1 \times S^2)$$

can be realised as a two-sided minimal hypersurface of index one in a closed Riemannian 4-manifold of positive Ricci curvature.

The metric in Theorem B is constructed as follows: The starting point is the construction of J.-P. Sha and D.-G. Yang [SY91], which provides a metric of positive Ricci curvature on the space obtained from

$M \times S^1$ by performing $(\ell+1)$-surgeries along the S^1-factor, resulting in the manifold $\Sigma_0 M \#_\ell (S^2 \times S^{n-1})$. We then show that a reflection on the S^1-factor of $M \times S^1$ induces an isometric action on this space with fixed point set $M\#(-M)\#_\ell(S^1 \times S^{n-1})$, which is a totally geodesic hypersurface, such that the induced metric has positive scalar curvature. Finally, we establish a deformation result (Proposition 2.3) that shows that under these hypotheses the metric on the ambient manifold can be deformed so that the hypersurface is minimal of index one while keeping the ambient Ricci curvature positive.

Remark 1.1. *Following Chodosh-Li-Stryker in* [CLS24, Remark B.3]*, the deformation theorem in Proposition 2.3 can also be used to construct a closed Riemannian* 4*-manifold* (X^4, g) *with* $\mathrm{Ric} > 0$ *that admits a complete, two-sided, stable minimal hypersurface immersed in it.*

We note that the radius r of the circle factor in the initial manifold $M \times S^1$ can be chosen arbitrarily small. As a consequence, since varying r only affects the hypersurface in the region where the surgeries are performed, its area has a uniform positive lower bound for all r. The same holds for the diameter of the ambient manifold, while the volume of the ambient manifold converges to 0 as $r \to 0$.

In view of Corollary C, the following question arises naturally:

Question 1.2. *Let N be closed, oriented* 3*-manifold that admits a Riemannian metric of positive scalar curvature. Can N be realised as a minimal hypersurface of index one within a closed Riemannian 4-manifold of positive Ricci curvature?*

By the work of R. Schoen and S.-T. Yau [SY79] and G. Perelman [Per02, Per03a, Per03b], see also [Mar12], a closed, oriented 3-manifold N admits a Riemannian metric of positive scalar curvature if and only if it is the connected sum of finitely many spherical space forms and copies of $S^1 \times S^2$. Besides the examples in Corollary C, to the best of our knowledge, the only additional examples are given by spherical space forms. This follows from [Zho24], where the author constructs complete Riemannian manifolds with nonnegative Ricci curvature which are isometric to a cone over S^3/Γ outside a compact set for any spherical space form S^3/Γ, combined with a standard doubling and smoothing argument based on Perelman's gluing lemma from [Per97].

This paper is organised as follows: In Section 2 we establish a deformation result to obtain hypersurfaces of index one. In Section 3 we then recall the construction of Sha–Yang and use the deformation result to prove Theorem B.

Acknowledgements. The second named author would like to thank the Faculty of Mathematics of the University of Vienna for its hospitality while parts of this work were carried out.

2. Preliminaries

In this section, we will gather some preparatory results about metrics of positive Ricci curvature with a totally geodesic hypersurface.

Recall that for a Riemannian metric on a cylinder $(-\delta,\delta)\times N$ of the form dt^2+g_t, where N is a manifold and g_t a smoothly varying family of Riemannian metrics on N, the second fundamental form of the hypersurface $\{t\}\times N$ with respect to the unit normal ∂_t is given by

$$\mathrm{II}=-\frac{1}{2}g_t'\,. \tag{1}$$

Here g_t' denotes the t-derivative of g_t, i.e. for $u,v\in T_xN$ we have

$$g_t'(u,v)=\frac{\partial}{\partial t}g_t(u,v)\,. \tag{2}$$

The expression for II e.g. follows from [Pet16, Section 3.2.1], where we use that the Hessian $\mathrm{Hess}f$ of the function $f(t,x)=t$ on $(-\delta,\delta)\times N$ is given by $\mathrm{Hess}f=-\frac{1}{2}g_t'$.

