

## NON-NEGATIVE RICCI CURVATURE ON CLOSED MANIFOLDS UNDER RICCI FLOW

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**ABSTRACT.** In this short paper we show that non-negative Ricci curvature is not preserved under Ricci flow for closed manifolds of dimensions four and above, strengthening a previous result of Knopf for complete non-compact manifolds of bounded curvature. This brings down to four dimensions a similar result Böhm and Wilking have for dimensions twelve and above. Moreover, the manifolds constructed here are Kähler manifolds and relate to a question raised by Xiuxiong Chen.

### 1. INTRODUCTION

Ricci flow is a flow of Riemannian metrics designed to improve a given initial Riemannian metric and has been a powerful tool in understanding geometry and topology of manifolds. A nice and important feature of this flow is that it preserves the “non-negativity” of various curvature conditions.

The flow was introduced on closed manifolds by Richard Hamilton in [20]. He proved that non-negative scalar curvature is preserved by the flow in all dimensions. He also showed that the flow preserves non-negative Ricci and non-negative sectional curvatures on three dimensional closed manifolds. Later, in [21], Hamilton proved that Ricci flow also preserves the non-negativity of the Riemann curvature tensor in all dimensions. Also in higher dimensions, independent work of Huisken, Margerin and Nishikawa (see [16, Section 7.1]) demonstrated the invariance of certain sets defined by pinching conditions. Haiwen Chen, in [11], showed that 2-non-negativity of the Riemann tensor is preserved in all dimensions. In dimension four, Hamilton showed in [22] that positive isotropic curvature is invariant under Ricci flow, and it has recently been proved independently by Brendle and Schoen [5] and by Huy T. Nguyen [25] that this holds in dimensions five and above. In their important paper [5], Brendle and Schoen also proved that each of the properties of  $M^n \times \mathbb{R}$  and  $M^n \times \mathbb{R}^2$  having non-negative isotropic curvature is preserved by Ricci flow for all  $n \geq 4$ . Critically, the latter property is weaker than pointwise  $\frac{1}{4}$ -pinched sectional curvature, but stronger than non-negative sectional curvature. In related work, Brendle found that if  $M^n \times \mathbb{S}^2(1)$  has non-negative isotropic curvature, then this remains so when the metric on  $M$  is evolved by the Ricci flow (where  $\mathbb{S}^2(1)$

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denotes a two-sphere of constant curvature 1); we refer the reader to [4]. Böhm and Wilking in [3] provided a method for constructing new invariant curvature conditions from known ones. Wilking later announced in [33] an interesting result that provides a unified approach for understanding previously discovered invariant curvature conditions.

Ricci flow was extended to complete non-compact manifolds with bounded curvature by Wan-Xiong Shi in [31] and to Kähler manifolds by Huai-Dong Cao in [9] and by Shigetoshi Bando in [1]. Similar invariant curvature conditions were later found for these flows too. In [30], Shi proved that non-negative Ricci curvature is preserved by the flow on complete manifolds of three dimensions. Bando and Mok, in [27] and [1], proved that non-negative bisectional curvature (which is a stronger assumption than and implies non-negative Ricci curvature; see [19]) on closed Kähler manifolds is also an invariant curvature condition, and this later was extended by Shi in [32] to complete non-compact Kähler manifolds with bounded curvature.

There has also been work showing that some curvature conditions are not preserved in general. In [26], Lei Ni exhibited complete non-compact Riemannian manifolds with bounded non-negative sectional curvature of dimensions four and above where the Ricci flow does not preserve the non-negativity of the sectional curvature. Dan Knopf later showed in [23] examples of complete non-compact Kähler manifolds of bounded curvature and non-negative Ricci curvature whose Kähler-Ricci evolutions immediately acquire Ricci curvature of mixed sign, for each complex dimension above one. Finally, in [2], Böhm and Wilking showed an example of a closed twelve dimensional manifold where Ricci flow evolves an initial homogeneous metric of positive sectional curvature to metrics with mixed Ricci curvature. By taking products with spheres, they settled that positive Ricci curvature is not invariant under the flow for dimensions twelve and above.

