On the blow-up of
four-dimensional Ricci flow singularities

By *Davi Máximo* at Austin

Abstract. In this paper we prove a conjecture by Feldman–Ilmanen–Knopf (2003) that the gradient shrinking soliton metric they constructed on the tautological line bundle over $\mathbb{CP}^1$ is the uniform limit of blow-ups of a type I Ricci flow singularity on a closed manifold. We use this result to show that limits of blow-ups of Ricci flow singularities on closed four-dimensional manifolds do not necessarily have non-negative Ricci curvature.

1. Introduction

Suppose (M, g) is a closed Riemannian manifold. One can evolve the metric g by

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc}(g),$$

a (weakly) parabolic system known as *Ricci flow*. Under certain curvature conditions on the initial g, it is possible to prove normalized convergence to a round metric (see Hamilton [18], Brendle–Schoen [6], Böhm–Wilking [5]) and thus say a lot about the topology of M. But for a large set of initial metrics the flow will become singular in finite time before converging to any smooth limiting metric.

This happens when the curvature blows up on certain regions of the manifold and, indeed, the standard short-time existence result for Ricci flow says that if the flow becomes singular at some finite time $T < \infty$, then $\lim_{t \nearrow T} \max_{x \in M} |\mathrm{Rm}(x, t)| = \infty$. Actually, one can prove that this happens if, and only if, $\limsup_{t \nearrow T} \max_{x \in M} |\mathrm{Rc}(x, t)| = \infty$, see Šešum [26].

In order to analyze these singularities, one follows the conventional wisdom of singular analysis from non-linear PDEs and does a *blow-up* at the singularity using the scaling symmetry of the equation. Depending on how much compactness is at hand, one can extract a *singularity model* from a sequence of such blow-ups, which will usually have a better geometry than the original Ricci flow. Moreover, if one has a good knowledge about the possible singularity models, then one is able to understand the structure of the singularity formation and see how to perform surgery while controlling the geometry and the topology of the manifold, thus arriving at the so-called *Ricci flow with surgeries*.

The above blow-up analysis in three-dimensional Ricci flow has been proved quite successful by the work of Hamilton (e.g. see [19]) and Perelman ([24, 25]) on the Poincaré and

Geometrization conjectures, and a lot of their theory carries on to higher dimensional settings, which is an area of active research.

Indeed, for an n-dimensional Ricci flow $g(t)$ on a maximal time interval $[0, T)$ with $T < \infty$, it follows from Hamilton–Perelman's theory that one can choose a sequence of points $p_i \in M$ and times $t_i \nearrow T$ with

$$\lambda_i = |\mathrm{Rm}|(p_i, t_i) = \sup_{x \in M, t \le t_i} |\mathrm{Rm}|(x, t) \longrightarrow \infty$$

such that the rescaled flows

$$g_i(t) = \lambda_i g \left(t_i + \frac{t}{\lambda_i} \right)$$

will converge (in a suitable sense and up to subsequence) to a complete Ricci flow $(N, g_\infty(t))$, which one calls *singularity model*. Moreover, if the singularity is of *type I*, i.e., the curvature blows up like

$$\limsup_{t \nearrow T} \max_M |\mathrm{Rm}|(T - t) < \infty,$$

by the work of Enders–Müller–Topping in [13] the above limit $(N, g_\infty(t))$ will be a non-flat *gradient shrinking soliton*, that is a self-similar Ricci flow, where the metric g_∞ evolves only by scaling and diffeomorphism. This means there will exist a time dependent function f defined on N such that one can obtain the Ricci flow $g_\infty(t)$ from pullbacks of an initial metric $g_\infty(0)$ by diffeomorphisms ϕ_t of N generated by $\frac{1}{T-t}\nabla f$, that is,

$$g_\infty(t) = (T - t)\phi_t^* g_\infty(0),$$

and one can check this is the case whenever the soliton equation

$$(1.1) \qquad \mathrm{Rc}(g_\infty) + \nabla\nabla f = \frac{1}{2} g_\infty$$

is satisfied at some point in time (see Cao [10] for a survey on metrics satisfying the above equation).

In dimension three, the *Hamilton–Ivey pinching* estimate (see [19,20]) roughly states that if the flow has a region with very negative sectional curvature, then the most positive sectional curvature is much larger still. This implies that limits of blow-ups of three-dimensional Ricci flows will have non-negative sectional curvature and thus drastically constrains the singularities that can appear, making three-dimensional Ricci flow with surgeries plausible. In higher dimensions, one has that such limits will have non-negative scalar curvature by the work of Chen [11], but this type of estimate is lacking for more useful curvature conditions.

In this paper we prove that certain such estimates *cannot* exist for Ricci curvature in dimension four, i.e., that limits of blow-ups of four-dimensional Ricci flows do not necessarily have non-negative Ricci curvature. We achieve this by solving a question left by Feldman–Ilmanen–Knopf in [14], which we state soon.

Let M be $\mathbb{CP}^2$ blown-up at one point and L be $\mathbb{C}^2$ blown-up at zero. We invite the reader not familiar with these spaces to see Appendix A before reading what follows.

We will think of M as $\mathbb{C}^2\setminus\{0\}$ with one $\mathbb{CP}^1$ glued at 0 (the section Σ_0) and another at ∞ (the section Σ_∞) and of L as $\mathbb{C}^2\setminus\{0\}$ with a $\mathbb{CP}^1$ glued only at 0. Both M and L are line bundles over $\mathbb{CP}^1$, M with line $\mathbb{CP}^1$ and L with line $\mathbb{C}$. One can consider *Kähler metrics* on these manifolds and evolve them by Ricci flow.

Indeed, let $g(t)$ be a one-parameter family of Riemannian metrics on the manifold M evolving by Ricci flow[1]

$$(1.2) \qquad \frac{\partial}{\partial t} g = -\mathrm{Rc}(g).$$

Assume further that $g(0)$ is Kähler. It is a noted fact that $g(t)$ will remain Kähler with regard to the same complex structure and the flow is thus called Kähler–Ricci flow, see e.g. Cao [8]. The Kähler class $[\omega(t)]$ of the metric $g(t)$ will evolve by

$$(1.3) \qquad \partial_t[\omega(t)] = -[\mathrm{Rc}(\omega)] = -c_1(M),$$

where $c_1(M)$ is the first Chern class of the complex surface M. In particular,

$$(1.4) \qquad [\omega(t)] = [\omega(0)] - t c_1(M).$$

Moreover, on M, the cohomology classes of the divisors $[\Sigma_0]$ and $[\Sigma_\infty]$ span the class $H^{1,1}(M; \mathbb{R})$ so any Kähler class $[\omega]$ can be written uniquely as

$$[\omega] = b[\Sigma_\infty] - a[\Sigma_0]$$

for constants $0 < a < b$, and the first Chern class satisfies $c_1(M) = -[\Sigma_0] + 3[\Sigma_\infty]$. This and equation (1.4) give

$$(1.5) \qquad [\omega(t)] = b(t)[\Sigma_\infty] - a(t)[\Sigma_0]$$

for $a(t) = a(0) - t$ and $b(t) = b(0) - 3t$. Thus, if initially $b(0) > 3a(0)$, then $a(t) \to 0$ as $t \nearrow T = a(0)$ and the class $[\omega(T)]$ will not be Kähler. This will mean that the $\mathbb{CP}^1$ of the section $[\Sigma_0]$ has collapsed to a point and thus Ricci flow must have become singular at a time no later than $t = T$.