In particular, the hypersurface $\{t\}\times N$ is totally geodesic if and only if $g_t'=0$.

Lemma 2.1. *Let (N^{n-1},g_N) be a closed Riemannian manifold. Then the following statements are equivalent:*

(i) *For all $\varepsilon>0$ there exists $\delta>0$ and a metric $g^\varepsilon=dt^2+g_t^\varepsilon$ on $(-\delta,\delta)\times N$ of positive Ricci curvature such that $\{0\}\times N$ is totally geodesic, $g_0^\varepsilon=g_N$, and $\mathrm{Ric}(\partial_t,\partial_t)=\varepsilon$ at $t=0$.*

(ii) *(N,g_N) has non-negative scalar curvature.*

Proof. For any metric g of the form dt^2+g_t on $(-\delta,\delta)\times N$ such that $\{0\}\times N$ is totally geodesic the Ricci curvatures at $t=0$ are given as follows:

$$\mathrm{Ric}(\partial_t,\partial_t)=-\frac{1}{2}\mathrm{tr}_{g_0}g_0''\,, \tag{3}$$

$$\mathrm{Ric}(v,\partial_t)=0\,, \tag{4}$$

$$\mathrm{Ric}(u,v)=\mathrm{Ric}^{g_0}(u,v)-\frac{1}{2}g_t''(u,v) \tag{5}$$

for $u,v\in TN$, see e.g. [Pet16, Section 3.2.1].

Now suppose $\varepsilon>0$ and $g=g^\varepsilon$ as in (i). Then it follows that

$$\varepsilon=-\frac{1}{2}\mathrm{tr}_{g_0}g_0''=\sum_{i=1}^{n-1}\left(\mathrm{Ric}^g(e_i,e_i)-\mathrm{Ric}^{g_0}(e_i,e_i)\right)>-\mathrm{scal}^{g_0}\,, \tag{6}$$

where (e_i) is a local orthonormal frame of (N,g_N). Since $g_0=g_N$, it follows that $\mathrm{scal}^{g_N}>-\varepsilon$ for all ε, i.e. $\mathrm{scal}^{g_N}\geq 0$.

Conversely, assume that $\mathrm{scal}^{g_N}\geq 0$ and let $\varepsilon>0$. For $\delta>0$ we then define the metric $g^\varepsilon=dt^2+g_t^\varepsilon$ on $(-\delta,\delta)\times N$, where

$$g_t^\varepsilon=g_N+t^2h \tag{7}$$

and

$$h = \mathrm{Ric}^{g_N} - \frac{\mathrm{scal}^{g_N} + \varepsilon}{n-1} g_N \,. \tag{8}$$

Note that g_t^ε is a Riemannian metric for δ (and hence $|t|$) sufficiently small. We then have $g_0^\varepsilon = g_N$, $g_0^{\varepsilon\prime} = 0$, and $g_0^{\varepsilon\prime\prime} = 2h$. In particular, $\{0\} \times N$ is totally geodesic. Further, the Ricci curvatures at $t = 0$ are given by

$$\mathrm{Ric}(\partial_t, \partial_t) = -\mathrm{tr}_{g_N} h = \varepsilon \,, \tag{9}$$

$$\mathrm{Ric}(v, v) = \mathrm{Ric}^{g_N}(v, v) - h = \frac{\mathrm{scal}^{g_N} + \varepsilon}{n-1} g_N(v, v) > 0 \tag{10}$$

and $\mathrm{Ric}(\partial_t, v) = 0$ for every $v \in TN \setminus \{0\}$. In particular, the metric g^ε has positive Ricci curvature at $t = 0$ and therefore also on $(-\delta, \delta) \times N$ if we choose δ sufficiently small. This shows that g^ε is a metric as in (i). ☐

Remark 2.2. *An observation similar to the one employed in the proof of the implication from (ii) to (i) in Lemma 2.1 appears in* [CLS24, Appendix B.2].