Starting on [14] and later with [15], Xiuxiong Chen and Gang Tian answered affirmatively the following important question on Ricci flow: on a compact Kähler-Einstein manifold, does the Kähler-Ricci flow converge to a Kähler-Einstein metric if the initial metric has positive bisectional curvature? They remarked that what their argument really needed was for the Ricci curvature to be positive along the Kähler-Ricci flow. Since the positivity of Ricci curvature being preserved under the Ricci flow was unknown, they used the fact that the positivity of the bisectional curvature is preserved and that it implies positive Ricci curvature. Since their work, others have tried to remove or weaken this strong assumption of positive bisectional curvature, and we refer the interested reader to [28], [29], [12], [13]. In 1988, Richard Hamilton and Huai-Dong Cao announced a proof that positive orthogonal bisectional curvature is preserved along the flow, and that this suffices to guarantee that non-negative Ricci is preserved. More recently, Xiuxiong Chen published a proof in [12] that non-negative Ricci curvature is preserved on a Kähler solution, assuming its bisectional curvature remains positive as long as it exists. Moreover, under the same assumption, he proved that if the initial Ricci curvature is not positive, then the lower bound of Ricci will increase with time and actually reach zero as time goes to infinity. Motivated by his nice result in [12], Chen asked later in [13] if some form of lower bound of the Ricci curvature would be preserved under Ricci flow at least for closed Kähler manifolds.

The purpose of this short paper is to show that this is *not* the case in general. For each complex dimension greater than one, we construct a closed Kähler manifold with non-negative Ricci curvature whose evolution under Ricci flow immediately acquires Ricci curvature of mixed sign. Let  $M$  denote  $\mathbb{CP}^2$  blown up at one point (in the next section, following Calabi in [6], we realize  $M$  as the total space of a bundle  $\mathbb{CP}^1 \hookrightarrow M \rightarrow \mathbb{CP}^1$ ; see [17, Section 2.7.2] for more details). We prove the following:

**Theorem 1.** *There exist Kähler metrics on  $M$  with non-negative Ricci curvature that immediately acquire mixed Ricci curvature when evolved by Ricci flow.*

*Remark 2.* By taking products of  $M$  with spheres, we lower Böhm and Wilking's previous result in [2] to real dimension four and above. Observe that these products will not be Kähler manifolds anymore.

Theorem 1 can actually be generalized to all complex dimensions above one:

**Theorem 3.** *For  $n \geq 2$ , let  $M_n$  be  $\mathbb{CP}^n$  blown up at one point. Then, there exist Kähler metrics on  $M_n$  with non-negative Ricci curvature that immediately acquire mixed Ricci curvature when evolved by Ricci flow.*

*Remark 4.* Examples found in Theorems 1 and 3 can be thought of as compactifications of the ones found by Knopf in [23].

*Remark 5.* It is also possible to construct Kähler metrics on  $M$  where positive Ricci curvature is preserved by Ricci flow. An example is the soliton metric found independently by Koiso [24] and Cao [7].

*Remark 6.* A three-manifold  $M^3$  has non-negative Ricci curvature if and only if  $M^3 \times \mathbb{R}$  has non-negative isotropic curvature. Our result shows that non-negative Ricci curvature does not behave well in higher dimensions, so Brendle and Schoen's observation that non-negative isotropic curvature on  $M^n \times \mathbb{R}$  is invariant suggests that this property is the "right" generalization to higher dimensions.

The rest of this paper is organized as follows. We first deal with the critical case  $n = 2$  and begin by proving Theorem 1 in Sections 2 and 3: in Section 2, we review  $U(2)$ -invariant metrics on  $\mathbb{C}^2 \setminus \{0\}$  and construct an initial metric on  $M$  with non-negative Ricci curvature, and in Section 3 the proof of Theorem 1 is finished. Finally, in Section 4 we discuss the general case stated in Theorem 3.

## 2. AN INITIAL METRIC OF NON-NEGATIVE RICCI CURVATURE

In this section we consider rotationally symmetric Kähler metrics on  $\mathbb{C}^2 \setminus \{0\}$  and derive an initial metric of non-negative Ricci curvature on the twisted line bundle  $\mathbb{CP}^1 \hookrightarrow M \rightarrow \mathbb{CP}^1$  of Calabi [6, Section 3]. Metrics of this sort were also used by Koiso in [24], Cao in [7, 8] and Feldman, Ilmanen and Knopf in [18] to construct examples of gradient Ricci solitons. For convenience of the reader, we start following Calabi in [6] and realizing  $M_n$  as the total space of a bundle  $\mathbb{CP}^1 \hookrightarrow M \rightarrow \mathbb{CP}^{n-1}$  (for more details about this construction, see [17, Section 2.7.2]).