In [14], Feldman–Ilmanen–Knopf conjectured that indeed, at least for $U(2)$-invariant metrics, if $b(0) > 3a(0)$, then a type I singularity will develop along Σ_0 precisely at time $t = T$ and the blow-up limit of such singularity is the gradient shrinking soliton they have constructed on L, the *FIK soliton*.

Since their work, a lot of investigation has been done on Kähler–Ricci flow of general Kähler manifolds. Of relevance to this paper are Tian–Zhang [33], Song–Weinkove [29], and the more recent Song [27]. When restricted to the Kähler–Ricci flow of $U(2)$-invariant metrics of M as above, [33] gives the singular time to be exactly $T = a(0)$, and in [29] the authors prove, among other things, that the singularity at $t = T$ will develop only along Σ_0, with $g(T)$ being a smooth metric on $M \setminus \Sigma_0$. Finally, Song [27] proved that such a singularity is type I, and using the compactness at hand, he argued that limits of blow ups of the flow will subconverge in the Cheeger–Gromov–Hamilton sense to a complete non-flat gradient shrinking Kähler–Ricci soliton on a manifold homeomorphic to $\mathbb{C}^2$ blown up at one point. Moreover, the isometry group of this soliton contains the unitary group $U(n)$; but since the complex structure might jump in the limit (see e.g. [12]), one is not able to argue using Cheeger–Gromov–Hamilton convergence that this soliton is in fact the FIK soliton.

In this paper we complete the proof of the Feldman–Ilmanen–Knopf conjecture for a large set of initial metrics:

---

[1] In the context of Kähler geometry, Ricci flow appears in the literature as $\partial_t g = -\mathrm{Rc}(g)$ instead of the usual $\partial_t g = -2\mathrm{Rc}(g)$. This only changes things by scaling.

Theorem 1.1. *Let $g(t)$ be metrics on the manifold M evolving by Kähler–Ricci flow (1.2). For a large open set [2] of $U(2)$-invariant initial metrics $g(0)$ belonging to the Kähler class $b(0)[\Sigma_\infty] - a(0)[\Sigma_0]$ with $b(0) > 3a(0)$, one has the following:*

(i) *the flow is smooth until it becomes singular at time $T = a(0)$,*

(ii) *at $t = T$ the flow develops a type I singularity in the region Σ_0 and $g(T)$ is a smooth Riemannian metric on $M \setminus \Sigma_0$,*

(iii) *parabolic dilations of $g(t)$ converge uniformly on any parabolic neighborhood [3] of the singular set Σ_0 to the evolution of the FIK soliton.*

As we have pointed out, part (i) follows from a now standard general result of [33] and part (ii) is proved by putting together the results in [29] and [27]. Our proof of part (iii) is based on comparison principle techniques applied to the evolving metric potentials, and thus gives a Kähler limit with respect to the same original complex structure. We first prove convergence to the FIK metric in $C^{0,1}$ topology without making any type I blow up assumptions. To prove higher regularity without making further restrictions on the class of initial data, we then use the type I blow up for the scalar curvature proved in [27].

We remark that comparison principle techniques have also been used on a large body of work on yet another type of Ricci flow singularity, *neckpinches*, see Angenent–Knopf [3, 4], Angenent–Caputo–Knopf [1], Angenent–Isenberg–Knopf [2], and Gu–Zhu [16].

For the reader interested in more results concerning singularity analysis of the Kähler–Ricci flow we suggest looking also at Song–Weikove [30, 31], Song–Tian [28], Fong [15], and the references therein. In the latter article, Fong studies singularity formation for the case $b(0) < 3a(0)$. For general Kähler–Ricci flow lecture notes see Song–Weinkove [32].

Theorem 1.1 has the following two consequences. Since the FIK soliton has Ricci curvature of mixed sign near Σ_0:

Theorem 1.2. *Limits of blow-ups of Ricci flow singularities on closed four-dimensional manifolds do not necessarily have non-negative Ricci curvature.*

Moreover, after constructing a metric on M with strictly positive Ricci curvature that satisfies the conditions of Theorem 1.1, we will have:

Corollary 1.3. *Positive Ricci curvature is not preserved by Ricci flow in four dimensions or higher.*

As mentioned before, the above results shows a contrast between Ricci flow in dimensions three and four. Moreover, Corollary 1.3 is related to a previous result of the author [23], and also a result on lower bounds for Ricci curvature under the Kähler–Ricci flow by Zhang [34], which provides other examples that imply the same result stated in the corollary.

---

[2] See Definition 3.6 in Section 3 for a precise statement on the set of initial metrics.
[3] This notion is made precise in Section 3 too. See also Remark 3.11.

Knopf, to whom I am deeply grateful. I also wish to warmly thank Jian Song and Ben Weinkove for hepful comments on an earlier version of this work, and to acknowledge the NSF for its support.

2. *U(2)*-invariant Kähler metrics

In this section we consider rotationally symmetric Kähler metrics on $\mathbb{C}^2\backslash\{0\}$ to derive Kähler metrics on any given Kähler class of the complex surface M, following an ansatz introduced by Calabi [7, Section 3].

Let g be a $U(2)$-invariant Kähler metric on $\mathbb{C}^2\backslash\{0\}$, the latter with complex coordinates $z = (z^1, z^2)$. Define $u = |z^1|^2$, $v = |z^2|^2$, and $w = u + v$.