Using Lemma 2.1, we prove the main result of this section:

Proposition 2.3. *Let (M^n, g_0) be a closed Riemannian manifold of positive Ricci curvature and let $N^{n-1} \subseteq M$ be an embedded totally geodesic two-sided hypersurface. Suppose that g_0 induces a metric of non-negative scalar curvature on N. Then, for any $\varepsilon > 0$, there exists a deformation g_t, $t \in [0, 1]$, of g_0 with the following properties:*

- (i) *The metric g_t has positive Ricci curvature for all t;*
- (ii) *The deformation g_t is constant outside an arbitrarily small neighbourhood of N;*
- (iii) *The induced metric on N remains unchanged and N is totally geodesic for all g_t;*
- (iv) *The normal Ricci curvatures of g_1 at N satisfy $\mathrm{Ric}^{g_1}(\nu_N, \nu_N) = \varepsilon$, where ν_N denotes a unit normal vector field at N.*

Proof. By considering normal coordinates around N, since $N \subseteq M$ is two-sided, we can identify a neighbourhood of N with $(-\delta, \delta) \times N$ for some $\delta > 0$ and write the metric g_0 in this neighbourhood as $g_0 = ds^2 + h_s$, where h_s is a family of metrics on N. The condition of N being totally geodesic is then equivalent to $h_s' = 0$ at $s = 0$.

Hence, for $g_N = g_0|_N$, the metrics g^ε from Lemma 2.1 and g_0 coincide up to first order at $\{0\} \times N$. The existence of the required deformation therefore follows from [Wra02, Theorem 1.10], see also [BH22]. ☐

Remark 2.4. *With the help of Proposition 2.3 one can answer in the affirmative the question raised in* [CLS24, Remark B.3].

3. Proof of Theorem B

We start with the construction of Sha–Yang [SY91] to construct a metric of positive Ricci curvature on the connected sum $\Sigma_0 M \#_\ell (S^2 \times S^{n-1})$, where M admits a metric of positive Ricci curvature g. We address the reader also to [And92] for a different approach to similar constructions.

For the construction we assume that there are $(\ell+1)$ pairwise disjoint embeddings of discs

$$\bigsqcup_{\ell+1} D^n_{r_2} \subseteq (M, g)\,, \tag{11}$$

each equipped with the induced metric of a geodesic ball of some radius $r_2 > 0$ in the round sphere S^n of radius 1. This can always be achieved by local deformations around $(\ell+1)$ points while preserving positive Ricci curvature, see [Wra02, Theorem 1.10], [BH22] or [Rei24, Lemma 4.2].

We now consider the product metric $r_1^2 ds_1^2 + g$ on $S^1 \times M$. Note that this metric has non-negative Ricci curvature. The connected sum $\Sigma_0 M \#_\ell (S^2 \times S^{n-1})$ is then obtained by performing surgery along the corresponding embeddings $\bigsqcup_{\ell+1} (S^1 \times D^n_{r_2}) \subseteq (S^1 \times M)$, that is, we remove the interior of each copy of $S^1 \times D^n_{r_2}$ in $S^1 \times M$ and glue in a copy of $D^2 \times S^{n-1}$ along the resulting boundary component $S^1 \times S^{n-1}$. For a proof that this space results in $\Sigma_0 M \#_\ell (S^2 \times S^{n-1})$ as claimed see e.g. [Rei24, Lemma 2.7].

From a metric perspective, the approach taken in [SY91] consists of considering normal coordinates around the centre of each $D^n_{r_2}$ and viewing the metric on each embedded $S^1 \times D^n_{r_2}$ as a doubly warped product metric with constant warping function for the S^1-factor, i.e.

$$r_1^2 ds_1^2 + (dt^2 + \sin^2(t) ds_{n-1}^2) = dt^2 + r_1^2 ds_1^2 + \sin^2(t) ds_{n-1}^2 \tag{12}$$

for $t \in [0, r_2]$. To perform the surgery, this metric now gets replaced by a different doubly warped product metric

$$dt^2 + h(t)^2 ds_1^2 + f(t)^2 ds_{n-1}^2 \tag{13}$$

on $[t_0, r_2] \times S^1 \times S^{n-1}$ for some $t_0 < r_2$ and smooth warping functions $h, f \colon [t_0, r_2] \to [0, \infty)$ that coincide with the previous warping functions near $t = r_2$, i.e.:

(i) $h(r_2) = r_1$ and all derivatives of h vanish at $t = r_2$;
(ii) $f(t) = \sin(t)$ near $t = r_2$.

