LENGTH OF A CLOSED GEODESIC IN 3-MANIFOLDS OF POSITIVE SCALAR CURVATURE

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Abstract. Let M be a closed 3-dimensional Riemannian manifold with positive scalar curvature, $R_g \geq 6$. We show that M contains a non-trivial closed geodesic of length less than 22500. This confirms a conjecture of M. Gromov in dimension 3.

1. Introduction

Closed geodesics are basic objects of study in Riemannian Geometry, appearing in other fields of mathematics, such as Topology and Dynamical Systems, as well as Physics.

Let M^n be a closed Riemannian manifold. The existence of a periodic geodesic was established in 1951 by A. Fet and L. Lusternik by using Morse theory on the space ΛM^n of closed piecewise differentiable curves on M^n [LF51]. It is natural to want to estimate the length of a shortest closed geodesic in terms of other geometric parameters of M^n, such as its volume, diameter and curvature. Generally speaking, it is easier to bound the length of a shortest closed geodesic on non-simply connected closed Riemannian manifolds. For example, it is easy to see that the length of the shortest closed geodesic on non-simply connected closed Riemannian manifolds is always bounded above by twice the diameter of the manifold. In 1949 K. Loewner found a sharp area upper bound for the length of the shortest closed geodesic on a Riemannian 2-torus, followed by 1951 result of C. Pu establishing a sharp area upper bound for the length of the shortest closed geodesic on a Riemannian real projective plane. These results gave rise to Systolic Geometry. See [CK03], [Kat07] for a survey of the field.

In his foundational paper [Gro83], M. Gromov asked if there exists a constant $c(n)$, such that the length of a shortest closed geodesic, $l(M^n)$ can always be bounded from above by $c(n) \cdot vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ denotes the volume of M^n. In the same paper he proved his famous systolic inequality for essential manifolds: a class of manifolds, which, in particular, include closed Riemannian manifolds that admit a metric on non-positive sectional curvature. Recently, the constant in this inequality was improved to $c(n) = n$ by A. Nabutovsky in [Nab22], building on results from [Pap20], [LLNR22]. One can likewise ask whether $l(M^n)$ can always be uniformly bounded in terms of the diameter of M^n. Note that it was demonstrated by F. Balacheff, C. Croke and M. Katz in [BCK09], unlike the non-simply connected case, when M^n is simply connected it is not always possible to bound $l(M^n)$ by twice the diameter.

When M^n is a simply connected closed Riemannian manifold, it is difficult to establish curvature-free upper bounds for $l(M^n)$. In fact, the only case when such upper bounds are known to exist for a simply connected manifold is that of a Riemannian 2-sphere. The first such upper bounds were established by C. Croke in [Cro88]. They were subsequently improved by A. Nabutovsky and the third author in [NR02] and independently by S. Sabourau in [Sab04] to $4d$ and $8\sqrt{A}$, where d is the diameter and A is the area of the manifold. The area bound was further improved to $4\sqrt{2A}$ by the third author in [Rot06]. Also, in [AVP22] I. Adelstein and F. Vargas Pallete proved that on any non-negatively curved Riemannian 2-sphere, there exists a closed geodesic of length at most $3d$.

A classical theorem of Toponogov [Top59] states that simple closed geodesics in a Riemannian two-sphere with sectional curvature $K_g \geq 1$ have length at most 2π. In dimension $n > 2$, curvature bounds are helpful in finding upper bounds for $l(M^n)$ (see [NR06], [WZ22]), and to have a positive curvature (sectional, *e.g* [BTZ82]) seems to be particularly useful. For a closed Riemannian manifold M^n with a positive Ricci curvature, $Ric \geq \frac{n-1}{r^2}$, it was proven by the third author that $l(M^n) \leq 8\pi nr$ in [Rot24]. The bound was improved to πnr by H.-B. Rademacher in [Rad24].

The main result of our paper provides a bound of this nature in dimension 3, replacing the lower Ricci curvature bound with a much weaker assumption of lower scalar curvature bound.

Theorem A. *Let (M^3, g) be a closed manifold with scalar curvature $R_g \geq 6$. Then M contains a non-trivial closed geodesic of length at most* 22500.

This confirms a conjecture of Gromov [Gro18, Conjecture (a)] in dimension 3. The bound in Theorem A is not sharp but it reveals an underlying rigidity in the space of 3-manifolds with $R_g \geq 6$, a condition which is flexible enough to allow for complicated geometric behavior such as splines (gravitational wells) and drawstrings, see *e.g.* [KX23].

Our proof will rely on the existence theorem of Fet and Lusternik. Their agument can be summarized as follows.

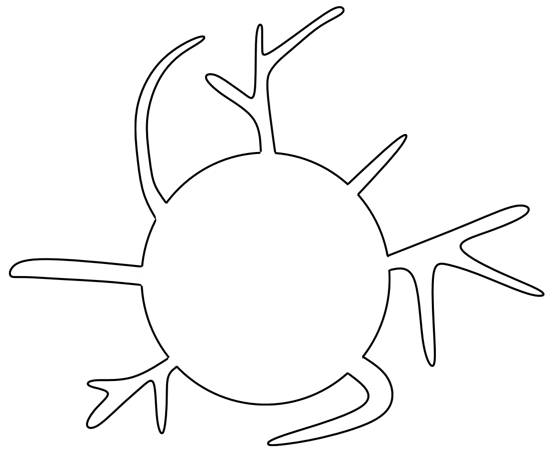

FIGURE 1. A 3-sphere with many splines and $R \geq 6$

Let ΩM^n be the space of piecewise differentiable curves on M^n. From homotopy theory equations we know that there exists an integer q, such that $\pi_q\Omega(M^n) \neq \{0\}$, while $\pi_q(M^n) = \{0\}$. Let us consider a non-contractible map $f : S^q \longrightarrow \Omega M^n$. We will try to deform f to a map whose image lies in the space of constant curves $\Omega_0 M^n$ along the integral curves of the negative gradient of the energy functional on ΩM^n. We should not be able to do it as it would contradict the non-contractibility of f. Thus, we obtain a closed geodesic as an obstruction to this extension.

From this proof we can see that if we can estimate the length of curves in a non-contractible sphere $S^q \subset \Omega M^n$ in terms of some geometric parameters of M^n then we will immediately obtain an upper bound for the length of a shortest closed geodesic.

Note that it is not always possible to obtain such sweepout. For example, in [Lio13], the first author constructed a sequence of metrics on S^2 for which optimal sweepouts cannot be controlled solely in terms of the diameter of the sphere. The construction is based on a result of S. Frankel and M. Katz, in which they have defined a family of metrics on a 2-disk with fixed diameter, but such that they require an uncontrollable length increase to contract the boundary of the disk to a point (see [FK93]). Likewise, O. Alshawa and H. Y. Cheng have constructed a sequence of metrics on S^3 of fixed diameter and volume where the optimal sweepouts must have arbitrarily long curves (see [AC25]).

Hence, one requires curvature assumptions to guarantee existence of a sweepout by short closed curves. In [LZ18] it was shown that 3-manifolds with positive Ricci curvature and bounded volume admit a sweepout by short 1-cycles. Note, however, that the minimax method applied to the space of 1-cycles may not produce a geodesic, but only a stationary geodesic net.

It may also be possible to obtain a short closed geodesic on an n-dimensional sphere even if the sweepout by short curves does not exist using the following dichotomy: either one obtains a sweepout or shows that the existence of a sweepout by short curves is obstructed precisely by the existence of short closed geodesics of Morse index less than $n-1$.

However, in this paper we show that a sweepout of a Riemannian 3-sphere by short closed curves exists and, thus, the closed geodesic we obtain will be a so called min-max geodesic with Morse index 2 for generic Riemannian metrics.

Theorem B. *Let (S^3, g) be a Riemannian 3-sphere with scalar curvature $R_g \geq \Lambda_0 > 0$. Then there exists a non-contractible map $F : (I^2, \partial I^2) \longrightarrow (\Omega S^3, \Omega_0 S^3)$ with the length of $F(x)$ bounded by $\frac{22500\sqrt{6}}{\sqrt{\Lambda_0}}$ for all x. Hence, (S^3, g) contains a non-trivial closed geodesic γ of length at most $\frac{22500\sqrt{6}}{\sqrt{\Lambda_0}}$ and Morse index ≤ 2. Moreover, if the metric g is bumpy then γ has Morse index 2.*

1.1. **Proof overview.** In Section 2 we use topological classification of compact 3-manifolds with positive scalar curvature to reduce Theorem A to the case of 3-spheres. In the rest of the paper we construct a non-contractible 2-parameter family of short closed curves, proving Theorem B, which via Morse theory on the free loop space [Bot82] implies Theorem A for $M \cong S^3$.

Our construction of a family of short closed curves on the 3-sphere can be roughly described as the following dimension reduction argument (with one important caveat we explain below): first we construct a family of "small" 2-spheres foliating M, then we construct a sweepout of each 2-sphere in the foliation by short curves with the property that the sweepout changes continuously as we vary the 2-spheres.

Now we explain the caveat to the strategy described above. It has to do with the notion of "smallness" for 2-spheres that we need to construct a sweepout by short closed curves. It is shown in [LNR15] that we can construct a sweepout of a 2-sphere by curves controlled in terms of the area and diameter of the 2-sphere (but neither area only nor diameter only bounds are sufficient). Hence, the strategy in the previous paragraph can be realized if we could prove the existence of a foliation of M by 2-spheres of small area and *intrinsic* diameter. We state this result as a conjecture below in terms of tree-foliations (see Definition 3.3).

Conjecture 1.1. *There exists a constant $C > 0$ with the following property. Let (S^3, g) be a Riemannian 3-sphere with scalar curvature $R_g \geq \Lambda_0 > 0$. Then there exists a tree-foliation $\{\Sigma_t\}_{t\in T}$ of (S^3, g) with*

(1) $\mathrm{diam}(\Sigma_t) \leq \frac{C}{\sqrt{\Lambda_0}}$;

(2) $\mathrm{Area}(\Sigma_t) \leq \frac{C}{\Lambda_0}$.

Here $\mathrm{diam}(\Sigma_t) = \sup_{x,y}\{dist_{\Sigma_t}(x, y) : x, y \in \Sigma_t\}$ *denotes the diameter of Σ_t in the intrinsic metric on Σ_t.*

In [LM], Mean Curvature Flow with Surgery was used to show that a 3-sphere with positive scalar curvature admits a tree-foliation by 2-spheres of controlled area and *ambient* diameter. Unfortunately, this is not enough to guarantee existence of a sweepout by short curves. To any 2-sphere Σ embedded in M one can attach three very long "fingers" (or "spikes") that wrap around Σ inside M, so that they stay in a small neighbourhood of Σ. This operation increases the diameter of Σ in the ambient metric by an arbitrarily small amount, but any sweepout of such a sphere by closed curves will contain a curve that climbs up and down one of the spikes and hence will have very large length (examples like this of Riemannian metrics on the 2-sphere resembling a three-legged starfish were considered in [Sab04, Remark 4.10], [Lio14, Fig. 1]).