**2.1. Calabi's bundle construction.** Cover the projective space  $\mathbb{CP}^{n-1}$  with its usual  $n$  charts  $\varphi_\alpha : \mathcal{U}_\alpha \rightarrow \mathbb{C}^{n-1}$ , where  $\mathcal{U}_\alpha = \{[x_1 : x_2 : \dots : x_n] \in \mathbb{CP}^{n-1} : x_\alpha \neq 0\}$  and  $\varphi_\alpha : [x_1 : x_2 : \dots : x_n] \mapsto \left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_n}{x_\alpha}\right)$ . We then can write  $\mathbb{CP}^{n-1} = (\bigsqcup_{\alpha=1}^n \varphi_\alpha(\mathcal{U}_\alpha)) / \simeq$ , where, for example,

$$\varphi(\mathcal{U}_1) \ni (z_1, \dots, z_{n-1}) \simeq \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_{n-1}}{z_1}\right) \in \varphi_2(\mathcal{U}_2).$$

After formally identifying  $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ , we define for each positive integer  $k$  the  $k$ -twisted bundle

$$\mathcal{F}_k^n = \left( \bigsqcup_{\alpha=1}^n (\mathcal{U}_\alpha \times \mathbb{CP}^1) \right) / \sim$$

where  $\mathcal{U}_\alpha \times \mathbb{CP}^1 \ni ([x_1 : \dots : x_n]; \xi) \sim ([y_1 : \dots : y_n]; \eta) \in U_\beta \times \mathbb{CP}^1$  if, and only if,  $[x_1 : \dots : x_n] = [y_1 : \dots : y_n]$  and  $\eta = \left(\frac{y_\beta}{x_\alpha}\right)^k \xi$ . Moreover, let  $S_0 = \{[x_1 : \dots : x_n]; 0\}$  and  $S_\infty = \{[x_1 : \dots : x_n]; \infty\}$ , which are two global sections of  $\mathcal{F}_k^n$ . Set  $\widehat{\mathcal{F}}_k^n = \mathcal{F}_k^n \setminus (S_0 \cup S_\infty)$  and let  $\Psi : \mathbb{C}^n \setminus \{0\} \rightarrow \widehat{\mathcal{F}}_k^n$  be the map

$$\Psi : (x_1, x_2, \dots, x_n) \mapsto ([x_1 : \dots : x_n]; x_\alpha^k)$$

if  $x_\alpha \neq 0$ . Because  $([x_1 : \dots : x_n]; x_\alpha^k) \sim [x_1 : \dots : x_n]; x_\beta^k$  whenever  $x_\alpha \neq 0$  and  $x_\beta \neq 0$ , the map  $\Psi$  is well defined. Furthermore, one can check that  $\Psi$  is  $k$ -to-one and surjective.

In particular, when  $k = 1$ ,  $\Psi$  is one-to-one so  $\mathcal{F}_1^n = M_n$  can be thought of as  $\mathbb{C}^n \setminus \{0\}$  with  $\mathbb{CP}^{n-1}$ 's glued at 0 (the section  $S_0$ ) and  $\infty$  (the section  $S_\infty$ ). We next look for suitable metrics on  $\mathbb{C}^n \setminus \{0\}$  that can be extended to  $M_n$ .

**2.2.  $U(2)$ -invariant Kähler metrics on  $\mathbb{C}^2 \setminus \{0\}$ .** Let  $g$  be a  $U(2)$ -invariant Kähler metric on  $\mathbb{C}^2 \setminus \{0\}$ , the latter with complex coordinates  $z = (z^1, z^2)$ . Define  $u = |z^1|^2$ ,  $v = |z^2|^2$  and  $w = u + v$ .