To achieve the change in the topology, i.e. to replace $S^1 \times D^n$ by $D^2 \times S^{n-1}$, the functions h and f satisfy different boundary conditions at $t = t_0$. More precisely, they satisfy the following:

(iii) The function h is an odd function at $t = t_0$ with $h'(t_0) = 1$ (in particular, $h(t_0) = 0$);

(iv) The function f is an even function at $t = t_0$ with $f(t_0) > 0$ (in particular, $f'(t_0) = 0$).

With these boundary conditions satisfied, we obtain a smooth metric on $D^2 \times S^{n-1}$, see e.g. [Pet16, Proposition 1.4.7]. For a construction of such functions that results in a metric of positive Ricci curvature, we refer to [SY91, Lemma 1] or [Rei24, Lemma 3.3].

Next, we construct a totally geodesic embedding $M\#(-M)\#_\ell(S^1 \times S^{n-1}) \subseteq \Sigma_0 M\#_\ell(S^2 \times S^{n-1})$. For that, we first note that the involution on $S^1 \times M$ that is a reflection on S^1 and the identity on M is isometric with respect to the product metric $r_1^2 ds_1^2 + g$ with fixed point set $S^0 \times M = M \sqcup (-M)$.

By definition, each embedded copy of $S^1 \times D^n_{r_2} \subseteq S^1 \times M$ is preserved by the involution, which, in normal coordinates on $D^n_{r_2}$ is given by a reflection of the S^1 factor in $[0, r_2] \times S^1 \times S^{n-1}$ and the identity on the remaining factors.

Since this involution is isometric for any metric of the form (13), it also defines an isometric involution after preforming the surgery, i.e. on $\Sigma_0 M\#_\ell(S^2 \times S^{n-1})$. Its fixed point set, which is necessarily totally geodesic, is the space obtained from $M \times S^0$ by removing $\bigsqcup_{\ell+1} D^n_{r_2} \times S^0$ and gluing in $(\ell + 1)$ copies of $D^1 \times S^{n-1}$. In other words, it is the manifold obtained by doubling $M \setminus \bigsqcup_{\ell+1} D^n$ along its boundary, which is diffeomorphic to $M\#(-M)\#_\ell(S^1 \times S^{n-1})$.

To apply Proposition 2.3 it remains to verify that the induced metric has positive scalar curvature.

Lemma 3.1. *For $r_1 > 0$ sufficiently small there exists a choice of warping functions h, f such that they satisfy the boundary conditions (i)–(iv), the metric (13) has positive Ricci curvature, and such that the induced metric on $M\#(-M)\#_\ell(S^1 \times S^{n-1}) \subseteq \Sigma_0 M\#_\ell(S^2 \times S^{n-1})$ has positive scalar curvature.*

Proof. We follow the construction of [Rei24, Lemma 3.3]. The initial step consists of defining $f\colon [0, \infty) \to (0, \infty)$ as the solution of the initial value problem

$$f(0) = 1\,, \tag{14}$$

$$f'(0) = 0\,, \tag{15}$$

$$f'' = \frac{\alpha\lambda_0^2}{2} f^{-\alpha-1}\,, \tag{16}$$

where $\lambda_0 \in (\cos(r_2), 1)$ and $\alpha \in (n-2, \frac{n-2}{\lambda_0^2})$. The function h is then defined by

$$h = \frac{2}{\alpha\lambda_0^2} f'\,. \tag{17}$$

It is shown in the proof of [Rei24, Lemma 3.3] that these functions satisfy the boundary conditions (iii) and (iv) with $t_0 = 0$ and that the resulting metric (13) has positive Ricci curvature. In addition, f satisfies $f''>0$, $f\geq 1$, and $f'(t)\to\lambda_0$ as $t\to\infty$, and h satisfies $h''<0$ and $h'>0$.