While we are not able to prove Conjecture 1.1, or prove a bound for any other geometric invariant of 2-spheres in a foliation that would imply existence of a sweepout by short curves of each sphere, we found a way to control a different geometric invariant that we call *L-shortness*. L-shortness guarantees that for every closed curve γ on a surface Σ embedded in M there exists a continuous family of arcs of length at most L in M connecting the points of γ to a fixed point $\gamma(0)$. Note that the curves may not necessary lie on Σ. We then use deformation techniques that were developed in [NR] to prove that L-shortness of an embedded 2-sphere Σ implies that an arbitrary sweepout of Σ can be deformed to a family of closed curves of length controlled in terms of L and, moreover, this deformation can be made continuously with respect to deformations of Σ.

Hence, our proof can be summarized as follows. First, in Section 3 we construct a tree-foliation of M by L-short 2-spheres for L bounded above in terms of the lower bound for scalar curvature of M. Then in Section 4 we consider an arbitrary family of sweepouts of these

2-spheres (together forming a two-parameter family of closed curves in M) and show that L-shortness can be used to deform this two-parameter family of (possibly very long) curves to a family of curves of controlled length. In order to prove existence of a tree-foliation by L-short 2-spheres in Section 3 we develop a certain combinatorial analog of the Mean Curvature Flow that works by deforming a given mean convex surface along certain minimal 2-discs in a way that decreases the area by a definite amount after each deformation.

1.2. **Acknowledgments.** Y. L. was supported by NSERC Discovery grant. R.R. was supported by NSERC Discovery grant. R. R. gratefully acknowledges the support from the Princeton IAS Summer Collaborators program in 2024 and the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the Fall 2024 semester, where part of this paper was completed. Part of this paper was completed while Y. L. and R. R. were at the Metric Geometry trimester program at the Hausdorff Research Institute for Mathematics; they are grateful to the Institute for its hospitality. D.M. was partially supported by NSF grant DMS-1910496. Part of this paper was written while R. R. was on a sabbatical leave. During this time she was partially supported by the Simons Foundation International Award SFI-MPS-SFM-00006548.

2. 3-manifolds non-diffeomorphic to the 3-sphere

We first observe that is enough to prove Theorem A for orientable manifolds. If (M^3, g) is non-orientable and γ is a short geodesic in the orientable double cover of M, then the image of γ under the covering map is a short geodesic in (M, g). Suppose (M^3, g) is a closed orientable 3-manifold with scalar curvature $R \geq 6$. By the topological classification of 3-manifolds with positive scalar curvature ([SY79], [GL83], [Per02], [Per03a], [Per03b]) M must be diffeomorphic to a connect sum of

$$(S^3/\Gamma_1)\# \cdots \#(S^3/\Gamma_k)\#(S^2 \times S^1)\# \cdots \#(S^2 \times S^1)$$

where $\Gamma_1, \ldots, \Gamma_k$ are finite subgroups of $SO(4)$. Thus, assuming M is non-diffeomorphic to the 3-sphere, it implies that M is either reducible or a spherical space form S^3/Γ, for some nontrivial Γ. We consider the former case first:

Theorem 2.1. *Suppose (M^3, g) is a closed reducible 3-manifold with scalar curvature $R_g \geq 6$. Then M contains a non-trivial closed geodesic of Morse index at most one and length at most $\frac{400\pi}{3}$.*

Proof. Since M is reducible there exists an embedded sphere S in M that does not bound a 3-ball. By [MSY82] we can minimize in the isotopy class of S to obtain a stable minimal 2-sphere Σ. By classical area and diameter estimate for stable minimal surfaces in positive scalar curvature, *e.g*, [MN12, Proposition A.1] and [LM, Corollary 2.4], we have that Σ has area $\text{Area}(\Sigma) \leq \frac{4\pi}{3}$ and intrinsic diameter $\text{diam}(\Sigma) \leq \frac{2\pi}{3}$.

By [LNR15, Main Theorem A] there exists a sweepout of Σ by a 1-parameter family of closed curves $\{F(t)\}_{t\in[0,1]}$, $F : ([0,1], \partial[0,1]) \to (\Omega\Sigma, \Omega_0\Sigma) \subset (\Omega M, \Omega_0 M)$ of length at most

$$L(F(x)) \leq 200d \max\left\{1, \log\left(\sqrt{\frac{A}{d}}\right)\right\} \leq \frac{400\pi}{3}.$$

Here we use notation $L(\gamma)$ to denote the length of γ.

Since Σ does not bound a 3-ball it follows by Poincare Conjecture that Σ is not contractible in M and hence the family $\{F(x)\}$ is not contractible in $(\Omega M, \Omega_0 M)$. Thus, by [Bot82], it follows that there exists a closed geodesic of index less than or equal to 1 of length at most $\frac{400\pi}{3}$. □

If M is a spherical space form, then the problem reduces to the case when $M \cong S^3$, by mapping a closed geodesic from the universal cover.

3. Existence of Morse foliation by L-short spheres

In this section, we will show how to foliate a 3-sphere with 2-spheres that will have a special geometric property we call L-shortness. This property will be used in the next section to construct a two-parameter family of short closed curves.

Definition 3.1 (L-Shortness)**.** Let $\Sigma \subset M^3$ be an embedded surface. We will say that Σ is L-short in M if for every closed curve $\gamma : [0,1] \to \Sigma$ with $\gamma(0) = \gamma(1)$ there exists a smooth family of arcs $\{\alpha_t : [0,1] \to M\}_{t\in[0,1]}$ of length at most L, such that

- $\alpha_t(0) = \gamma(0)$;
- $\alpha_0 = \alpha_1$ is a constant curve;
- $\alpha_t(1) = \gamma(t)$.

Remark 3.2. Note that in the definition of L-shortness, the interiors of arcs α_t are not required to lie in Σ.

We will also need the following notion of tree-foliation, analogous to Definition 3.6 in [LM].

Definition 3.3. Let $U \subset M^3$ be a subset with ∂U a disjoint union of embedded spheres. A family of surfaces $\{\Sigma_x\}_{x\in T}$ is called a tree foliation of U if there exist a tree T and continuous functions $f : M \to T$ and $g : T \to \mathbb{R}$, such that

- $g \circ f$ is a Morse function and T is the corresponding Reeb graph (that is, $T = M/\sim$ with $x \sim y$ if x and y belong to the same connected component of a fiber of $g \circ f$) ;
- for each connected component Σ of ∂U we have that $\Sigma = f^{-1}(v)$ for a vertex v of degree 1;
- for each edge $E \subset T$ we have that $\{f^{-1}(t) = \Sigma_t\}_{t\in E^\circ}$ is a smooth family of embedded 2-spheres.

Observe that the following properties of a tree foliation follow from the definition:

(1) T has vertices of degree 1 and 3;
(2) pre-image of a vertex of degree 1 is a point or a connected component of ∂U;
(3) pre-image of a vertex v of degree 3 satsifes $f^{-1}(v) = S_1 \cup S_2$, where S_i is homeomorphic to a sphere, $S_1 \cap S_2 = \{p\}$ and $f^{-1}(v) \setminus p$ is smooth and embedded.
(4) For a vertex v of degree 3 and each edge $E \subset T$ as t converges to $v \in \partial E$ there is smooth graphical convergence of Σ_t to a subset of $f^{-1}(v)$ away from the singular point $p \in f^{-1}(v)$.

The main result of this section is the following theorem.

Theorem 3.4. *Let M^3 be a 3-sphere with scalar curvature $R \geq 6$. There exists a tree foliation $\{\Sigma_t\}$ of M, such that each Σ_t is L-short for an $L < 4500$.*

3.1. **Area and diameter control implies L-shortness.** Using the following result from [LNR15], we show that an area and diameter upper bound for a 2-sphere implies it must be L-short.

Theorem 3.5. *Suppose Σ is a Riemannian 2-sphere or a 2-disc of area $\mathrm{Area}(\Sigma) \leq A$ and diameter $\mathrm{diam}(\Sigma) \leq d$ and γ_0, γ_1 are two paths connecting points $p, q \in \Sigma$. Then there exists a homotopy γ_t from γ_0 to γ_1 with length $L(\gamma_t) \leq 2(L(\gamma_1) + L(\gamma_0)) + 200d \max\{1, \log(\frac{\sqrt{A}}{d})\}$.*

Proof. The result follows if both γ_0 and γ_1 can be homotoped to the same length minimizing geodesic between p and q through curves with the the desired length bound. Hence, without any loss of generality, we may assume that γ_1 is a length-minimizing geodesic.

Subdivide γ_0 into N small arcs $\{\alpha_i\}_{i=0}^N$ with endpoints $\partial\alpha_i = \{x_i, x_{i+1}\}$ of length less than $\varepsilon < \mathrm{convrad}(\Sigma)$. After a small perturbation we can assume that each α_i is a geodesic arc. Let τ_i denote a length minimizing geodesic from p to x_i. Consider a sequence of curves $c_i = \tau_i \cup \bigcup_{j\geq i} \alpha_j$. Since α_i, τ_i and τ_{i+1} are minimizing geodesics, they do not intersect except at the endpoints. Hence, they bound a disc D_i. By [LNR15, Theorem 1.2], there exists a contraction of $\tau_i \cup \alpha_i \cup \tau_{i+1}$ to a point through based loops with length increase bounded by $L(\tau_i \cup \alpha_i) + L(\tau_{i+1}) + 200d \max\{1, \log(\frac{\sqrt{A}}{d})\}$. By Lemma 4.3 there exists a path homotopy from $\tau_i \cup \alpha_i$ to τ_{i+1} through curves of length at most

$$\min\{L(\tau_i \cup \alpha_i), L(\tau_{i+1})\} + L(\tau_i \cup \alpha_i) + L(\tau_{i+1}) + 200d \max\left\{1, \log(\frac{\sqrt{A}}{d})\right\}.$$

It follows that we have a sequence of homotopies from c_i to c_{i+1}, starting with $\gamma_0 = c_0$ and ending with $c_N = \gamma_1$, satisfying the desired length bound. $\square$

As a consequence of Theorem 3.5 we have the following lemma:

Lemma 3.6. *Suppose Σ is a 2-sphere or a 2-disc in M with intrinsic diameter d and area A. Then for every $\varepsilon > 0$ Σ is L-short for $L \leq 4d + 200d \max\{1, \log(\frac{\sqrt{A}}{d})\} + \varepsilon$.*

Proof. Let γ be a closed curve in Σ and $p \in \gamma$. We will construct a family of arcs connecting p to the points of γ and satsifying the desired length bound. Subdivide γ into small arcs γ_i by a sequence of pointa $\{p = a_0, ..., a_N = p\}$ with $\partial\gamma_i = \{a_{i-1}, a_i\}$, such that each γ_i has length bounded by $\delta < \frac{\varepsilon}{10}$. Consider a sequence of geodesics $\{\alpha_i\}_{i=1}^{N-1}$ connecting p to a_i and that are length minimizing in Σ. Note we have $L(\alpha^i) \leq d$. By Theorem 3.5 there exists a homotopy α_t^i from α_{i-1} to $\gamma_i \cup \alpha_i$ through curves of length bounded by

$$\begin{aligned} L(\alpha_t^i) &\leq 2(L(\alpha_{i-1} \cup \gamma_i) + L(\alpha_i)) + 200d \max\left\{1, \log(\frac{\sqrt{A}}{d})\right\} \\ &\leq 4d + 2\delta + 200d \max\left\{1, \log(\frac{\sqrt{A}}{d})\right\} \end{aligned}$$

By adding a family of subarcs of γ_i connecting the endpoint a_{i-1} of α_t^i to the points of γ_i we obtain a family of curves connecting p to the points of γ_i that interpolated between α^{i-1} and α^i. Together these homotopies give us the desired homotopy α_t. $\square$

3.2. **Surgery and L-shortness.** To construct our desired foliation, we will need to perform certain cut-and-paste surgeries on 2-spheres with controlled area and diameter. Unfortunately, these operations do not necessarily preserve the bound on intrinsic diameter. However, the next proposition shows how L-shortness can be preserved in appropriate cases.