Since g is a Kähler metric and the second de Rham cohomology group $H^2(\mathbb{C}^2\backslash\{0\}) = 0$, by the $\partial\bar{\partial}$-lemma one can find a global real smooth function $P : \mathbb{C}^2\backslash\{0\} \longrightarrow \mathbb{R}$ such that

$$(2.1) \qquad g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} P.$$

The further assumption of g being rotationally symmetric allows us to write $P = P(r)$, where $r = \log w$.[4] We then set $\varphi(r) = P_r(r)$ (we use a subscript for the derivative since later P will be regarded as a function of time as well) and compute from (2.1)

$$(2.2) \qquad g = [e^{-r}\varphi\delta_{\alpha\beta} + e^{-2r}(\varphi_r - \varphi)\bar{z}^\alpha z^\beta]dz^\alpha d\bar{z}^\beta,$$

and[5]

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \frac{1}{w^2} \begin{pmatrix} v\varphi + u\varphi_r & (\varphi_r - \varphi)\bar{z}^1 z^2 \\ (\varphi_r - \varphi)z^1\bar{z}^2 & u\varphi + v\varphi_r \end{pmatrix}.$$

Because $\det(g_{\alpha\bar{\beta}}) = e^{-2r}\varphi\varphi_r$, one can quickly note that a potential P on $\mathbb{C}^2\backslash\{0\}$ gives rise to a Kähler metric as in (2.1) if, and only if,

$$(2.3) \qquad \varphi > 0 \quad \text{and} \quad \varphi_r > 0.$$

Given a metric as above, Calabi's lemma [7, Section 3] tells us that g will extend to a smooth Kähler metric on the complex surface M if φ satisfies the following asymptotic properties − henceforth called *Calabi's conditions*:

(i) There exist positive constants a_0 and a_1 such that φ has the expansion

$$(2.4) \qquad \varphi(r) = a_0 + a_1 w + a_2 w^2 + O(|w|^3)$$

as $r \to -\infty$.

(ii) There exists a positive constant b_0 and a negative constant b_1 such that φ has the expansion

$$(2.5) \qquad \varphi(r) = b_0 + b_1 w^{-1} + b_2 w^{-2} + O(|w|^{-3})$$

as $r \to \infty$.

Remark 2.1. We note that $\varphi_r > 0$ for r finite, but $\varphi_r = 0$ for $r = \pm\infty$.

---

[4] Depending on the purpose of the computation, one coordinate might be preferable than the other and we will use both r and w in the rest of the paper.

[5] The matrix (g) is actually a 4×4 matrix: $(g) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{smallmatrix}\right)$.

Summarizing the above:

Lemma 2.2 (Calabi [7]). *Any potential φ satisfying conditions (2.3), (2.4) and (2.5) will give rise to a $U(2)$-invariant Kähler metric on M. Moreover, this metric will belong in Kähler class $b_0[\Sigma_\infty] - a_0[\Sigma_0]$, and satisfy $|\Sigma_0| = \pi a_0$ and $|\Sigma_\infty| = \pi b_0$.*

2.1. Curvature terms. On Kähler manifolds, the Ricci tensor is given locally by

$$\mathrm{Rc}_{\alpha\bar\beta} = -\frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta} \log \det g.$$

In particular, for g as in (2.2), one has globally

$$(2.6) \qquad \mathrm{Rc}_{\alpha\bar\beta} = e^{-r}\psi \delta_{\alpha\beta} + e^{-2r}(\psi_r - \psi)\bar z^\alpha z^\beta$$

where $\psi = -\partial_r(\log \det g) = 2 - \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r}$. From equations (2.2) and (2.6), we compute the eigenvalues of the Ricci curvature endomorphism[6]

$$(2.7) \qquad \lambda_1 = \frac{\psi}{\varphi} \qquad \text{with eigenvector } U = \bar z^2 \frac{\partial}{\partial z^1} + \bar z^1 \frac{\partial}{\partial z^1},$$

$$\lambda_2 = \frac{\psi_r}{\varphi_r} \qquad \text{with eigenvector } V = z^1 \frac{\partial}{\partial z^2} + z^2 \frac{\partial}{\partial z^2}.$$

and, in particular, the scalar curvature

$$(2.8) \qquad R(r,t) = \frac{2}{\varphi}\left(2 - \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r}\right) + \frac{2}{\varphi_r}\left[\left(-\frac{\varphi_r}{\varphi}\right)_r + \left(-\frac{\varphi_{rr}}{\varphi_r}\right)_r\right].$$

Using Calabi's conditions we find for r near $-\infty$

$$(2.9) \qquad \lambda_1 = \frac{1}{a_0} + O(e^r),$$

$$\lambda_2 = -\frac{1}{a_0} - \frac{2a_2}{a_1^2} + O(e^r),$$

and for r near $+\infty$

$$(2.10) \qquad \lambda_1 = \frac{3}{b_0} + O(e^{-r}),$$

$$(2.11) \qquad \lambda_2 = \frac{1}{b_0} + \frac{2b_2}{b_1^2} + O(e^{-r}).$$

Moreover, for the Riemann curvature, a direct computation shows

$$(2.12) \quad R_{\alpha\bar\beta\gamma\bar\delta} = e^{-4r}\left[-\varphi_{rrr} + 4\varphi_{rr} - 2\varphi_r + 2\varphi - 4\frac{\varphi_r^2}{\varphi} + \frac{\varphi_{rr}^2}{\varphi_r}\right]\bar z^\alpha z^\beta \bar z^\gamma z^\delta$$

$$+ e^{-3r}\left[\varphi_r - \varphi_{rr} - \varphi + \frac{\varphi_r^2}{\varphi}\right](\bar z^\alpha z^\beta \delta^{\gamma\delta} + \bar z^\alpha \delta^{\beta\gamma} z^\delta$$

$$+ \delta^{\alpha\beta}\bar z^\gamma z^\delta + \delta_{\alpha\delta} z^\beta \bar z^\gamma) + e^{-2r}[-\varphi_r + \varphi](\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}).$$

---

[6] The map $\mathrm{Rc} : TM \longrightarrow TM$ is obtained by raising one index.

2.2. The effect of Ricci flow. From equations (2.2) and (2.6), one can see that $g(t)$ evolves by Ricci flow $\partial_t g = -\mathrm{Rc}(g)$ if, and only if, φ evolves by $\varphi_t = -\psi$, that is,

$$(2.13) \qquad \varphi_t = \frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - 2.$$

Remark 2.3. Equation (2.13) looks alarming from the PDE point of view, as it might degenerate. On the other hand, we recall that for potentials φ that yield Kähler metrics on $\mathbb{C}^2 \backslash \{0\}$ we have $\varphi_r > 0$ and this condition is preserved, so (2.13) is parabolic.