We now analyse the scalar curvatures of the fixed points set of the involution on each copy of $D^2\times S^{n-1}$, i.e. on $D^1\times S^{n-1}$. Here the metric is given by

$$dt^2+h(t)^2ds_0^2+f(t)^2ds_{n-1}^2=dt^2+f(t)^2ds_{n-1}^2\,, \tag{18}$$

where we extended the interval $[0,\infty)$ to $\mathbb{R}$ and defined $f(-t)=f(t)$ for $t<0$.

Recall that the scalar curvature of such a warped product metric is given by

$$\mathrm{scal}=-2(n-1)\frac{f''}{f}+(n-1)(n-2)\frac{1-f'^2}{f^2}\,, \tag{19}$$

see e.g. [Pet16, Section 4.2.3]. As shown in the proof of [Rei24, Lemma 3.3], we have the following:

$$\frac{f''}{f}=\frac{\alpha\lambda_0^2}{2}f^{-\alpha-2}\,, \tag{20}$$

$$\frac{1-f'^2}{f^2}=\frac{1-\lambda_0^2+\lambda_0^2f^{-\alpha}}{f^2}\,. \tag{21}$$

Hence, we obtain, by using $f\geq 1$,

$$\begin{aligned}\mathrm{scal}&=(n-1)f^{-\alpha-2}\left(-\alpha\lambda_0^2+(n-2)(\lambda_0^2+(1-\lambda_0^2)f^{\alpha})\right)\\&\geq(n-1)f^{-\alpha-2}\left(-\alpha\lambda_0^2+(n-2)\right)\\&>0\,,\end{aligned} \tag{22}$$

where we used that $\alpha\lambda_0^2<n-2$.

It remains to choose $t_1>0$ with $f'(t_1)>\cos(r_2)$ and modify the functions f and h near t_1 to satisfy the boundary conditions (i) and (ii). Here we proceed along the same lines as in [Rei24, Lemma 3.3] and modify f to be of the form $f(t)=N\sin(\frac{t-t'}{N})$ for some $N,t'>0$ near $t=t_1$, and so that all derivatives of h vanish at $t=t_1$ while preserving the positivity of the Ricci curvature of the metric (13). The linear dependence of the scalar curvature of the induced metric on the second derivatives of f, as well as the fact that it is a linear combination of the terms $-\frac{f''}{f}$ and $\frac{1-f'^2}{f^2}$ ensure in the same way that it remains positive throughout the construction.

Finally, the boundary conditions (i) and (ii) are achieved by appropriately rescaling the metric, i.e. replacing h and f by $\frac{1}{N}h(\cdot N)$ and $\frac{1}{N}f(\cdot N)$, and shifting their domain by t'. Note that a smaller value of $h(t_1)=r_1$ can always be achieved by choosing a larger value of t_1, so we can achieve condition (i) for all r_1 sufficiently small. □

Hence, we obtain the following result:

Proposition 3.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$ with positive Ricci curvature. Then, for all $\ell \in \mathbb{N}$, there exists a metric g_ℓ of positive scalar curvature on $M\#(-M)\#_\ell(S^1 \times S^{n-1})$ such that for all $\varepsilon > 0$ there is a metric $\bar{g}_\ell^\varepsilon$ of positive Ricci curvature on $\Sigma_0 M\#_\ell(S^2 \times S^{n-1})$ and a totally geodesic embedding*

$$(M\#(-M)\#_\ell(S^1 \times S^{n-1}), g_\ell) \subseteq (\Sigma_0 M\#_\ell(S^2 \times S^{n-1}), \bar{g}_\ell^\varepsilon) \tag{23}$$

with normal Ricci curvatures $\mathrm{Ric}^{\bar{g}_\ell^\varepsilon}(\nu) = \varepsilon$, where ν is a unit normal vector field of $M\#(-M)\#_\ell(S^1 \times S^{n-1})$.