Proposition 3.7. *Let $\varepsilon < \operatorname{convrad}(M)$*[1]*. Suppose Σ_1 is an embedded sphere with $\operatorname{diam}(\Sigma_1) \leq l_0$ and $\operatorname{Area}(\Sigma_1) \leq l_0^2$. Let B denote a closed ball of radius ε. Suppose $\Sigma_2 \subset M$ is an embedded sphere, such that $S = \Sigma_2 \cap B$ satisfies the following:*

(1) $\Sigma_2 \setminus \Sigma_1 \subset S$;
(2) For each connected component D of S we have $\partial D \subset \partial D'$, where D' is a connected component of $\Sigma_1 \cap B$.

Then Σ_2 is L-short for $L \leq 204 l_0 + 3\varepsilon$.

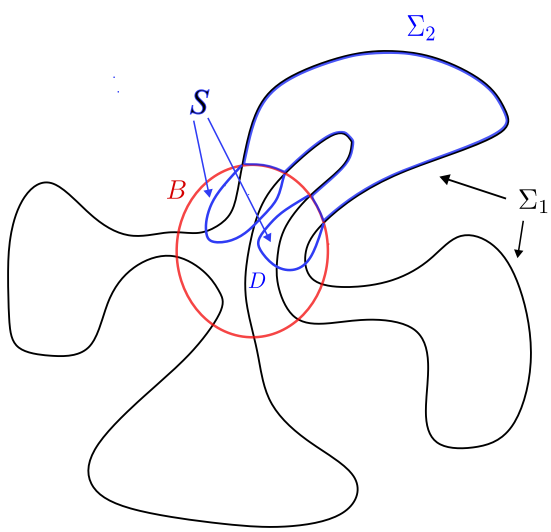

FIGURE 2. L-short Σ_2 after cut-and-paste surgeries inside B

Proof. We will prove a slightly stronger result then L-shortness. We will show that given any point $p \in \Sigma_2$ and a curve $\gamma : [0,1] \to \Sigma_2$, there exists a family of arcs α_t connecting p to $\gamma(t)$ and, moreover, if $\gamma(0) = \gamma(1) = \{p\}$, then we can take $\alpha_0 = \alpha_1 = \{p\}$. We will assume that $p \in \Sigma_1 \cap \Sigma_2$; the case $p \in \Sigma_2 \setminus \Sigma_1 \subset B$ is a straightforward modification of the argument below.

Let $\gamma \subset \Sigma_2$ be a path. Subdivide γ into segments γ_i of length $< \delta < \varepsilon$ and let $\{a_i\}_{i=0}^N$ denote the endpoints of $\{\gamma_i\}$, $\partial \gamma_i = \{a_i, a_{i+1}\}$, with $a_0 = \gamma(0), a_N = \gamma(1)$. Without any loss of generality we can

[1]Throughout the paper, $\operatorname{convrad}(M)$ denotes the convexity radius of M.

assume that γ intersects ∂S transversally and that $\gamma \cap \partial S \subset \{a_i\}$. We now construct a collection of arcs α_i from p to a_i. We consider the following two cases:

(1) a_i is contained in the interior of $\Sigma_2 \setminus S \subset \Sigma_1$; we connect a_i to p by a curve $\alpha_i \subset \Sigma_1$ of length $\leq \operatorname{diam}(\Sigma_1) \leq l_0$ in Σ_1;
(2) a_i is contained in a connected component D of S, see Figure 3. Let b_i be a point in $D \cap \partial B \in \Sigma_1$. We connect a_i to point b_i by a geodesic (in the ambient metric) $\tau_i \subset B$ of length at most ε; we connect b_i to p by a curve β_i of length $\leq \operatorname{diam}(\Sigma_1) \leq l_0$ in Σ_1; we set $\alpha_i = \tau_i \cup \beta_i$.

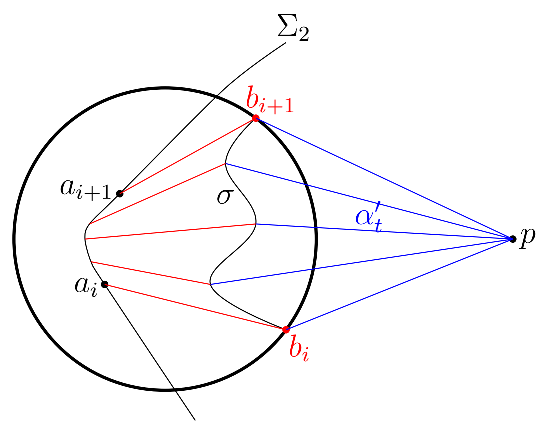

FIGURE 3. Case 2: a_i is contained in a connected component D of S

In case (2), if $a_i \in \partial B$, then we set $b_i = a_i$.

We now claim that there exists a homotopy of arcs with fixed endpoints p and a_{i+1} interpolating between $\alpha_i \cup \gamma_i$ and α_{i+1} through curves of length $\leq 204 l_0 + 2\delta$. Putting these deformations together we obtain a homotopy from α_0 to α_1 with desired properties.

If $a_i, a_{i+1} \in \Sigma_1$, then existence of desired homotopy follows from Theorem 3.5.

Suppose a_i, a_{i+1} lie in a connected component D of $S = \Sigma_2 \cap B$ and let D' denote a connected component of $\Sigma_1 \cap B$ with $\partial D \subset \partial D'$. Let $\sigma \subset D'$ be a curve connecting b_i to b_{i+1}. Note that we don't have a bound on the length of σ. By Lemma 3.6 there exists a family of arcs $\alpha'_t \subset \Sigma_1$ of controlled length interpolating between β_i and β_{i+1} with endpoints in σ. Since σ lies inside the ball B there exists a family of arcs connecting σ to α_{i+1} through curves of length $\leq 2\varepsilon$. This gives the desired family of arcs α_t. $\square$

Definition 3.8. Let $V \subset M$ be a domain with smooth boundary and $\Sigma \subset \partial V$ be an embedded 2-sphere. Let $\{D_i\}_{i=1}^k$ be a collection of disjoint embedded 2-discs in V, satisfying:

- $Int(D_i) \subset M \setminus \Sigma$;
- $\partial D \subset \Sigma$ is a disjoint union of embedded closed curves.

A family of embedded surfaces $\{\Sigma_t\}$ is called an ε-neck-pinching along $\{D_i\}$ if it is obtained from Σ by deforming the small cylinders $N_\varepsilon(\partial D_i) \cap \Sigma$ along D_i until they pinch.

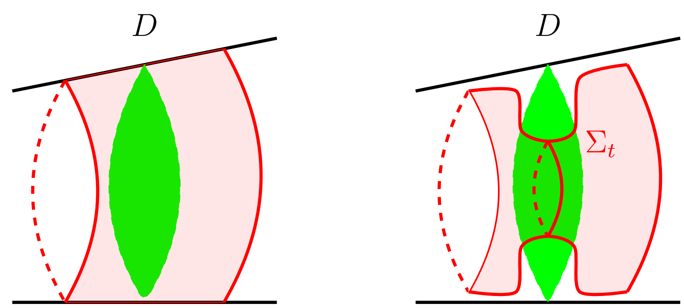

FIGURE 4. ε-neck-pinching along a disk D

More precisely, given an $\varepsilon > 0$, we realize ε-neck-pinching as a tree-foliation $\{\Sigma_t\}_{t \in T}$ of a region V', $N_{\varepsilon/2}(\Sigma \cup \bigcup D_i) \cap V \subset V' \subset N_\varepsilon(\Sigma \cup \bigcup D_i) \cap V$. The set of vertices of T can be decomposed $\mathcal{V} = \{v_j\}_{j=0}^{k+1} \cup \{w_l\}_{l=1}^k$, such that:

(1) Vertices $\{v_j\}_{j=0}^k$ have degree 1, $\Sigma_{v_0} = \Sigma$ and $\bigsqcup_{j=1}^{k+1} \Sigma_{v_j} = \partial V' \setminus \Sigma$;
(2) Vertices $\{w_l\}_{l=1}^k$ have degree 3.

Moreover, if Σ is minimal or mean convex, and D_i are minimal embeddings, then $\partial V' \setminus \Sigma$ is a collection of embedded spheres with mean curvature pointing inside $V \setminus V'$.

The next proposition shows how L-shortness behaves under ε-neck-pinching.

Proposition 3.9. *Suppose Σ is an L-short embedded sphere and $\{D_i\}_{i=1}^k$ a collection of disjoint embedded discs with $\partial D_i \subset \Sigma$, satisfying:*

(a) $\mathrm{L}(\partial D_i) < l_1$,
(b) $\mathrm{diam}(D_i) \le l_2$,
(c) $\mathrm{Area}(D_i) \le l_2^2$,

Suppose $\{\Sigma_t\}$ is an ε-neck-pinching family along $\{D_i\}$. Then Σ_t is L'-short for $L' \le L + l_1 + 204 l_2 + 4\varepsilon$.

Proof. Let γ be a curve on Σ_t. We will construct a family of short arcs α_t connecting points of γ to a point $p \in \gamma$.

Let $S_t = \Sigma \cap N_\varepsilon(\Sigma)$. We will fist suppose that $p \in S_t$. After a small perturbation, we may assume that γ intersects ∂S_t transversally and let A_t^i denote the connected components of $\Sigma_t \setminus S_t$ that lies in the small neighbourhoods $N_\varepsilon(D_i)$.

For each connected component $a_j \subset A_t^i$ of $\gamma \setminus S_t$, let b_j be a curve in $\Sigma \cap N_\varepsilon(\partial D_i)$ connecting the endpoints ∂a_j. It follows from Lemma 3.6 that there exists a continuous family of arcs α_s^j connecting $a_j(s)$ to $b_j(s)$, starting and ending on a constant map, and of length bounded by $l_1 + 204l_2 + 2\varepsilon$. (Even though a_j do not lie on the discs D_j, they belong to an ε-neighborhood of D_j, so, the result follows by projecting a_j onto D_j and connecting points of the projection a_j' to a_j by arcs of length $< \varepsilon$.)