The Kähler class will evolve as in (1.5) and

$$\varphi(-\infty, t) = a(t) = a_0 - t, \quad \varphi(+\infty, t) = b(t) = b_0 - 3t,$$

so the flow will become singular no later than $t = a(0)$. In fact, a general result in Kähler–Ricci flow, see [33], says that the Kähler–Ricci flow exists and is smooth up until the first time $t = T$ where the Kähler class $[w(T)]$ ceases to be Kähler. In our case, because $b(0) > 3a(0)$, this happens precisely at $T = a(0)$. Furthermore, since $|\Sigma_0| = \pi a(t)$, the section Σ_0 will vanish when $t = T$ and, as it turns out, $g(T)$ is still smooth in $M \backslash \Sigma_0$, see [29].

The following scale invariant estimate is an immediate consequence of the maximum principle and will be useful later.

Lemma 2.4. *Let* $F = \frac{\varphi_r}{\varphi}$. *Then,* $0 \leq F(r, t) \leq \max\{\max F(\cdot, 0), 1\}$.

Proof. Because of (2.3), $F \geq 0$. To prove the upper bound, let r_0 be a local spatial maximum of F, at which we must have

$$F_{rr}(r_0) \leq 0 \quad \text{and} \quad F_r(r_0) = 0.$$

We then compute the evolution of F

$$F_t = \frac{F_{rr}}{\varphi_r} + \frac{F_r}{\varphi}\left(1 - \frac{F_r}{F^2}\right) + 2\frac{F}{\varphi}(1 - F),$$

and thus note that

$$F_t(r_0) = \frac{1}{\varphi}\left[\frac{F_{rr}}{F} + 2F(1 - F)\right] \leq \frac{2}{\varphi}F(1 - F).$$

So, at a local spatial maximum where $F(r_0) > 1$ (respectively ≥ 1), $F(x, t)$ is decreasing (respectively non-increasing) in time, and the lemma follows since $F \equiv 0$ in the boundary points $r = -\infty, +\infty$. $\qquad \square$

2.3. Soliton metrics on L and M. Following [14], consider a $U(2)$-invariant gradient shrinking soliton metric on L or M, normalized such that $|\Sigma_0| = \pi$. Then it must have a potential φ satisfying

$$(2.14) \qquad \frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - C\varphi_r + \varphi - 2 = 0,$$

for some constant C and with $\lim \varphi = 1$ as $r \to -\infty$ and another asymptotic condition as $r \to +\infty$, depending on whether one is looking for a soliton on M or L.

In the case one is looking on M, by the independent work of Koiso in [21] and Cao in [9], C must be a constant between $\frac{1}{2}$ and 1 and there will exist only one such potential, modulo translations in r. This soliton has positive Ricci curvature, as Cao observed in [9], and satisfies $a_0 = 1, b_0 = 3$.

For a soliton on L, Feldman–Ilmanen–Knopf proved that the C must be $\sqrt{2}$ and that there exists only one such potential, again modulo translations in r. For our purposes, it is relevant to note that this soliton has Ricci curvature of mixed sign: $\lambda_2 < 0$ for r near $-\infty$. In fact, using the soliton equation (2.14) and the expression (2.7) for the eigenvalue one finds that $\lambda_2 = 1 - \sqrt{2}\frac{\varphi_{rr}}{\varphi_r}$ and since $\lim_{r \to -\infty} \frac{\varphi_{rr}}{\varphi_r} = 1$, we have that λ_2 is negative near the section Σ_0.

3. Dilation variables, type I blow-up, and convergence modulo diffeomorphisms

3.1. Dilation Variables.   What we know so far about the singularity formation is that it occurs along the section Σ_0, which shrinks to a point by [29], and that it is type I [27]. Hence it will be useful to use *parabolically dilated variables* that allow us to *zoom in on Σ_0 in a type I fashion*.

Given an evolution $\varphi(r, t)$ with singular time T as before, we define the *dilated time* variable $\tau = -\log(T - t)$, the *dilated spatial* variable $\rho = r + \tau$ (which correspond to complex coordinates $\zeta = e^{\tau/2}z$) and set

(3.1) $$\phi(\rho, \tau) = e^\tau \varphi(r(\rho, \tau), t(\tau)).$$

The function $\phi(\rho, \tau)$ evolves by

(3.2) $$\phi_\tau = \frac{\phi_{\rho\rho}}{\phi_\rho} + \frac{\phi_\rho}{\phi} - \phi_\rho + \phi - 2$$

and is a Kähler potential on M in complex coordinates ζ. To see that, we note that it must satisfy $\phi, \phi_\rho > 0$ for all τ, and the Calabi conditions (2.4) and (2.5) – or, alternatively, that ϕ represents the metric $\bar{g}(\tau)$ on M equivalent to $g(t)$ scaled by $e^\tau = \frac{1}{T-t}$ and pulled back by the diffeomorphism $z \longrightarrow e^{-\tau/2}z$.

Now that ρ, τ are defined, we explain what we mean by zoom in Σ_0 in a type I fashion, first on the level of the potential and then on the level of the metric.

For fixed ρ, we let $t \nearrow T$ and thus have $\tau \nearrow +\infty$ and $(r(\rho, \tau), t(\tau)) \to (-\infty, T)$. The dilation

$$\phi(\rho, \tau) = \frac{1}{T - t}\varphi(\rho - \tau, t)$$

is then a type I (note the factor $\frac{1}{T-t}$) zoom in on how φ is going to zero along Σ_0, since $\varphi(\rho - \tau, t) \to \varphi(-\infty, T) = 0$.

Geometrically, the metric $\bar{g}(\tau)$ given by $\phi(\rho, \tau)$ is just $\frac{1}{T-t}g(t)$ modified by diffeomorphism, and if $g(t)$ has a type I singularity at $t = T$, $\bar{g}(\tau)$ has bounded curvature as $\tau \nearrow +\infty$. Moreover,

$$|\Sigma_0|_{\bar{g}(\tau)} = \pi,$$
$$|\Sigma_\infty|_{\bar{g}(\tau)} = [(b_0 - 3a_0)e^\tau + 3]\pi,$$

which indicates that as $\tau \nearrow +\infty$, Σ_∞ is being blown away and suggests that the metrics $\bar{g}(\tau)$ on M are becoming more and more like metrics on L where $|\Sigma_0| = \pi$, as one recalls from Appendix A.

Our goal is to prove that indeed the limit of these metrics will be modeled by the evolution of the FIK soliton metric on L constructed in [14]. But there are major difficulties that do not allow us to prove directly such convergence by working with ρ, τ. One comes from the fact that the FIK potential is not stationary in these coordinates; in fact, the soliton is still moving by diffeomorphisms, or more precisely, it is translating in the ρ variable. The other is the fact that there actually exist a whole family of FIK potentials, generated by the translations in r. These difficulties lead us to the approach presented in the next section.