Proof. We use the construction described above, i.e. we perform $(\ell+1)$ surgeries on $S^1 \times M$ equipped with the product metric $r_1^2 ds_1^2 + g$ for some $r_1 > 0$, which has non-negative Ricci curvature. Note that the induced metric on the embedded totally geodesic hypersurface $S^0 \times M$ is given by g, which has positive Ricci curvature, and hence also positive scalar curvature.

By Lemma 2.1, by choosing r_1 sufficiently small, we can now perform $(\ell+1)$ surgeries on this space, i.e. remove $(\ell+1)$ pairwise disjoint copies of $S^1 \times D^n_{r_2}$ for $r_2 > 0$ sufficiently small, and smoothly replace each one with a copy of $D^2 \times S^{n-1}$, equipped with a metric of strictly positive Ricci curvature. In addition, the scalar curvature of the induced metric on the hypersurface $D^1 \times S^{n-1} \subseteq D^2 \times S^{n-1}$ is strictly positive.

Hence, we overall obtain a metric of non-negative Ricci curvature on $\Sigma_0 M\#_\ell(S^2 \times S^{n-1})$ with points of strictly positive Ricci curvature, and a totally geodesic embedding of $M\#(-M)\#_\ell(S^1 \times S^{n-1})$ whose induced metric has positive scalar curvature. We now slightly deform the metric on $\Sigma_0 M\#_\ell(S^2 \times S^{n-1})$ to have globally strictly positive Ricci curvature. Here we apply the deformation results of Ehrlich [Ehr76, Theorem 5.1], which show that such a deformation is possible C^4-close to the original metric.

As explained in [Ehr76, p. 20], this deformation can be arranged to preserve the isometry group of the original metric. In particular, since the embedding $M\#(-M)\#_\ell(S^1 \times S^{n-1}) \subseteq \Sigma_0 M\#_\ell(S^2 \times S^{n-1})$ is the fixed point set of the isometric involution induced be a reflection on S^1, it remains the fixed point set of this involution along the deformation. As a consequence, it remains a totally geodesic hypersurface. In addition, since the deformation is C^4-close, the induced metric on $M\#(-M)\#_\ell(S^1 \times S^{n-1})$ again has positive scalar curvature.

Finally, we apply the deformation of Proposition 2.3 to obtain the required metric. □

End of proof of Theorem B. For a fixed $\ell \in \mathbb{N}$, let g_ℓ be the Riemannian metric on $N_\ell := M\#(-M)\#_\ell(S^1 \times S^{n-1})$ obtained from the application of Proposition 3.2. The Laplacian Δ_{g_ℓ} has discrete eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots . \tag{24}$$

For any $0 < \varepsilon < \lambda_1$, Proposition (3.2) also gives an embedding

$$(M\#(-M)\#_\ell(S^1 \times S^{n-1}), g_\ell) \subseteq (\Sigma_0 M\#_\ell(S^2 \times S^{n-1}), \bar{g}_\ell^\varepsilon) \tag{25}$$

that is totally geodesic and hence minimal. Since $\mathrm{Ric}^{\bar{g}_\ell^\varepsilon}(\nu) = \varepsilon$, the Jacobian operator of N_ℓ reduces to (c.f. [CM11, Definition 1.31]):

$$\mathrm{L}_{N_\ell} = \Delta_{g_\ell} + \mathrm{Ric}^{\bar{g}_\ell^\varepsilon}(\nu) + |\mathrm{I\!I}_{N_\ell}|^2 = \Delta_{g_\ell} + \varepsilon.$$

Thus, the Morse index of L_{N_ℓ} is equal to the number of eigenvalues of Δ_{g_ℓ} less than ε. By (24), that number must be 1. □

Remark 3.3. *Since in the above proof we can choose ε arbitrary, and since the sequence of eigenvalues of Δ_{g_ℓ} diverges, we can also realise arbitrarily high indices with the same argument (however, depending on the multiplicities of the eigenvalues, not every index can be realized).*

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest or associated data in our manuscript.

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