Let $\gamma' \subset S'$ be a curve obtained from γ by replacing each a_j with b_j. Let γ'' denote the curve obtained by projecting γ' to Σ. Since Σ is L-short, there exists a family of arcs α_t' connecting p to $\alpha_t'(1) = \gamma''(t)$ of length bounded by $L + \varepsilon$. Adding a small arc to $\alpha_t'(1)$ in the tubular neighbourhood of Σ to connect it to $\gamma'(t)$ and then adding arcs α^j gives the desired family of arcs with length bounded by $l_1 + 204l_2 + 4\varepsilon$. $\square$

Remark 3.10. We will also need the following local version of Proposition 3.9: Given (M^3, g), let $r > 0$ be real number smaller than the convexity radius of M. Suppose Σ is an L-short 2-sphere and D is an embedded disk contained inside of a ball $B_r \cap \Sigma$. Then, there exists an ε-neck-pinching family $\{\Sigma_t\}$ along D such that Σ_t is L'-short for $L' \leq L + 2r$. The proof follows by a straightforward modification of the proof of Proposition 3.9.

3.3. **Discrete MCF with surgery.** We don't know how to control L-shortness of surfaces in Mean Curvature Flow. To overcome this issue we describe a combinatorial version of the flow. In this version, we repeatedly replace portions of the surface in a small ball by a certain minimizer in the isotopy class until the surface converges to a stable minimal surface.

We need the following slight modification of the notion of geometrically prime regions from [LM].

Definition 3.11. A 3-manifold V with non-empty boundary is geometrically prime if

(1) V is diffeomorphic to S^3 with some 3-balls removed;
(2) there are no closed embedded minimal surfaces in the interior of V;

(3) there exists a closed connected component Σ (that we'll call "large") of ∂V that is either mean convex or minimal of Morse index 1;
(4) $\partial V \setminus \Sigma$ is either empty or a disjoint union of stable minimal 2-spheres.

Definition 3.12. Let U be an open set with mean convex boundary and $\Sigma \subset cl(U)$ be a surface with $\partial\Sigma \subset \partial U$. By the Theorem of W. Meeks, L. Simon and S.T. Yau [MSY82] (see also [CDL03, Theorem 7.3]) there exists a minimal surface $S \subset U$ with $\partial S = \partial\Sigma$ obtained by minimizing area among surfaces isotopic to Σ with fixed boundary and possibly with some necks pinched. We call such S an isotopy minimizer for Σ in U. Let $\mathcal{A}(\Sigma, U)$ denote the area of an isotopy minimizer for Σ in U.

The next lemma shows how to foliate the region between a surface Σ and a new surface $\hat{\Sigma}$ obtained from Σ by replacing with an isotopy minimizer in a small ball.

Lemma 3.13. *Let U be a geometrically prime region in (M, g) and Σ its large connected component of ∂U. Let $r, \varepsilon \in (0, \mathrm{convrad}(M)/2]$, $\varepsilon < \frac{1}{100}$ and $p \in \Sigma$. Suppose $\partial B_r(p)$ interesects Σ transversally and let S be an isotopy minimizer for $\Sigma \cap B_r(p)$ in $U \cap B_r(p)$. Then there exist a mean convex surface $\hat{\Sigma}$ and subset V of U lying between Σ and $\hat{\Sigma}$, such that V admits a Morse function $g : V \longrightarrow [0,1]$ with the following properties:*

(1) $\Sigma = g^{-1}(0)$;
(2) $\hat{\Sigma} = g^{-1}(1)$, *and it satisfies the area estimate:*
$$\mathrm{Area}(\hat{\Sigma}) \leq \mathrm{Area}(\Sigma \setminus B_r(p)) + \mathcal{A}(\Sigma, B_r(p) \cap U);$$
(3) $g^{-1}(t) \setminus B_{r+2\varepsilon}(p)$ *is a family of graphical surfaces over Σ with $C^{2,\alpha}$-norm less than 2ε.*

Proof. Consider first the case where $\partial(\Sigma \cap B_r(p))$ is isotopic to the minimizer $S \cap B_r(p)$ through isotopies inside the ball $B_r(p)$ with fixed boundary, as shown in Figure 5.

Since $S \cup (\Sigma \setminus B_r(p))$ is a 2-sphere with corners (along $\partial B_r(p)$), Σ is mean convex, and $S \cap B_{r_0}(p)$ is minimal, we can flow $S \cup (\Sigma \setminus B_r(p))$ by mean curvature flow for a small time $t_0 > 0$, obtaining a family of mean convex surfaces Σ_t terminating at $\hat{\Sigma}$. Let $\hat{S}$ denote $\hat{\Sigma} \cap B_r(p)$. By interior estimates for mean curvature flow, *e.g.* [EH91][Theorem 4.2], if we choose $t_0 = t_0(\varepsilon)$ sufficiently small, $\hat{\Sigma}$ will satisfy a bound on its $C^{2,\alpha}$-norm as in property (3). Additionally, since $S \cup (\Sigma \setminus B_r(p))$ has

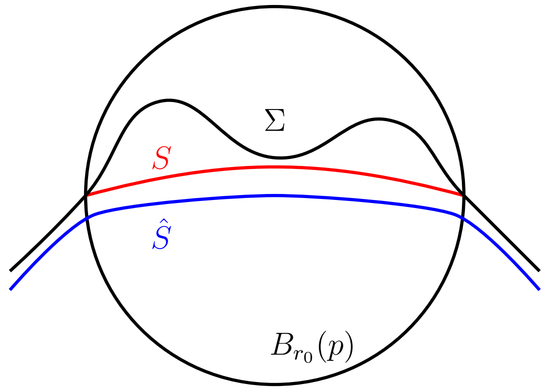

FIGURE 5. $S \cap B_r(p)$ and $\Sigma \cap B_r(p)$ are isotopic.

mean curvature pointing away from Σ, the monotonicity of area along the flow ensures that $\hat{\Sigma}$ satisfies the inequality in (2). The desired Morse function $g : V \longrightarrow [0,1]$ can then be constructed by noting that V is diffeomorphic to $\hat{\Sigma} \times [0,1]$, and one can patch a foliation of $V \cap B_r(p)$ with the foliation of $V \setminus B_r(p)$ by the surfaces Σ_t outside of $B_r(p)$.

By the Definition 3.12 of isotopy minimizers we have that the topology of S can defer from the topology of $\Sigma \cap B_{r_0}(p)$ only by finitely many neck-pinches. If follows that $n_c(S)$, the number of connected components of S, is greater than or equal to $n_c(\Sigma \cap B_{r_0}(p))$, the number of connected components of $\Sigma \cap B_{r_0}(p))$. Moreover, $n_c(S) = n_c(\Sigma \cap B_{r_0}(p))$ only if S and $\Sigma \cap B_{r_0}(p)$ are isotopic in $B_{r_0}(p)$. The proof now proceeds by induction on $k = n_c(S) - n_c(\Sigma \cap B_{r_0}(p))$.

Assume we have proved the Lemma for a fixed value of $k \geq 0$. Suppose the minimizer S had $n_c(\Sigma \cap B_{r_0}(p)) + k + 1$ connected components (Figure 6 illustrates the case $n_c(S) = 2$, $n_c(\Sigma \cap B_{r_0}(p)))$.

By Definition 3.12 there exists a disc $D' \subset U$ with $\gamma = \partial D' \subset \Sigma \cap B_r(p)$ with the property that D' separates two connected components of S. By Meeks-Yau [MY82] we can minimize in the class of embedded discs that are isotopic to D' in $U \cap B_r(p) \setminus S$ with fixed boundary γ to obtain an embedded disc D.

By Remark 3.10, disc D can be used to deform Σ by a neck-pinching family, which is L'-short, with $L' \leq L + 2r$. After the deformation, we reduce the number of boundary components of Σ by one. We can now apply the inductive assumption to the new boundary and region we obtained. $\square$

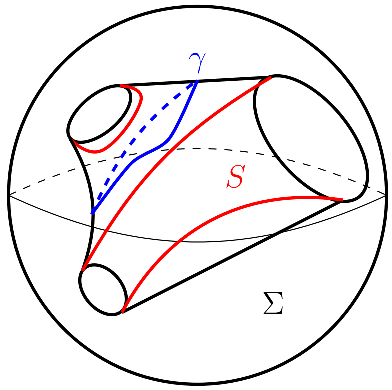

FIGURE 6. S and Σ are not smoothly isotopic.

Remark 3.14. By Remark 3.10, if in addition to the hypothesis of Lemma 3.13 one assumes the surface Σ is L-short, then for all ε sufficiently small we can conclude that surfaces $g^{-1}(t)$ will be L'-short, for an $L' \leq L + 3r$.

Given a geometrically prime region N, let Σ_i denote the stable minimal boundary components of N. For $A > 0$, we consider the set of stationary integral varifolds of the form:

$$\mathcal{V}_{N,A} = \left\{ V = \sum_j a_{i_j} \Sigma_{i_j},\ a_{i_j} \in \mathbb{N},\ ||V||(N) \leq A \right\}.$$

Since the coefficients $a_{i_j} \geq 0$, the varifold measure of a given $V = \sum_j a_{i_j} \Sigma_{i_j}$ is given by:

$$||V||(N) = \sum_j a_{i_j} \operatorname{Area}(\Sigma_{i_j}),$$

and, so, for every $A > 0$, the set $\mathcal{V}_{N,A}$ is finite.

The next Proposition shows that, in a geometrically prime region, using isotopies and neck-pinches in a small ball of fixed radius, one can always decrease the area on an embedded sphere Σ by a definite amount unless Σ, as a varifold, lies in a small neighborhood of $\mathcal{V}_{N,A}$. We will use the $\mathbf{F}$ distance function of varifolds from Pitts [Pit81], a metrization of varifold weak topology.

Proposition 3.15. *Let $N \subset M$ be a geometrically prime region. Given $A > 0$ and $\eta_0 > 0$, there exist $r_0 \in (0, \operatorname{convrad}(M)/2)$ and $\varepsilon_0 > 0$ with the following properties. Let Σ be an embeddded sphere and U connected component of $N \setminus \Sigma$, such that*

- *∂U is mean convex;*

- Area$(\Sigma) \leq A$;
- $\mathbf{F}(\Sigma, \mathcal{V}_{N,A}) > \eta_0$.

Then there exists a ball $B_{r_0}(p)$*, such that* $\text{Area}(\Sigma \cap B_{r_0}(p)) - \mathcal{A}(\Sigma \cap B_{r_0}(p), U \cap B_{r_0}(p)) > \varepsilon_0$.

Proof. We argue by contradiction. Suppose the result does not hold. Then there exists a sequence $\Sigma_i \subset U_i$, such that

$$\text{Area}(\Sigma_i \cap B_{r_0}(p)) - \mathcal{A}(\Sigma \cap B_{r_0}(p), U_i \cap B_{r_0}(p)) \leq 2^{-i} \tag{1}$$

for all $p \in N$.

Since the areas of Σ_i are bounded by A there exists a subsequence of $\{|\Sigma_i|\}$ that converges to a varifold V. We claim that V is a stationary varifold. Indeed, otherwise, there exists a vector field W supported in $B_{r_0}(p) \cap U_i$ and an isotopy $F(\cdot, t)$ of N generated by W (the isotopy is equal to identity outside $B_{r_0}(p) \cap U_i$) that reduces areas of Σ_i for all sufficiently large i. Observe that the condition (1) implies that after taking a subsequence we may assume that $\{\Sigma_j\}$ is $1/j$-a.m. in the sense of [CDL03, Definition 3.2]. By [CDL03, Theorem 7.1] the support of V is a smooth minimal surface. □

We will also need a lemma that allows us to foliate regions bounded by spheres of very small area by L-short spheres.