3.2. Equations in ϕ, τ variables. Let $\tau = -\log(T - t)$ and $\rho = r + \tau$ be the dilation variables introduced above. Because $\phi_\rho > 0$ along the flow, we can actually write ρ as a function of ϕ at any fixed time τ and thus consider the function

$$(3.3) \qquad y(\phi, \tau) = \phi_\rho(\rho, \tau),$$

which, for any given τ, is defined on the inverval $[1, (b_0 - 3a_0)e^\tau + 3]$, satisfying

$$y(1) = y((b_0 - 3a_0)e^\tau + 3) = 0$$

and $y(\phi)$ positive otherwise.

We next find the evolution equation of $y(\phi, \tau)$. First we note: since $\phi_\rho = y$, we have $\phi_{\rho\rho} = y_\rho = y_\phi y$ and $\phi_{\rho\rho\rho} = y_{\phi\phi}y^2 + y_\phi^2 y$. And we then compute $\partial_\tau\big|_\rho y$, which in our notation means *the derivative of y with respect to τ while fixing ρ.*

$$\partial_\tau\big|_\rho y = \phi_{\rho\tau}$$

$$= \frac{\phi_{\rho\rho\rho}}{\phi_\rho} - \left(\frac{\phi_{\rho\rho}}{\phi_\rho}\right)^2 + \frac{\phi_{\rho\rho}}{\phi} - \left(\frac{\phi_\rho}{\phi}\right)^2 + \phi_\rho - \phi_{\rho\rho}$$

$$= y_{\phi\phi}y + \frac{y_\phi y}{\phi} - \left(\frac{y}{\phi}\right)^2 + y - y_\phi y.$$

Finally, since $\partial_\tau\big|_\phi y = \partial_\tau\big|_\rho y - y_\phi \phi_\tau$, we have

$$(3.4) \qquad \partial_\tau\big|_\phi y = y_{\phi\phi}y + \frac{y_\phi y}{\phi} - \left(\frac{y}{\phi}\right)^2 + y - y_\phi y - y_\phi \phi_\tau$$

$$= y_{\phi\phi}y + (2 - \phi - y_\phi)y_\phi + y\left(1 - \frac{y}{\phi^2}\right).$$

Remark 3.1. Because of Lemma 2.4, $\frac{y}{\phi}$ is uniformly bounded in time.

The advantages of these variables are two-fold. First of all, they do not see translations in the ρ variable and thus the whole family of FIK potentials are represented by just one stationary potential $\mathcal{Y} = \mathcal{Y}(\phi^{\text{FIK}})$. Secondly, the non-linearities of (3.4) are mild when compared with (3.2), and this allow us to develop a barrier method based on the comparison principle in the next section.

Remark 3.2. In fact, one can check that all the FIK potentials satisfy the same equation when written using the coordinates as above

$$\mathcal{Y}(\phi) = \phi_\rho = \frac{\phi(\phi - 2) + \sqrt{2}(\phi - 1) + 1}{\sqrt{2}\phi},$$

and also that $\partial_\tau\big|_\phi \mathcal{Y} = 0$.

3.3. Comparison and convergence modulo diffeomorphisms for a large class of potentials.

In this section we use techniques based on the comparison principle to prove an important step towards Theorem 1.1, which is convergence modulo diffeomorphisms for a large class of potentials.

Notation. In what follows we will write just ∂_τ as short for $\partial_\tau\big|_\phi$. In particular, ∂_τ and ∂_ϕ are commuting derivatives.

The first thing we prove is that for any initial data satisfying the conditions of Theorem 1.1, one always has a C^1 bound.

Lemma 3.3. *For any data $y(\phi, \tau)$ coming from Ricci flow as in Theorem* 1.1, *the derivative y_ϕ is uniformly bounded in time.*

Proof. Since at the boundary y_ϕ takes values 1 or -1, we only need to deal with interior spatial maxima and minima. We write the evolution equation for y_ϕ:

$$(3.5) \qquad y_{\phi\tau} = y y_{\phi\phi\phi} + [2 - \phi - y_\phi] y_{\phi\phi} - 2\frac{y y_\phi}{\phi^2} + \frac{2y^2}{\phi^3}.$$

If y_ϕ has a negative local spatial minimum at ϕ_0, then $y_{\phi\phi}(\phi_0) = 0$, $y_{\phi\phi\phi}(\phi_0) \geq 0$, and thus

$$y_{\phi\tau}(\phi_0) = y y_{\phi\phi\phi}(\phi_0) - 2\frac{y y_\phi}{\phi^2}(\phi_0) + \frac{2y^2}{\phi^3}(\phi_0) > 0,$$

so y_ϕ is uniformly bounded from below.

Finally, let ϕ_0 be a local spatial maximum of y_ϕ, so that $y_{\phi\phi}(\phi_0) = 0$ and $y_{\phi\phi\phi}(\phi_0) \leq 0$. By Remark 3.1, $\frac{y}{\phi} < C$ for some constant C independent of time. Suppose $y_\phi(\phi_0) > C$, then

$$y_{\phi\tau}(\phi_0) = y y_{\phi\phi\phi}(\phi_0) - 2\frac{y y_\phi}{\phi^2}(\phi_0) + \frac{2y^2}{\phi^3}(\phi_0)$$

$$\leq 2\frac{y}{\phi^2}\left[-y_\phi + \frac{y}{\phi}\right] < 0.$$

We hence conclude y_ϕ is uniformly bounded from above too and the lemma is proved. □

Remark 3.4. The above lemma, together with Remark 3.1, says that the evolution (3.4) has bounded coefficients on any compact interval $[1, \phi_0]$.

Remark 3.5. Because $y_\phi = \frac{\phi_{\rho\rho}}{\phi_\rho}$, the lemma will be useful to prove C^2 bounds for ϕ.

We next construct upper and lower barriers that for "most" initial data will trap the solution to evolution (3.4) and squeeze it to the FIK potential $\mathcal{Y}(\phi) = \frac{\phi(\phi-2)+\sqrt{2(\phi-1)+1}}{\sqrt{2}\phi}$.