Since  $g$  is a Kähler metric and the de Rham cohomology group  $H^2(\mathbb{C}^2 \setminus \{0\}) = 0$ , by the  $\partial\bar{\partial}$ -lemma one can find a global real smooth function  $P : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{R}$  such that<sup>1</sup>

$$(2.1) \quad g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} P.$$

The further assumption of  $g$  being rotationally symmetric allows us to write  $P = P(r)$ , where  $r = \log w$ . We then set  $\varphi(r) = P_r(r)$  (we use subscript for the derivative since later  $P$  will be regarded as a function of time as well) and compute from (2.1)

$$(2.2) \quad g = [e^{-r} \varphi \delta_{\alpha\beta} + e^{-2r} (\varphi_r - \varphi) \bar{z}^\alpha z^\beta] dz^\alpha d\bar{z}^\beta.$$

Moreover,<sup>2</sup>

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \frac{1}{w^2} \begin{pmatrix} v\varphi + u\varphi_r & (\varphi_r - \varphi) \bar{z}^1 z^2 \\ (\varphi_r - \varphi) z^1 \bar{z}^2 & u\varphi + v\varphi_r \end{pmatrix},$$

and thus we find that  $\det(g_{\alpha\bar{\beta}}) = e^{-2r} \varphi \varphi_r$ . From these two past relations we can rapidly observe that a potential  $P$  on  $\mathbb{C}^2 \setminus \{0\}$  gives rise to a Kähler metric as in

<sup>1</sup>We are following the notation in [17, Chapter 2].

<sup>2</sup>The matrix  $(g)$  is actually a  $4 \times 4$  matrix,  $(g) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ , where  $A = \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}$ .

(2.1) if, and only if,

$$(2.3) \quad \varphi > 0 \text{ and } \varphi_r > 0.$$

For computing the Ricci curvature, recall that on Kähler manifolds the complex Ricci tensor  $R_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}}$  is given by

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log \det g.$$

To put this expression in a more useful form we follow [23] and define

$$(2.4) \quad G = -\log \det g = 2r - \log \varphi - \log \varphi_r,$$

$$(2.5) \quad \psi = G_r = 2 - \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r}.$$

One next calculates

$$(2.6) \quad R_{\alpha\bar{\beta}} = e^{-r} \psi \delta_{\alpha\beta} + e^{-2r} (\psi_r - \psi) \bar{z}^\alpha z^\beta,$$

and thus Ricci can be represented as<sup>3</sup>

$$\begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix} = \frac{1}{w^2} \begin{pmatrix} v\psi + u\psi_r & (\psi_r - \psi)\bar{z}^1 z^2 \\ (\psi_r - \psi)z^1 \bar{z}^2 & u\psi + v\psi_r \end{pmatrix}.$$

Also as in [23] we will say that a  $(1,0)$  tensor  $W$  is an eigenvector of the complex Ricci tensor corresponding to the eigenvalue  $\lambda$  if  $R_{\alpha\bar{\beta}}W^\alpha = \lambda g_{\alpha\bar{\beta}}W^\alpha$ . Hence understood, the eigenvalues of Rc are

$$\begin{aligned} \lambda_1 &= \frac{\psi}{\varphi} \quad \text{with eigenvector} \quad U = \bar{z}^2 \frac{\partial}{\partial z^1} + \bar{z}^1 \frac{\partial}{\partial z^2}, \\ \lambda_2 &= \frac{\psi_r}{\varphi_r} \quad \text{with eigenvector} \quad V = z^1 \frac{\partial}{\partial z^2} + z^2 \frac{\partial}{\partial z^1}. \end{aligned}$$

**2.3. A metric of non-negative Ricci curvature.** To find a metric of non-negative Ricci curvature on a Calabi line bundle  $\mathbb{CP}^1 \hookrightarrow M \rightarrow \mathbb{CP}^1$ , we want a  $U(2)$ -invariant Kähler potential  $P$  on  $\mathbb{C}^2 \setminus \{0\}$  with the following properties:

- (1)  $\varphi > 0$  everywhere;
- (2)  $\varphi_r > 0$  everywhere;
- (3)  $\psi > 0$  everywhere;
- (4)  $\psi_r \geq 0$  everywhere;
- (5)  $g$  extends smoothly to a complete metric by adding a  $\mathbb{CP}^1$  at  $z = 0$ ; and
- (6)  $g$  extends smoothly to a complete metric by adding a  $\mathbb{CP}^1$  at  $z = \infty$ .