Lemma 3.16. *Let M be a Riemannian 3-manifold and $\delta > 0$. There exists $\varepsilon(M, \delta) > 0$ with the following property. Let $U \subset M$ be a subset of M with $Vol(U) < Vol(M \setminus U)$ and $\partial U = \Sigma$ an L-short mean convex 2-sphere with* $\text{Area}(\Sigma) < \varepsilon$.

Then there exists a tree-foliation $\{\Sigma_t\}_{t \in T}$ of U, starting with $\Sigma_{t_0} = \Sigma$, by L'-short surfaces for $L' \leq L + \delta$.

Proof. Fix $r \in (0, \frac{\delta}{2})$, so that r is smaller than the convexity radius of M and every ball of radius r is 2-bilipschitz diffeomorphic to a Euclidean ball of the same radius. Suppose $\varepsilon > 0$ is such that $\frac{\varepsilon}{r} < \frac{r}{100}$.

Let $K = \lfloor \frac{2\,\text{diam}(M)}{r} \rfloor$. Let $p \in \Sigma$. Fix a point $p \in \Sigma$. By coarea inequality we can find balls $\{B_{r_i}(p)\}_{i=1}^K$ with $r_i \in [ir/5, (i+1)r/5]$ and

$$\text{L}(\partial B_{r_i}(p) \cap \Sigma) \leq \frac{r}{20}.$$

Let γ denote a connected component of $\Sigma \cap \partial B_{r_i}(p)$. From the length estimate we have that γ is contained in a ball $B_{r/40}(q)$ for some $q \in \gamma$. By [MY82] there exists an embedded disc $D_\gamma \subset U$ that minimizes area among all discs in U with boundary γ. It follows from the isoperimetric inequality and monotonicity formula for minimal surfaces that, assuming ε is sufficiently small, we have

$$D_\gamma \subset U \cap B_{r/4}(q) \tag{2}$$

Consider a connected component C_k of $\Sigma \setminus \bigcup_i \partial B(p, r_i)$ that does not contain p. Since Σ is a 2-sphere there will be exactly one connected component of ∂C_k (let's call it γ_1) that is closer to p than all other connected components of ∂C_k. By construction, we have that for every point $x \in C_k$ we have

$$dist_M(x, \gamma_1) < \frac{2r}{5} \tag{3}$$

Let $\Sigma_k = C_k \cup \bigcup_j D_{\gamma_j}$, where γ_j denotes the connected components of ∂C_k. From (2) we have that each D_{γ_j} is contained in a ball of radius $\leq \frac{r}{4}$ centered around a point on ∂D_{γ_j}. Combining this with (3) we obtain that $\Sigma_k \subset B_r(p')$ for some $p' \in \gamma_1$. Since $r > 0$ is less than the convexity radius of M any surface contained in $B_r(p')$ is L-short for $L < 2r$. Let V_k denote the region bounded by Σ_k in $B_r(p')$. By Alexander's theorem there exists a diffeomorphism $\Phi : B(1) \to V_k$, where $B(1)$ denotes the unit ball in $\mathbb{R}^3$. Then

$$S_t = \Phi(\{(x, y, x) : x^2 + y^2 + z^2 = (1-t)^2\})$$

defines a foliation of V_k by L-short 2-spheres.

By Remark 3.10, there exists a neck-pinching family starting on Σ along discs $\{D_\gamma\}$ that is L'-short for $L' \leq L + \delta$. $\square$

3.4. **Proof of Theorem 3.4.** By approximation, it is enough to prove the result for generic metrics (in the sense of Baire Category). Thus, by a result of White [Whi91], we may assume the metric g is such that all of its closed embedded minimal surfaces have no nontrivial Jacobi fields. As in the proof of [LM, Theorem 2.7], we decompose M as a union of geometrically prime regions $\{M_i\}$ with disjoint interiors, and since M is diffeomorphic to the 3-sphere, for each i, the large boundary component S_i of ∂M_i is a minimal 2-sphere of Morse index 1 and all other boundary components are stable minimal 2-spheres. Since $R \geq 6$, by classical diameter and area estimates for minimal surfaces (cf., [LM, Theorem 2.1, Corollary 2.4]), the large boundary components S_i have area at most 4π and diameter at most $\frac{4\pi}{3}$, and all other components have area at most $\frac{2\pi}{3}$ and diameter $\frac{2\pi}{3}$.

We will define a tree-foliation of each M_i by L-short spheres for an $L < 4500$. It is clear that this implies existence of an L-short tree-foliation of M.

Now we describe the construction of the tree-foliation for M_i. We start by using the first eigenfunction of the stability operator, as in the proof of [MN12, Lemma 3.2] to deform the large boundary component S_i to a surface Σ^0 in the interior of M_i via an area-decreasing deformation in the tubular neighbourhood of S_i. Note that Σ^0 will have mean curvature pointing away from S_i at every point. Using Proposition 3.15, we then define a sequence of area-decreasing deformations and surgeries. At the end of each step, we obtain a (possibly disconnected) surface $\Sigma^k = \bigsqcup_j \Sigma^k_j$ of area

$$\mathrm{Area}(\Sigma^k) \leq \mathrm{Area}(\Sigma^0) - k\varepsilon_0.$$

with mean curvature pointing away from S_i (we allow for number of connected components of Σ^k to possibly increase with k).

The construction proceeds by induction on k. Let $l_0 = 14.5$ and $\eta_0 > 0$ be a small constant depending on M_i to be specified later. Let $r_0 \in (0, \frac{\mathrm{convrad}(M)}{2})$ and ε_0 be constants depending on η_0 that satisfy the conclusions of Proposition 3.15. Without any loss of generality can assume $r_0, \varepsilon_0 < 0.1$. We suppose that we have constructed a tree foliation of $U_{k-1} \subset M_i$, such that the following holds:

(*i*) $M_i \backslash U_{k-1} = \bigsqcup V_j$, where each V_j is a geometrically prime region;
(*ii*) the large boundary component Σ^{k-1}_j of V_j satisfies $\mathrm{diam}(\Sigma^{k-1}_j) \leq l_0$ and $\mathrm{Area}(\Sigma^{k-1}_j) \leq \mathrm{Area}(\Sigma_0) - (k-1)\varepsilon_0$;
(*iii*) Σ^{k-1}_j is at a distance greater than η_0 in $\mathbf{F}$ varifold distance from every stable minimal sphere in ∂M_i.

The new surface Σ^k will be obtained by applying the following procedure to each connected component Σ^{k-1}_j of Σ^{k-1}. Since $\mathrm{diam}(\Sigma^{k-1}_j) \leq l_0$ and, by hypothesis (*ii*),

$$\mathrm{Area}(\Sigma^{k-1}_j) \leq \mathrm{Area}(\Sigma_0) - (k-1)\varepsilon_0 < \mathrm{Area}(\Sigma_0) \leq 4\pi < l_0^2,$$

Lemma 3.6 gives that Σ^{k-1}_j is L-short for $L \leq 2958$. Using hypothesis (*iii*), we apply, for each j, Proposition 3.15 to find a ball $B_{r_0}(p)$ and surface $S \subset V_j$, so that $S \setminus B_{r_0}(p) = \Sigma^{k-1}_j \setminus B_{r_0}(p)$ and $S \cap B_{r_0}(p)$ is an isotopy minimizer for $\Sigma^{k-1}_j \cap B_{r_0}(p)$ in $B_{r_0}(p)$, satisfying

$$\mathrm{Area}(S \cap B_{r_0}(p)) < \mathrm{Area}(\Sigma^{k-1}_j \cap B_{r_0}(p)) - \varepsilon_0.$$

By Lemma 3.13, there exists a surface $\hat{\Sigma}^{k-1}_j$ such that

$$\mathrm{Area}(\hat{\Sigma}^{k-1}_j) \leq \mathrm{Area}(\Sigma^{k-1}_j \setminus B_r(p)) + \mathrm{Area}(S \cap B_{r_0}(p)) \leq \mathrm{Area}(\Sigma^0) - k\varepsilon_0.$$

Since each Σ^{k-1}_j is L-short with $L = 2958$, it follows from Remark 3.14 that the Morse foliation described in Lemma 3.13, which spans the region between $\hat{\Sigma}^{k-1}_j$ and Σ^{k-1}_j, has leaves that are L-short for $L = 2958 + 3r_0$.

Now we come to the key difficulty in the proof: the operation of replacing a part of the surface with an isotopy minimizer inside a small ball (which is described in Proposition 3.15 and Lemma 3.13) may potentially increase the intrinsic diameter of the surface in a way we cannot control. We introduced the notion of L-short surfaces to deal specifically with this problem.

For each connected component $\hat{S}$ of $\hat{\Sigma}_j^{k-1}$, we now consider two possibilities: if the intrinsic diameter d of $\hat{S}$ satisfies $d \leq l_0 = 14.5$, then $\hat{S}$ bounds a new geometrically prime region, which we label $\Sigma_{j'}^k$, and it satisfies the hypotheses of the inductive step. If $d > 14.5$, we pick points p_0 and q_0 at a distance d in $\hat{S}$ and consider boundaries of geodesic balls $\partial B_{p_0}^{\hat{S}}(r) = \partial\{dist_{\hat{S}}(x,p) \leq r : x \in \hat{S}\}$, $r \in (0,d]$. For each integer $l = 1, \ldots, \lceil d \rceil$, consider the region $A_l = B_{p_0}^{\hat{S}}(l) \setminus B_{p_0}^{\hat{S}}(l-1)$. Since $\text{Area}(\hat{S}) < 4\pi$, by the coarea formula, for each l, there exists a radius $r_l \in (l-1, l)$, such that $\partial B_{p_0}^{\hat{S}}(r_l)$ is a finite collection of embedded smooth curves of total length at most 2π. Observe that $\partial B_{p_0}^{\hat{S}}(r_l)$ may not be connected, and thus we denote its connected components as $\gamma_{l,1}, \gamma_{l,2}, \ldots, \gamma_{l,k_l}$. This way, each connected component of $\hat{S} \setminus \bigcup_{l,i} \gamma_{l,i}$ has area at most 4π and boundary curves of length at most 2π. For such component C, we glue in area-minimizing disks D_m along the boundary curves of C, and perform neck-pinching isotopies along each D_m. By the filling radius estimate, each disk D_m has diameter at most

$$\frac{2\pi}{3} + \frac{\mathrm{L}(\partial D_m)}{2} + \frac{2\pi}{3} \leq 7\pi/3$$

and since the area of $\hat{S}$ is at most 4π, we have $\text{Area}(D_m) \leq \frac{\text{Area}(\hat{S})}{2} \leq 2\pi$. Thus, by Proposition 3.9, the surfaces obtained in the process of neck-pinching will all be L-short for

$$L \leq 2958 + 3r_0 + 2\pi + 204\frac{7\pi}{3} + 4\varepsilon_0 < 4500$$

After pinching, each connected component E is obtained by a small area-decreasing deformation of $C \cup \bigcup_{m \in N_C} D_m$, where N_C denotes the set of indices with the property $\partial D_m \subset \partial C$ for all $m \in N_C$, which is a sphere of area at most $\text{Area}(\Sigma_0) - k\varepsilon_0 < 4\pi$ and diameter $d < 14.5$. First we explain why the area bounds hold. Let $\tilde{D}_m \subset \hat{S}$ denote a disc in $\hat{S}$ with $\partial \tilde{D}_m = \partial D_m$ and such that the interior of C is disjoint from the interior of $\tilde{D}_m$. Since D_m is area minimizing in the class of discs isotopic to $\tilde{D}_m$ with fixed boundary we have

$$\mathrm{Area}(E) \leq \mathrm{Area}(C \cup \bigcup_{m \in N_C} \tilde{D}_m) = \mathrm{Area}(\hat{S}) \leq \mathrm{Area}(\Sigma_0) - k\varepsilon_0 < 4\pi$$