Definition 3.6 (Metrics in the class $\mathcal{C}$). Let $\mathcal{C}$ be the class of all initial $U(2)$-invariant metrics on M belonging to the Kähler class $b(0)[\Sigma_\infty] - a(0)[\Sigma_0]$ with $b(0) > 3a(0)$, and such that, moreover, the parabolic blow up ϕ of the potential φ satisfies, when writing ϕ_ρ in ϕ coordinates as before,

$$y(\phi, 0) > \mathcal{Y}(\phi) - \frac{1}{5}\phi^2.$$

Let us remark that $\mathcal{Y}(\phi) - \frac{1}{5}\phi^2$ will be exactly the initial barrier that we will use for the evolution (3.4), thus we are restricting ourselves to potentials that are initially strictly above it. This barrier is mostly negative, but is positive and small (strictly less than 0.06) on a small neighborhood of $\phi = 2$. This implies that a large family of initial data belongs to the class $\mathcal{C}$.

Proposition 3.7. *For any initial data $y(\phi, 0)$ in the class $\mathcal{C}$, one has that $y(\phi, \tau)$ remains in the class $\mathcal{C}$ and converges uniformly on compact subsets to $\mathcal{Y}(\phi)$ as $\tau \nearrow \infty$.*

We note that the "elliptic" operator in (3.4)

$$\mathcal{E}[y] = y_{\phi\phi}y + (2 - \phi - y_\phi)y_\phi + y\left(1 - \frac{y}{\phi^2}\right)$$

can be written as a linear plus a quadratic part

$$\mathcal{E}[y] = \mathcal{L}[y] + \mathcal{Q}[y],$$

where $\mathcal{L}[y] = (2 - \phi)y_\phi + y$ and $\mathcal{Q}[y] = yy_{\phi\phi} - y_\phi^2 - \frac{y^2}{\phi^2}$. The non-linearity of $\mathcal{E}[\cdot]$ is such that for given functions $y(\phi), s(\phi)$:

$$\begin{aligned}
\mathcal{E}[y + s] &= \mathcal{L}[y + s] + \mathcal{Q}[y + s] \\
&= \mathcal{L}[y] + \mathcal{L}[s] + \mathcal{Q}[y] + \mathcal{Q}[s] + \mathcal{M}[y, s] \\
&= \mathcal{E}[y] + \mathcal{E}[s] + \mathcal{M}[y, s],
\end{aligned}$$

where $\mathcal{M}[y, s] = sy_{\phi\phi} + ys_{\phi\phi} - 2y_\phi s_\phi - 2\frac{ys}{\phi^2}$ is bilinear in y and s. This mild non-linearity suggests the barrier approach.

Proof of Proposition 3.7. Let

$$\mathcal{Y}(\phi) = \frac{\phi(\phi - 2) + \sqrt{2}(\phi - 1) + 1}{\sqrt{2}\phi}$$

be the FIK potential and consider $s(\phi, \tau) = -\lambda(\tau)\phi^2$, where $\dot{\lambda} = -\delta\lambda$ for some $\delta \in \mathbb{R}$. We compute $(\partial_\tau - \mathcal{E})(\mathcal{Y} + s)$:

$$\begin{aligned}
(\partial_\tau - \mathcal{E})[\mathcal{Y} + s] &= \partial_\tau s - \mathcal{L}[s] - \mathcal{Q}[s] - \mathcal{M}[\mathcal{Y}, s] \\
&= \lambda\left(\delta\phi^2 + \mathcal{L}[\phi^2] - \lambda\mathcal{Q}[\phi^2] + \mathcal{M}[\mathcal{Y}, \phi^2]\right),
\end{aligned}$$

and once substituting $\mathcal{L}[\phi^2], \mathcal{Q}[\phi^2]$, and $\mathcal{M}[\mathcal{Y}, \phi^2]$, we have

$$(3.6) \qquad (\partial_\tau - \mathcal{E})[\mathcal{Y} + s] = \lambda\left((\delta + 3\lambda - 1)\phi^2 + 2(2 - \sqrt{2})\phi - 3\phi^{-1}(2 - \sqrt{2})\right).$$

Let $y(\phi, \tau)$ be a solution coming from Ricci flow. Suppose that $y(\phi, 0)$ belongs to the class $\mathcal{C}$ of initial data.

Choose $\lambda(0) = 1/5$ and let δ be a positive number smaller than 10^{-6} to be fixed. Then, by (3.6) the function

$$y_1(\phi, \tau) = \mathcal{Y}(\phi) - \frac{1}{5}e^{-\delta\tau}\phi^2$$

satisfies $(\partial_\tau - \mathcal{E})[y_1] < 0$ for all times $\tau > 0$, i.e., $y_1(\phi, \tau)$ is a subsolution to our evolution problem.

By the comparison principle (Appendix B), if a solution y of $(\partial_\tau - \mathcal{E})y = 0$ initially starts above y_1, then y will stay above y_1 for all later times, as long as the boundary data behave as such.

A general solution y that initially belongs to class $\mathcal{C}$ satisfies the assumption

$$y(\phi, 0) > y_1(\phi, 0)$$

and has the boundary conditions

$$y(1) = y((b_0 - 3a_0)e^\tau + 3) = 0.$$

It is clear that $y(1) = 0 > y_1(1)$ for all times. Also

$$y_1(\phi, \tau) = \mathcal{Y} - \frac{1}{5}\exp(-\delta\tau)\phi^2 < \phi - \frac{1}{5}\exp(-\delta\tau)\phi^2 < 0$$

if $\phi > 5\exp(\delta\tau)$ (here we are using that $\mathcal{Y}(\phi)$ is always below ϕ). Because

$$(b_0 - 3a_0)e^\tau + 3 > 5\exp(\delta\tau),$$

if δ is chosen to be small enough, then we have $y((b_0-3a_0)e^\tau+3) = 0 > y_1((b_0-3a_0)e^\tau+3)$. Thus, the subsolution y_1 stays below y at the boundary, and therefore everywhere, for all later times $\tau > 0$. In particular, because λ is decreasing in magnitude, this implies $y(\phi, \tau)$ belongs to $\mathcal{C}$ for all later times.

Furthermore, for the same initial data $y(\phi, 0) \in \mathcal{C}$, we choose

$$y_2(\phi, \tau) = \mathcal{Y} + \lambda_0 e^{-\tau/2}\phi^2$$

where λ_0 is big enough so that $y_2(\phi, 0) = \mathcal{Y} + \lambda_0\phi^2 > y(\phi, 0)$. Moreover, by equation (3.6) we will have that $(\partial_\tau - \mathcal{E})[y_2] > 0$ for all $\tau > 0$. Moreover, one can check that the boundary data of y_2 stay above those of y for all times. Thus, again using comparison, y_2 stays above y for all later times.