To find such  $P$ , let us rewrite equation (2.5) as

$$(2.7) \quad [\log(\varphi\varphi_r)]_r = a,$$

where  $a = 2 - \psi$ . We next observe that we can formally solve the ODE above and find the formal solution ( $I(f)$  stands for an antiderivate of  $f$ )

$$(2.8) \quad \varphi(r) = \sqrt{I(2e^{I(a)})}.$$

In particular, if  $a$  is a constant function, we find that  $\varphi(r) = \sqrt{\frac{2}{a}e^{ar} + c}$  is a solution for (2.7). This ( $a \equiv 1$ ) in fact was the  $\varphi$  used in [23] and, as shown in this particular

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<sup>3</sup>The matrix  $(Rc)$  is actually a  $4 \times 4$  matrix,  $(Rc) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ , where  $R = \begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix}$ .

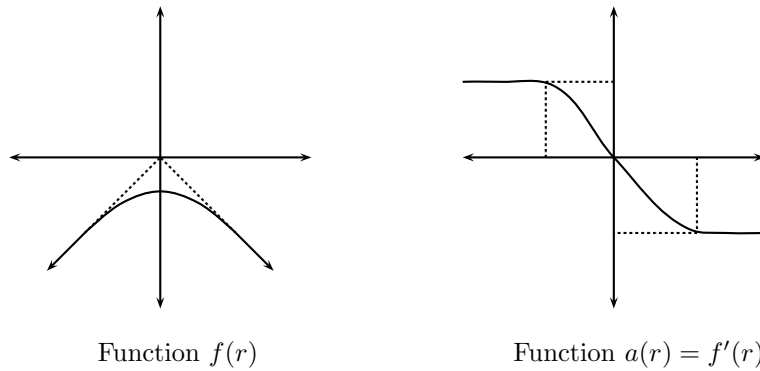


FIGURE 1

reference, it leads to potential  $P$  satisfying conditions (1) through (5) above, but not (6).

We consider instead a function  $a$  that is constant equal to 1 in a neighborhood of  $r = -\infty$ , constant equal to  $-1$  in a neighborhood of  $r = +\infty$ , and smooth on the whole real line.

To be precise, consider the function  $y(r) = -|r|$ . Because  $y''(r) = \delta_0(r)$ , it is possible to smooth  $y$  in a small neighborhood of  $r = 0$  and obtain a smooth function  $f(r)$  such that  $f''(r) \leq 0$ . Then, set  $a(r) = f'(r)$ ; the graph of the function  $a$  is shown on the right in Figure 1.

Since

$$\int_{\mathbb{R}} e^{f(x)} dx < \int_{\mathbb{R}} e^{-|x|} dx < +\infty,$$

the function  $F(r) = \int_{-\infty}^r e^{f(x)} dx$  is well defined and smooth. Thus, we find a solution to equation (2.7) for  $a$  chosen as above:

$$(2.9) \quad \varphi(r) = \sqrt{2F(r) + c},$$

where  $c$  is any arbitrary positive constant (since  $F(r) > 0$  everywhere). We claim that  $\varphi$  as above gives a potential  $P$  as desired.

First, it is clear that  $\varphi$  is positive everywhere, and since  $\varphi\varphi_r = e^f > 0$ ,  $\varphi_r$  is also positive everywhere. Thus  $\varphi$  satisfies properties (1) and (2) above; i.e.,  $P$  gives rise to a Kähler metric.

Moreover,  $\psi = 2 - a$ , so  $\psi > 0$  everywhere, since the function  $a$  takes values on the interval  $[-1, 1]$  only. Also (since  $f$  is concave)  $a$  is non-increasing, so  $\psi$  is non-decreasing and thus  $\psi_r \geq 0$ . Hence properties (3) and (4) are also satisfied; that is,  $g$  has non-negative Ricci curvature.

To show that  $\varphi$  satisfies conditions (5) and (6) it is convenient to compute  $F(r)$  when  $r$  is near  $\pm\infty$  and write explicitly

$$(2.10) \quad \varphi = \sqrt{2e^r + c}, \quad \text{for } r \text{ near } -\infty;$$

$$(2.11) \quad \varphi = \sqrt{-2e^{-r} + c + F_{\infty}}, \quad \text{for } r \text{ near } +\infty,$$

where  $F_{\infty} = \int_{\mathbb{R}} e^{f(x)} dx$  is a constant.