Now we explain the diameter bound. Let $p_1, p_2 \in E$. Observe that since S is a sphere there exists a unique connected component of ∂C that is closer to p than all other connected components γ of ∂C. We have $dist_{\hat{S}}(p_i, \gamma) < \frac{2\pi}{3} + 2$, and thus:

$$dist(p_1, p_2) < 2\left(\frac{2\pi}{3} + 4\right) + \frac{1}{2}\mathrm{L}(\gamma) < 14.5.$$

After doing this on each connected component Σ_j^{k-1} of Σ^{k-1}, and taking the union of new $\Sigma_{j'}^k$, we obtain the surface Σ^k.

Since for each Σ_j^k we have $\mathrm{Area}(\Sigma_j^k) \leq \mathrm{Area}(\Sigma_0) - k\varepsilon_0$, it is clear that the inductive process cannot go on forever. It stops when the large components of the geometrically prime regions still to be foliated do not satisfy the hypothesis (iii), that is, its $\mathbf{F}$-distance to the stable components of the boundary must be small. This implies that Σ_j^k is η_0-close in the varifold sense to a union Γ of some of the stable minimal boundary components of M_i (with some integer multiplicities). Since varifold convergence does not imply Hausdorff convergence, we argue as follows. Recall, both Σ_j^k and each connected component of Γ have bounds on area and diameter and thus are L-short for $L = 2958$. We consider equidistant surfaces $\Gamma_\delta = \partial N_\delta(\Gamma)$, $\delta > 0$, to Γ and their intersections with Σ_k. Given $\delta \in (0, r_0)$ small and $\varepsilon_1 < \frac{\delta^2}{100}$, we may choose $\eta_0 = \eta_0(\delta, \varepsilon_1)$ such that Σ_j^k being η_0-close to Γ in varifold sense implies that $\Sigma_j^k \setminus N_\delta(\Gamma)$ has area less than ε_1. Using the coarea inequality we can find $h \in (\delta, 2\delta)$, such that the intersection $\Sigma_j^k \cap \partial N_h(\Gamma)$ has length less than $\frac{\delta}{10}$. By the isoperimetric inequality each connected component of $\Sigma_j^k \cap \partial N_h(\Gamma)$ bounds a small area disc inside $\partial N_h(\Gamma)$. Starting with the innermost boundary curve we can apply Remark 3.10 to define neck-pinching families along these small discs. In the end, we obtain a tree foliation that starts on Σ_j^k and deforms it to a disjoint union of surfaces $\Sigma^L \cup \Sigma^S$, where $\Sigma^L \subset N_{2\delta}(\Gamma)$ and Σ^S satisfies $\mathrm{Area}(\Sigma^S) < \varepsilon_1$.

We apply Lemma 3.16 to foliated small regions bounded by connected components of Σ^S. Finally, we can foliate regions between Σ^L and Γ in an arbitrary way. All surfaces in the foliation will be $L + 2\delta$-short for $L < 4500$ since they are all contained in the δ-tubular neighborhood of a connected component of Γ, which is L-short.

4. Parametric sweepout of L-short spheres

Let M be a Riemannian 3-sphere with scalar curvature bounded below by 6. Let ΩM denote the space of closed, piecewise differentiable curves on M, and $\Omega_x M$, the subspace of closed curves of length at most x. (Note that $\Omega_0 M$ will then denote the space of constant curves, which can be identified with M itself.)

In this section we will construct a sweepout of M by "short" curves, *i.e.*, a map $f : D^2 \longrightarrow \Omega_x M$ where ∂D^2 is mapped to $\Omega_0 M$ and that is not contractible over 2-disks with fixed boundary. Here $x = 10^5$, which will immediately imply the proof of Theorem B.

Theorem 4.1. *Suppose $M = (S^3, g)$ is a Riemannien 3-sphere with scalar curvature ≥ 6. There exists a noncontractible family*

$$f : (D^2, \partial D^2) \to (\Omega_x M, \Omega_0 M),$$

where $x = 22500$.

To prove Theorem 4.1. we will fist define a family $F : (K, \partial K) \to (\Omega M, \Omega_0 M)$, where K is a 2-complex homeomorphic to a disc D^2, of potentially very long closed curves, but with the property that each curve $F(x)$ lies on an L-short 2-sphere. The construction of F proceeds as follows. By Theorem 3.4 there exists a tree-foliation parametrized by a tree T, $\{\Sigma_x\}_{x\in T}$, and each Σ_x is L-short with $L < 4500$.

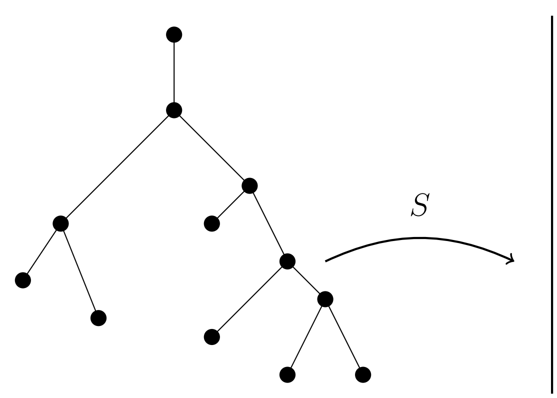

Figure 7. $S : T \longrightarrow \mathbb{R}$

Recall that a tree foliation $\{\Sigma_x\}_{x\in T}$ comes with maps $f : M \to T$ and $S : T \longrightarrow \mathbb{R}$, such that $\Sigma_x = f^{-1}(x)$ and $S \circ f$ is a Morse function.

We will next construct a CW complex K, corresponding to T in the following way. First for each edge e_i of T we will construct a rectangle $R_i = e_i \times [0, 1]$. We will now glue the rectangles that correspond to the neighboring edges of T in the following way. Consider an edge e_i and

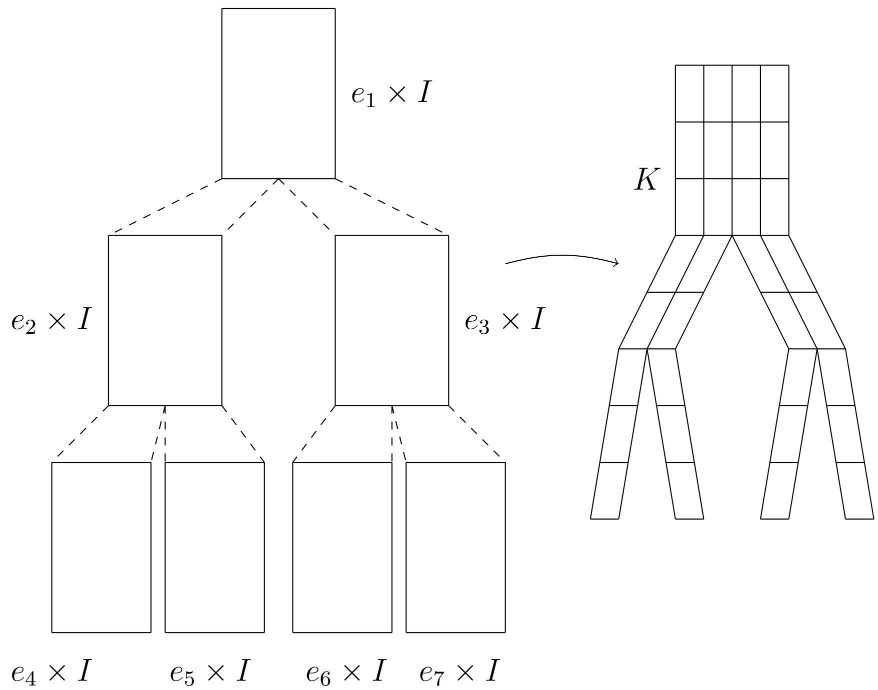

FIGURE 8. Constructing K

the two edges e_j and e_k situated directly below e_i. We will identify the lower edge of R_i with the upper edges of R_j and R_k via positive linear maps $p_j : [0,1] \longrightarrow [0, \frac{1}{2}]$, $p_2 : [0,1] \longrightarrow [\frac{1}{2}, 1]$. The resulting complex will be a disk, (see Fig. 8).

Let us now describe a possibly long sweepout $F : K \longrightarrow \Omega M$ by closed curves. The boundary of K will be mapped to constant curves, while for a given rectangle $e_i \times I$ in K and each horizontal segment $t \times I$, family $F(x), x \in (t, I)$, will be generated by a sweepout of a sphere Σ_t, which will vary continuously with the sphere.

We describe how this family is constructed in the neighbourhood of the bottom side of a rectangle $e_i \times I$, where it is glued to the top sides of rectangles $e_j \times I$ and $e_k \times I$. Let v denote the vertex of T where edges e_i, e_j, e_k meet. Relabeling if necessary we may assume that $\{\Sigma\}_{t \in e_i}$ undergoes a neck-pinch singularity at v. Let p denote the singular point of Σ_v. By the definition of tree foliation for $t \in e_i$ with $t \to v$ there exists a family of simple closed curves $\gamma_t \subset \Sigma_t$ converging to p with $L(\gamma_t) \to 0$. Let D_t^1 and D_t^2 denote the two connected components of $\Sigma_t \setminus \gamma_t$. We fix two continuous families of diffeomorphisms $\Phi_t^i : D(1) \to D_t^i$, $i = 1, 2$, where $D(1)$ denotes the unit disc in $\mathbb{R}^2$, with $\Phi_t^1|_{\partial D(1)} = \Phi_t^2|_{\partial D(1)}$. We define $F(t,s) = \Phi_t^1(\{x^2 + y^2 = (2s)^2\})$ for

$s \leq \frac{1}{2}$ and $F(t,s) = \Phi_t^2(\{x^2+y^2 = (2-2s)^2\})$ for $s > \frac{1}{2}$. Note that $F(t,\frac{1}{2}) = \gamma_t \to p$ as $e_i \ni t \to v$. For $t \in e_j$ very close to v we can continuously extend family $\{F(v,s)\}_{s\in[0,\frac{1}{2}]}$ to a sweepout of Σ_t; and for $t \in e_k$ very close to v we can continuously extend family $\{F(v,s)\}_{s\in[\frac{1}{2},1]}$ to a sweepout of Σ_t. The construction is illustrated in Fig. 9.