Hence we have proved that

(3.7) $$y_1(\phi, \tau) \leq y(\phi, \tau) \leq y_2(\phi, \tau),$$

and since on compact intervals for ϕ we have $y_1(\phi, \tau) \nearrow \mathcal{Y}(\phi)$ and $y_2(\phi, \tau) \searrow \mathcal{Y}(\phi)$ uniformly, we must have $y(\phi, \tau)$ converging to $\mathcal{Y}(\phi)$ uniformly on compact subsets and the proposition thus is proved.   □

Proposition 3.7 gives us uniform C^0-convergence for $y(\phi) = \phi_\rho$ on compact subsets. We next prove that one actually has uniform $C^{1,1}$-convergence. For that we will use the type I blow up of the scalar curvature proved by Song in [27], i.e., there exists a constant $C > 0$ such that

$$-C \leq R_{g(t)} \leq \frac{C}{T - t}.$$

In particular, for the dilated flow $\overline{g}(\tau)$ this implies

$$-Ce^{-\tau} \leq R_{\overline{g}(\tau)} \leq C,$$

and since $R_{\overline{g}}(\tau) = \frac{4}{\phi}\left(1 - y_\phi\right) - 2y_{\phi\phi}$, we must have $y_{\phi\phi}$ is uniformly bounded by Lemma 3.3.

Proposition 3.8. *For any initial data $y(\phi, 0)$ in the class $\mathcal{C}$, one has that $y(\phi, \tau)$ converges uniformly in the $C^{1,1}$ topology on compact subsets to $\mathcal{Y}(\phi)$ as $\tau \nearrow \infty$.*

Proof. Since we have uniform C^2 bounds for y on compact intervals, the $C^{1,1}$-convergence follows from the following standard argument and Proposition 3.7. Since the spatial derivative $y_{\phi\phi}$ is uniformly bounded in time, for any sequence of times, y_ϕ will converge uniformly up to subsequence. Moreover, since the convergence is uniform, the limit of y_ϕ along any such subsequence must be the spatial derivative $\mathcal{Y}_\phi$ of the stationary state $\mathcal{Y}$. Because the latter does not depend on the subsequence, we have that y_ϕ converges uniformly in time to the derivative $\mathcal{Y}_\phi$. Since y converges uniformly in C^1 while $y_{\phi\phi}$ is uniformly bounded, the proposition is then proved. □

Remark 3.9. For the reader's convenience, we point out the relation between the Riemann curvature and derivatives of y. By the rotational symmetry of $\bar{g}(\tau)$, we reduce our analysis to a point of the form $(z^1, z^2) = (\xi, 0)$, and use (2.12) to compute

$$(3.8) \qquad |\mathrm{Rm}(\bar{g}(\tau))| \leq 2|R_{1\bar{1}1\bar{1}}| + 2|R_{2\bar{2}2\bar{2}}| + 2|R_{1\bar{1}2\bar{2}}|$$

$$\leq 2\left|\frac{1}{\phi_\rho^2}\left(-\phi_{\rho\rho\rho} + \frac{\phi_{\rho\rho}^2}{\phi_\rho}\right)\right| + 4\left|\frac{-\phi_\rho + \phi}{\phi^2}\right| + 2\left|-\frac{\phi_{\rho\rho}}{\phi\phi_\rho} + \frac{\phi_\rho}{\phi^2}\right|$$

$$= 2\left|\frac{1}{\phi_\rho}\left(\frac{\phi_{\rho\rho}}{\phi_\rho}\right)_\rho\right| + \frac{4}{\phi}\left|-\frac{\phi_\rho}{\phi} + 1\right| + \frac{2}{\phi}\left|-\frac{\phi_{\rho\rho}}{\phi_\rho} + \frac{\phi_\rho}{\phi}\right|.$$

Moreover, since $\bar{g}(\tau) = \frac{1}{T-t}g(t)$, we note that by Song's [27] type I result there exists a uniform constant $C > 0$ such that

$$|\mathrm{Rm}(\bar{g}(\tau))| \leq C.$$

Recalling that $\frac{\phi_{\rho\rho}}{\phi_\rho} = y_\phi$, $\frac{1}{\phi_\rho}\left(\frac{\phi_{\rho\rho}}{\phi_\rho}\right)_\rho = y_{\phi\phi}$, and also that one has $\frac{\phi_\rho}{\phi}$ uniformly bounded by Lemma 2.4, we see from the above that type I blow up is what one really needs to establish a second derivative bound for y.

Combining Proposition 3.8 with Song's type I result [27], we have the following:

Theorem 3.10. *Let $g(0)$ be a metric on M belonging to the class $\mathcal{C}$. Then the Ricci flow (1.2) starting at $g(0)$ will develop a type I singularity along Σ_0. Moreover, parabolic dilations $\overline{g}(\tau)$ of $g(t)$ as in (3.1) will converge, modulo diffeomorphisms and uniformly on a corresponding time-dependent neighborhood of the singular region Σ_0, to one of the FIK solitons.*

Proof. We let φ be in the class of initial data $\mathcal{C}$ and consider the dilation variables ρ, τ as well as the dilated potential $\phi(\rho, \tau)$. We apply the following gauge-fixing construction: by the implicit function theorem, we can find a smooth function $C(\tau)$ such that if we define $\mu = \rho - C(\tau)$ and then define

$$\phi(\mu, \tau) = \phi(\mu - C(\tau), \tau),$$

we will have $\phi(0, \tau)$ constant in time, say equal to 2. Moreover, ϕ_μ is related to ϕ just as ϕ_ρ is related to ϕ, and thus we have by Proposition 3.8, $y(\phi)$ converges $C^{1,1}$ uniformly to $\mathcal{Y}(\phi)$, so that ϕ converges $C^{2,1}$ uniformly (in compact ϕ intervals, so in particular for $-\infty \leq \mu \leq 0$) to the unique ϕ^{FIK} that satisfies $\phi^{\mathrm{FIK}}(0) = 2$.

Finally, since we know that the singularity along Σ_0 is type I [27], the Riemann curvature $\mathrm{Rm}(\bar{g}(\tau))$ will be bounded uniformly in the region $-\infty \leq \mu \leq 0$. Shi's local estimates for higher derivatives of the Riemann curvature under Ricci flow then dictate that one actually has bounds of any higher order and that thus convergence is smooth.   □

A few remarks are now in order.