Calabi's lemma [6, Section 3] tells us that  $g$  will extend to a smooth Kähler metric on Calabi's twisted line bundle  $\mathbb{CP}^1 \hookrightarrow M \rightarrow \mathbb{CP}^1$  if  $\varphi$  satisfies the following asymptotic properties:

- (i) There exist positive constants  $a_0$  and  $a_1$  such that  $\varphi$  has the expansion

$$\varphi(r) = a_0 + a_1 w + a_2 w^2 + \mathcal{O}(|w|^3)$$

as  $r \rightarrow -\infty$ .

- (ii) There exist a positive constant  $b_0$  and a negative constant  $b_1$  such that  $\varphi$  has the expansion

$$\varphi(r) = b_0 + b_1 w^{-1} + b_2 w^{-2} + \mathcal{O}(|w|^{-3})$$

as  $r \rightarrow \infty$ .

For  $c > 0$ , by (2.10),  $\varphi$  admits the expansion near  $r = -\infty$ :

$$\varphi = \sqrt{c} + \frac{1}{\sqrt{c}}w - \frac{1}{2c^{3/2}}w^2 + \mathcal{O}(|w|^3).$$

Thus, by Calabi's lemma, property (5) is verified. And near  $r = \infty$ , by (2.11),  $\varphi$  admits the expansion

$$\varphi = \sqrt{c + F_\infty} - \frac{1}{\sqrt{c + F_\infty}}w^{-1} - \frac{1}{2(c + F_\infty)^{3/2}}w^{-2} + \mathcal{O}(|w|^{-3}),$$

so Calabi's lemma again guarantees that  $g$  satisfy property (6).

### 3. EFFECT OF THE KÄHLER-RICCI FLOW

We finally consider the Kähler-Ricci flow evolution  $(M, g(t))$ , where the initial Kähler potential  $P(r, 0)$  is taken as the Kähler potential  $P(r)$  constructed in the previous section.

Because  $g$  is rotationally symmetric, let us work at a point  $(\zeta, 0) \in \mathbb{C}^2 \setminus \{0\}$ , where  $\zeta \neq 0$  is arbitrary. Since

$$g|_{(\zeta, 0)} = \frac{1}{|\zeta|^2} \begin{pmatrix} \varphi_r & 0 \\ 0 & \varphi \end{pmatrix}$$

and

$$\text{Rc}|_{(\zeta, 0)} = \frac{1}{|\zeta|^2} \begin{pmatrix} \psi_r & 0 \\ 0 & \psi \end{pmatrix}$$

in the standard basis, the Kähler-Ricci flow equation

$$\frac{\partial}{\partial t} g = -\text{Rc}$$

will be satisfied if, and only if,  $\varphi$  evolves by  $\varphi_t = -\psi$ . This last assertion is reduced to

$$(3.1) \quad \varphi_t = \frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - 2.$$

Moreover, the function  $\psi$  (since  $\frac{\partial}{\partial r}$  and  $\frac{\partial}{\partial t}$  commute) must evolve by

$$(3.2) \quad \psi_t = \frac{\psi_{rr}}{\varphi_r} - \frac{\varphi_{rr}\psi_r}{\varphi_r^2} + \frac{\psi_r}{\varphi} - \frac{\varphi_r\psi}{\varphi^2}.$$

But near  $r = -\infty$  we have  $\psi(\cdot, 0) \equiv 1$ , and this reduces equation (3.2) to

$$(3.3) \quad \left. \frac{\partial}{\partial t} \psi \right|_{t=0} = -\frac{\varphi_r}{\varphi^2}$$

at the initial time. A final computation then shows that

$$(3.4) \quad \left. \frac{\partial}{\partial t} \psi_r \right|_{t=0} = \frac{e^r}{\varphi^5} (e^r - c),$$

which is strictly negative for  $r < \log c$ . Hence, the complex Ricci tensor must acquire a negative eigenvalue  $\lambda_2 < 0$  at all points close enough to  $r = -\infty$ . This completes the proof of Theorem 1.

#### 4. COMPLEX DIMENSION $n \geq 2$

In this section we expand our result to higher complex dimensions by proving Theorem 3.