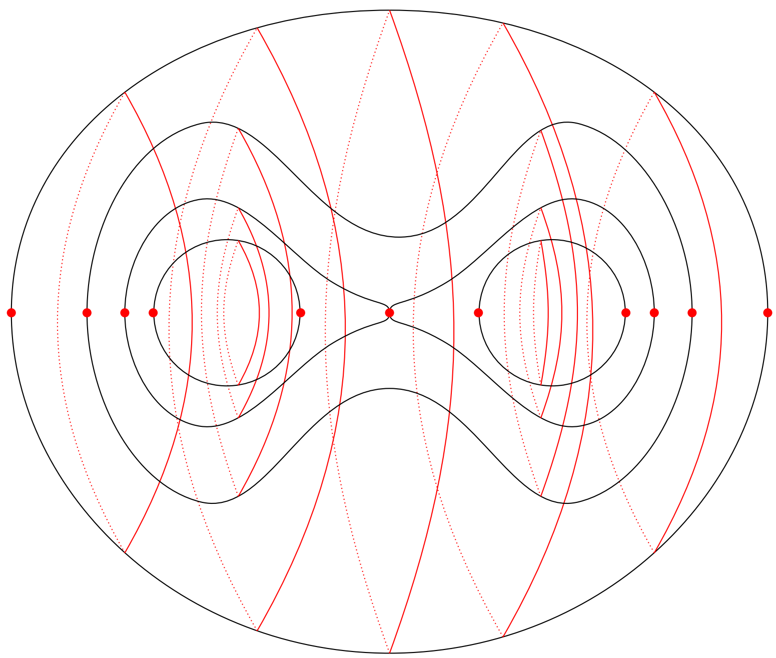

FIGURE 9. Sweepout of a sphere

Once we defined $F(x)$ for all $x \in U_v \times I$, where U_v denotes a neighbourhood of a vertex $v \in T$, we can extend to the rest of K, since for each edge e_i this amounts to interpolating between two homotopic families in $(\Omega\, M, \Omega_0\, M)$.

This finishes the construction of homotopically non-trivial map $F : (K, \partial K) \to (\Omega M, \Omega_0 M)$, where each curve $F(x)$ lies on an L-short sphere.

We will now show that this family can be homotopically deformed to a family of short closed curves. Let $\Pi_x M$ denote the space piecewise smooth maps from $[0,1]$ to M of length $\leq x$. Here and below $R(\delta)$ will denote any function of δ that tends to 0 as δ tends to 0.

Proposition 4.2. *Suppose $F : (K, \partial K) \to (\Omega M, \Omega_0 M)$ is a family of closed curves, such that each $F(x)$ lies on an L-short sphere.*

For every $\delta > 0$ there exists a map $H : (K \times [0,1], \partial K \times [0,1]) \to (\Pi_{5L+R(\delta)}M, \Pi_0 M)$ with the following properties:

(1) $H(x,0)$ is a constant map with image $F(x)(0)$;
(2) $H(x,t)(0) = F(x)(0)$ and $H(x,t)(1) = F(x)(t)$.

In other words, for fixed x, $H(x,t)$ is a 1-parameter family of arcs connecting $F(x)(0)$ to all other points of $F(x)$. $\{H(x,1)\}_{x\in K}$ is then a family of short closed curves.

Proof of Theorems 4.1 and Theorem B from Proposition 4.2. Let $F : (K, \partial K) \to (\Omega M, \Omega_0 M)$ be the non-contractible family of closed curves constructed above and let $H(x,t)$ be the corresponding family of arcs from Proposition 4.1. Since $H(x,1)(0) = H(x,1)(1)$, identifying the endpoints of $[0,1]$ we obtain a map $\tilde{F} : (K, \partial K) \to (\Omega M, \Omega_0 M)$, where $\tilde{F}(x)$ has the same image and length as $H(x,1)$. In particular, the length of $\tilde{F}(x)$ is bounded by $5L + R(\delta) \leq 22500$ for all $x \in K$.

We have that F is homotopic to $\tilde{F}$ with the homotopy given by the family of closed curves

$$h(x,t) = H(x,t) * F(x)|_{[t,1]}$$

Here $*$ denotes concatenation of two paths. This proves Theorem 4.1.

It follows from the Morse theory on the space of closed curves [Bot82] that there exists a closed geodesic γ in M of length less than $22500 + R(\delta)$. Choosing δ sufficiently small we obtain a bound of 22500. $\square$

In the rest of this section we prove Proposition 4.2.

The proof will follow from a construction that first appeared in [NR], where it was done in the case of based point loops or paths spaces on M. Here we modify this construction for the free loop space. The result will follow from the sequence of lemmas below.

Lemma 4.3. *Let us consider a digon formed by paths e_1, e_2 connecting points p_1, p_2 of lengths l_1, l_2 respectively, (see fig. 10 (a)). Suppose loop $\alpha = e_1 * \bar{e}_2$ based at p_1, (see fig. 10 (b)) can be contracted over loops α_τ based at p_1 of length at most l_3, (see fig. 10 (c)). Then e_1 is path-homotopic to e_2 over curves of length at most* $\min\{l_1, l_2\} + l_3$.

Moreover, a parametric version of this Lemma holds:

Suppose we are given a manifold X and families of paths $\{e_1^x\}_{x\in X}$, $\{e_2^x\}_{x\in X}$ connecting points $p_1(x), p_2(x)$ of lengths $l_1(x), l_2(x)$ respectively, and suppose there exists a family of contractions of loops $\alpha(x) = e_1^x * \bar{e}_2^x$ based at $p_1(x)$, over loops based at $p_1(x)$ of length at most $l_3(x)$. Then there exists a family of path homotopies of e_1^x to e_2^x over curves of length at most $\min_{i=1,2} \max_{x\in X}(l_i(x) + l_3(x))$.

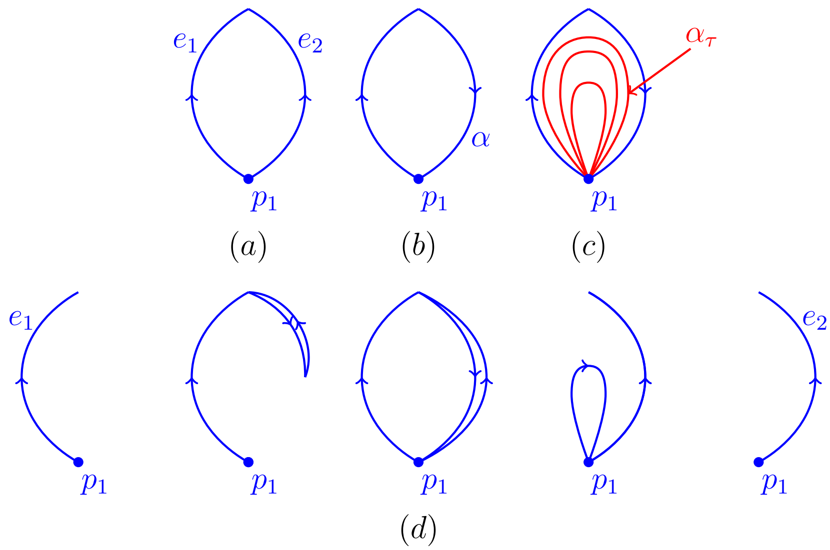

FIGURE 10. Constructing path homotopy

Proof. Assume we are in the non-parametric case first. Consider the following path-homotopy:

$$e_1 \sim e_1 * \bar{e}_2 * e_2 \sim \alpha_t * e_2 \sim e_2$$

(see fig. 10 (d)). Note that the length of curves in this homotopy is at most $l_2 + l_3$. Similarly, one can construct a homotopy of length at most $l_1 + l_3$.

Applying the same construction to parametric families we obtain the desired families of path homotopies. $\square$

Lemma 4.4. *Consider a quadrilateral formed by two pairs of opposite sides: $e_i = e_i(t)$ and $\sigma_i = \sigma_i(s)$, $i = 0, 1$, (see fig. 11 (a)). Let $l_i = length(e_i)$ and $m_i = length(\sigma_i)$. Consider a loop α based at $\sigma_0(\frac{1}{2}) = p$ corresponding to the concatenation $\bar{\sigma}_0|_{[\frac{1}{2},1]} * e_0 * \sigma_1 * \bar{e}_1 * \bar{\sigma}_0|_{[0,\frac{1}{2}]}$. (Here and below $\bar{\beta}$ will denote the curve β travelled in the opposite direction.) Suppose α is contractible to p along loops α_τ based at p of length at most l_3. Then there exists a one-parameter family of curves γ_s connecting $\sigma_0(s)$ with a corresponding $\sigma_1(s)$ of length at most $\min\{l_1, l_2\} + l_3 + 2(m_1 + m_2)$, such that $\gamma_0 = e_0$ and $\gamma_1 = e_1$.*

Moreover, the corresponding parametric version of this Lemma holds.

Proof. First note that $b_0 = \bar{\sigma}_0|_{[\frac{1}{2},1]} * e_0 * \sigma_1|_{[0,\frac{1}{2}]}$ is path-homotopic to $b_1 = \sigma_0|_{[\frac{1}{2},1]} * e_1 * \bar{\sigma}_1|_{[0,\frac{1}{2}]}$ by Lemma 4.3 over the curves of length at most $\min\{l_1, l_2\} + l_3 + 2(m_1 + m_2)$. One can, for example, consider the following path-homotopy: $b_0 \sim b_0 * \bar{b}_1 * b_1 \sim \alpha_\tau * b_1 \sim b_1$. Parametrize the curves in this path-homotopy by x to obtain a 1-parameter family of curves b_x.

Now, for each $s \in [0,1]$ let us connect the "opposite" points, $\sigma_0(s)$ with $\sigma_1(s)$ as follows. For $s \in [0, \frac{1}{2}]$ One can connect $\sigma_0(s)$ with $\sigma_1(s)$ by the curves $\bar{\sigma}_0|_{[s,0]} * e_0 * \sigma_1|_{[0,s]}$, while for $s \in [\frac{1}{2}, 1]$ (see fig. 11 (b)-(d)). We can connect the opposite points by curves $\sigma_0|_{[s,1]} * e_1 * \bar{\sigma}_1|_{[1,s]}$, (see fig. 11 (f)-(h)). Note that this will result in a possible discontinuity at $s = \frac{1}{2}$. However, we will fill this discontinuity by the family of curves defined in the paragraph above, (see fig. 11 (e)). Thus, the curves connecting $\sigma_0(\frac{1}{2})$ and $\sigma_1(\frac{1}{2})$ will not be unique and we can take $s = f(r)$, $r \in [0,1]$ for some non decreasing function $f(r)$.

Figure 11 demonstrates a family γ_s continuously connecting the "opposite" points of σ_1, σ_2 starting with e_0 and ending with e_1.

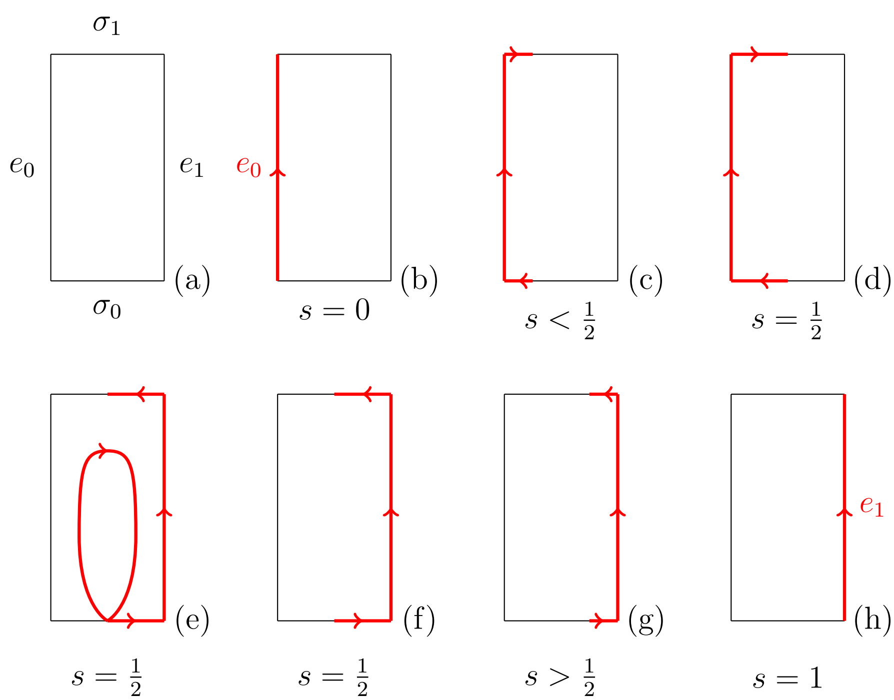

FIGURE 11. Constructing γ_s

We observe that the same construction can be done parametrically. $\square$

In this paper we will apply this lemma when m_1, m_2 are smaller than some small δ. In this case we will conclude that lengths of the 1-parameter family of curves constructed above is at most $l_3 + l_2 + R(\delta)$.

Definition 4.5. A one-parameter family of curves $G(s,t) = e_s(t)$, (see Fig. 12), $s, t \in I$ will be called a long δ-thin strip if the lengths of all transversal curves $\sigma_t(s) = G(s,t)$ are smaller than δ for all $t \in I$. We call the strip long, because for each $s \in I$ we don't have a control over the lengths of curves $e_s(t)$.

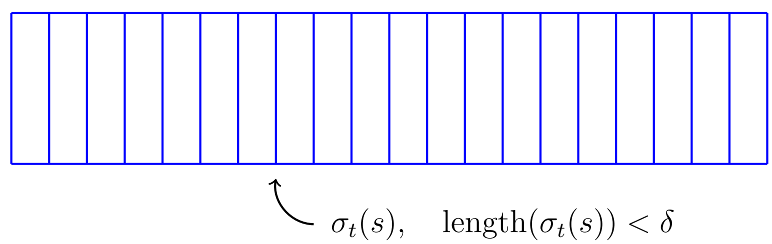

FIGURE 12. δ-thin strip $F : I \longrightarrow \Omega M$

Definition 4.6. $F : I^m \longrightarrow \Pi M$ will be called a long δ-thin strip, if there exists a continuous family of curves $\alpha_{\tilde{x}}$, connecting $\tilde{x} \in \partial I^m$ with the point $\tilde{p} = \left(\frac{1}{2}, ..., \frac{1}{2}\right)$ with the property that for each $t \in [0,1]$ the curve $S_t : \alpha_{\tilde{x}} \to M$ defined by $S_t(x) = F(x)(t)$ has length less than δ (see Figure 13).

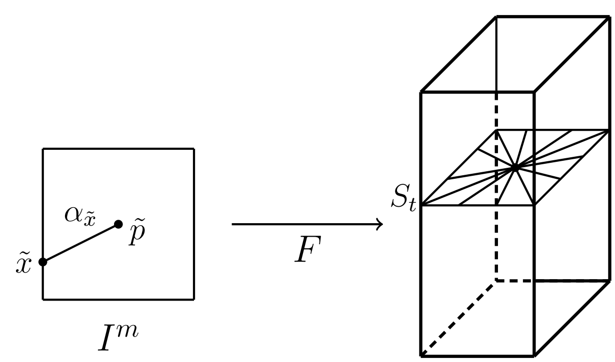

FIGURE 13. δ-thin strip $F : I^2 \longrightarrow \Omega M$

Lemma 4.7. *Let* $F : I \longrightarrow \Omega M, F(s) = e_s(t) = e_s$ *be a long δ-thin strip. Let* a_t^0, a_t^1 *be two one-parameter families of curves with the following properties:*

- $a_0^i = e_i(0)$;
- $a_t^i(0) = e_i(0)$;
- $a_t^i(1) = e_i(t)$.

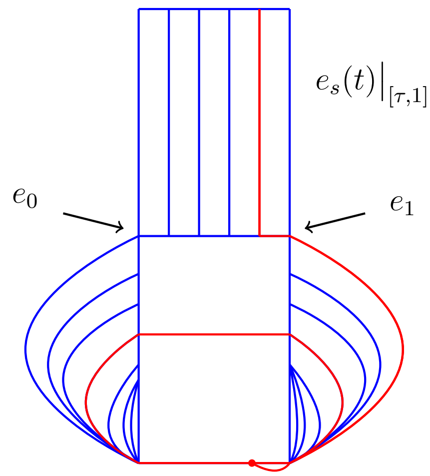

FIGURE 14. Homotopy between F and $\tilde{F}$

Suppose the maximal length of curves in both families is majorized by L. Then there exists a continuous family of curves $H : I \times I \longrightarrow \Pi_{3L+R(\delta)}M$, such that

- *$H(x,i) = a^i_x$ for $i = 0,1$.*
- *$H(x,t)$ is an arc of length $\leq 3L + R(\delta)$ connecting $F(x)(0)$ to $F(x)(t)$.*

Proof. Fix $t \in (0,1)$. Consider a quadrilateral formed by curves a^0_t, $e_{[0,1]}(t)$, a^1_t and $e_{[0,1]}(0)$. Applying Lemma 4.4 we obtain a family of arcs $H(t,x)$ interpolating between a^0_t and a^1_t. The result for all t follows by applying the parametric version of Lemma 4.4. $\square$

The statement of Lemma 4.7 can be generalized to higher dimensions in the following lemma.

Lemma 4.8. *Let $F : I^m \longrightarrow \Omega M$ be a long δ-thin strip. Suppose $H : \partial I^m \times [0,1] \longrightarrow \Pi M$ is a family of paths of length at most l, such that each path begins at $x = F(\tilde{x})(0)$ and ends at $e_{\tilde{x}}(t) = F(\tilde{x})(t)$. Suppose also that for $\tilde{p} = (\frac{1}{2},...,\frac{1}{2})$ there exists a one-parameter family of curves a^p_t connecting point $p = F(\tilde{p})(0)$ with the points $e_p(t) = F(p)(t)$ of length at most L. Then one can extend H to $I^m \times [0,1] \longrightarrow \Pi M$ with the same properties, so that the length of paths is at most $l+2L+R(\delta)$.*

Proof. For each $\tilde{x} \in \partial I^m$, consider the long δ-thin strip, with the boundary formed by the curves $e_x(t)|_{t\in[0,t^*]}$, $F(\alpha_{\tilde{x}})(0)$, $F(\alpha_{\tilde{x}})(t^*)$ and $e_p(t)|_{t\in[0,t^*]}$, (see fig. 15 (a)). Next consider a quadrilateral formed by $H(\tilde{x},t^*)$, $F(\alpha_{\tilde{x}})(t^*)$, $a^p_{t^*}$ and $F(\alpha_{\tilde{x}})(0)$, (see fig. 15 (b)). To each such quadrilateral apply Lemma 4.7. This will give us the required family

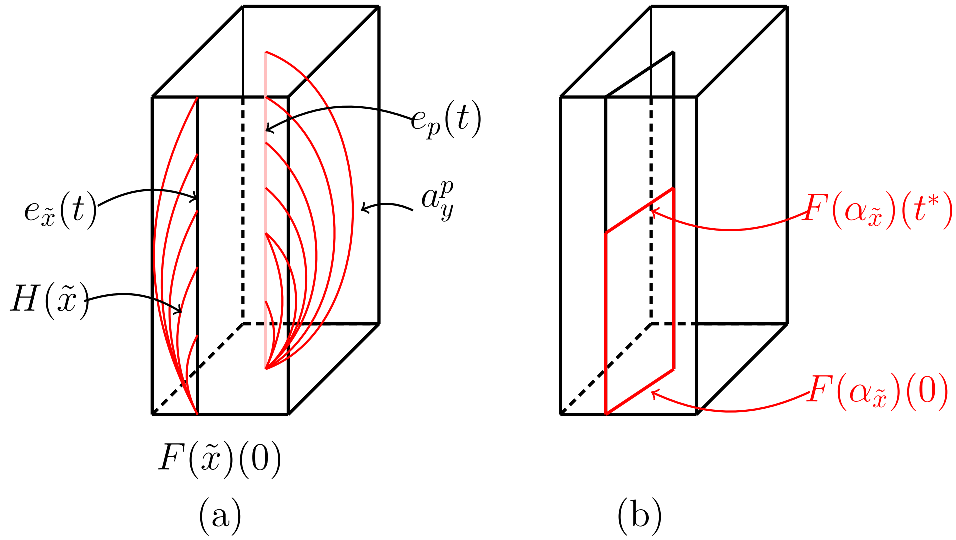

FIGURE 15. Extending H

of curves. Note that the resulting continuity of the family of curves follows from continuity of $F(\alpha_{\tilde{x}})(t)$ with respect to both x and t, which results in continuity of the quadrilaterals. The continuity of H implies that the homotopies that are constructed in Lemma 4.7 also vary continuously. □

Now we will use the Lemmas above to prove Proposition 4.2.

Let us partition each rectangle of R_i of K into N small rectangles rec_k by first subdividing each e_i of T into small sub-intervals using a partition $0 = x_0 < x_1 < ... < x_s = 1$ and next further subdividing each Σ_{x_r} $r \in \{0, ..., s\}$ so that each $F|_{rec_k}$ is a δ-thin long strip for all $k \leq N$.

The construction of H will be by induction on the skeleta of K.

Let v_i be a vertex in the subdivision of K that lies in the interior of K. $F(v_i) = \gamma_i(t)$, where $t \in [0,1]$ with $\gamma(0) = \gamma(1) = p_i$. Since each γ_i lies on an L-short sphere there exists a family of arcs a_t^i with $a_0^i = a_1^i$ equal to constant curves with image $F(v_i)(0)$, and a_t^i connecting $F(v_i)(0)$ to $F(v_i)((t))$. We define $H(v_i, t) = a_t^i$.

Applying Lemma 4.7 to each edge $[v_i, v_j]$ we extend H to the 1-skeleton of the triangulation. Finally, applying Lemma 4.8 for each rec_k we extend H the 2-skeleton. This finishes the proof of Proposition 4.2.

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