If  $P$  is now a  $U(n)$ -invariant Kähler potential on  $\mathbb{C}^n \setminus \{0\}$ , we still have formulas (2.1) and (2.2). The determinant of the matrix  $(g_{\alpha\bar{\beta}})$  is now  $\det(g_{\alpha\bar{\beta}}) = e^{-nr} \varphi^{n-1} \varphi_r$ , and, in particular, conditions in (2.3) are still necessary and sufficient. Formula (2.4) then becomes<sup>4</sup>

$$(4.1) \quad G = nr - (n-1) \log \varphi - \log \varphi_r,$$

so (2.5) is replaced by

$$(4.2) \quad \psi = G_r = n - (n-1) \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r},$$

but (2.6) remains the same. By rotational symmetry, we need to display the eigenvalues of Ricci curvature only at points of the form  $Z = (\zeta, 0, 0, \dots, 0) \in \mathbb{C}^n \setminus \{0\}$ , with  $\zeta \neq 0$  arbitrary. So we calculate

$$g|_Z = \frac{1}{|\zeta|^2} \begin{pmatrix} \varphi_r & & & \\ & \varphi & & \\ & & \ddots & \\ & & & \varphi \end{pmatrix}$$

and

$$\text{Rc}|_Z = \frac{1}{|\zeta|^2} \begin{pmatrix} \psi_r & & & \\ & \psi & & \\ & & \ddots & \\ & & & \psi \end{pmatrix}.$$

We next take  $\varphi$  as in (2.9). Hence  $\varphi$  and  $\varphi_r$  are both greater than zero everywhere. Moreover, we know that  $\psi \equiv n - a$ , so  $\psi > 0$  and, since  $a$  is non-increasing,  $\psi_r \geq 0$ .

Furthermore, expansions of  $\varphi$ ,

$$\varphi = \sqrt{c} + \frac{1}{\sqrt{c}} w - \frac{1}{2c^{3/2}} w^2 + \mathcal{O}(|w|^3),$$

near  $r = -\infty$ , and

$$\varphi = \sqrt{c + F_\infty} - \frac{1}{\sqrt{c + F_\infty}} w^{-1} - \frac{1}{2(c + F_\infty)^{3/2}} w^{-2} + \mathcal{O}(|w|^{-3}),$$

<sup>4</sup>Dan Knopf has drawn to my attention a small oversight in a corresponding formula for  $G$  in section 5 of [23]. As a consequence, a few gradient terms are missing in subsequent formulae of [23], yet these missing terms do not harm the argument made there.



near  $r = \infty$ , remain the same; so, by Calabi's lemma,  $\varphi$  gives rise to a Kähler metric on  $\mathbb{C}^n \setminus \{0\}$  that can be extended smoothly to  $M_n$ .

We next settle the evolution equations of  $\varphi$  and  $\psi$ :

$$\varphi_t = \frac{\varphi_{rr}}{\varphi_r} + (n-1) \frac{\varphi_r}{\varphi} - n,$$

and

$$\psi_t = \frac{\psi_{rr}}{\varphi_r} - \left( \frac{\varphi_{rr}}{\varphi_r^2} - \frac{n-1}{\varphi} \right) \psi_r - (n-1) \frac{\varphi_r \psi}{\varphi^2}.$$

Since  $\psi(\cdot, t) \equiv n-1$  on a neighborhood of  $r = -\infty$ , this last equation reduces to

$$\psi_t = -(n-1)^2 \frac{\varphi_r}{\varphi^2}.$$

A further computation then shows that

$$\left. \frac{\partial}{\partial t} \psi_r \right|_{t=0} = (n-1)^2 \frac{e^r}{\varphi^5} (e^r - c),$$

which, as before, is strictly negative for  $r < \log c$ . This concludes the proof of Theorem 3.

*Remark 7.* More generally, for each  $1 \leq k \leq n-1$ , one can construct Kähler metrics with non-negative Ricci curvature on  $k$ -twisted bundles  $\mathcal{F}_k^n$  that immediately acquire mixed Ricci sign under Ricci flow. This is achieved by considering the potential  $\varphi$  obtained when using the function  $ka(r)$  instead of  $a(r)$  and then using Calabi's lemma to glue back sections  $S_0$  and  $S_\infty$ .

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