Remark 3.11.   For any fixed $\rho \in \mathbb{R}$, consider the parabolic neighborhood

$$N(\rho) = \{z \in \mathbb{C}^2 : \rho(z) \leq \rho\} = \{z \in \mathbb{C}^2 : |z|^2 \leq e^\rho(T - t)\}.$$

Theorem 3.10 says that for metrics of the class $\mathcal{C}$, there exist diffeomorphisms Ψ_τ (corresponding precisely to the $C(\tau)$ change of gauge) such that as $\tau \nearrow \infty$

$$\Psi_\tau^* \bar{g}(\tau) \longrightarrow g_{\mathrm{FIK}}$$

uniformly on the neighborhood $\Psi_\tau^{-1}(N(\rho))$. In the next section, we will prove that the diffeomorphisms Ψ_τ^{-1} correspond to the one-parameter family of diffeomorphisms by which the FIK solitons move under Ricci flow, i.e., we will prove that for large τ one has asymptotically

$$C(\tau) = (\sqrt{2} - 1)\tau \pm \text{constant}.$$

This will conclude the proof of Theorem 1.1 that parabolic dilations $\bar{g}(\tau)$ of $g(t)$ converge to the flow of an FIK soliton uniformly on any parabolic neighborhood $N(\rho)$.

Remark 3.12.   If we write φ in non-logarithmic coordinates as $\varphi(r, t) = f(w, t)$, where $w = e^r$, and expand $\phi(\mu, \tau) = \phi(\mu - C(\tau), \tau) = e^\tau \varphi(\mu - C(\tau) - \tau, t)$ around $w = 0$ we get

$$\phi(\mu, \tau) = e^\tau f(e^{\mu - C(\tau) - \tau}, t)$$
$$= 1 + e^\mu e^{-C(\tau)} f_w(0, t) + \cdots.$$

Because ϕ must converge smoothly to FIK for fixed μ, we must have that[7]

$$f_w(0, t) \sim e^{C(\tau)}.$$

Remark 3.13.   Finally, we note that by Theorem 3.10 the scalar curvature at the singular region Σ_0 must blow up like

$$(3.9) \qquad\qquad R_{g(t)}\big|_{\Sigma_0} = \frac{4 - 2\sqrt{2}}{T - t} + O\big((T - t)^{\alpha - 1}\big),$$

for some $\alpha > 0$. In fact, by the barrier argument in Proposition 3.7, there must exist positive constants C_0 and δ_0 such that $|y - \mathcal{Y}| \leq C_0 e^{-\delta_0 \tau}$ holds uniformly in time on a fixed interval of the form $[1, \phi_0]$, and by Theorem 3.10, the higher derivatives of y are uniformly bounded in time on $[1, \phi_0]$. Moreover, we can use an interpolation inequality of the form (e.g., [22, Corollary 7.21])

$$(3.10) \qquad\qquad \|\partial_\phi(y - \mathcal{Y})\|_p \leq \epsilon \|\partial_{\phi\phi}(y - \mathcal{Y})\|_p + \frac{C}{\epsilon}\|y - \mathcal{Y}\|_p,$$

---

[7] This notation means that there exist constants $0 < C_1 < C_2$ such that
$$C_1 e^{C(\tau) - \tau} \leq f_w(0, t) \leq C_2 e^{C(\tau) - \tau}$$
as $\tau \nearrow \infty$, et cetera.

on $[1, \phi_0]$, where $p > 1$, $\| \cdot \|_p$ is the usual (spatial) L^p-norm, ϵ is any positive number, and C is a universal constant that does not depend on $y, \mathcal{Y}, \epsilon$, or p, and we can thus argue that whenever $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2} \neq 0$, by setting

$$\epsilon = \frac{\|y - \mathcal{Y}\|_p^{1/2}}{\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2}}$$

in (3.10), one has

$$\|\partial_\phi(y - \mathcal{Y})\|_p \leq (1 + C)\|y - \mathcal{Y}\|_p^{1/2}|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2},$$

and if $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2} = 0$, what we want follows directly from (3.10). Furthermore, because $[1, \phi_0]$ has finite measure, we can send p to infinity and pass the above inequality to the limit:

$$(3.11) \qquad \|\partial_\phi(y - \mathcal{Y})\|_\infty \leq (1 + C)\|y - \mathcal{Y}\|_\infty^{1/2}|\partial_{\phi\phi}(y - \mathcal{Y})\|_\infty^{1/2},$$

and since $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_\infty^{1/2}$ is uniformly bounded in time and $\|y - \mathcal{Y}\|_\infty \leq C_0 e^{-\delta_0 \tau}$ in $[1, \phi_0]$, we find positive constants C_1, δ_1 such that

$$\|y_\phi - \mathcal{Y}_\phi\|_\infty \leq C_1 e^{-\delta_1 \tau}.$$

Moreover, by bootstrapping the argument above, we can also find positive constants δ_2, C_2 such that on $[1, \phi_0]$:

$$\|y_{\phi\phi} - \mathcal{Y}_{\phi\phi}\|_\infty \leq C_2 e^{-\delta_2 \tau}.$$

Finally, since

$$(T - t)R_{g(t)}\big|_{\Sigma_0} = R_{\overline{g}(\tau)}\big|_{\Sigma_0} = -2y_{\phi\phi}(1, \tau),$$

and $\mathcal{Y}_{\phi\phi}(1) = \sqrt{2} - 2$, $e^{-\tau} = T - t$, (3.9) follows for $\alpha = \delta_2 > 0$.

4. End of proof of Theorem 1.1

In this section, we finish the proof of Theorem 1.1. This is done by using Theorem 3.10 and the remarks following it.

Let $\varphi(r, t)$ be a potential belonging to the class $\mathcal{C}$, $\phi(\rho, \tau)$ its corresponding dilated potential as in (3.1), and $\phi(\mu, \tau)$ and $C(\tau)$ as in Theorem 3.10. By Remark 3.11, and since (unnormalized) FIK potentials move under Ricci flow by the diffeomorphisms

$$z \longrightarrow e^{-\sqrt{2}\tau/2}z,$$

it is enough to prove that asymptotically for large τ one has

$$(4.1) \qquad C(\tau) = (\sqrt{2} - 1)\tau \pm \text{constant}.$$

Note that Remark 3.12 tells us that if we write φ in non-logarithmic coordinates as $\varphi(r, t) = f(w, t)$, we must have then

$$f_w(0, t) \sim e^{C(\tau)}.$$

This allows us to use estimate (3.9) on the blow-up of the scalar curvature along Σ_0 to study $C(\tau)$ for large τ. In fact, recalling the Ricci eigenvalues,

$$R_{g(t)}\big|_{\Sigma_0} = 2\lambda_1\big|_{\Sigma_0}(t) + 2\lambda_2\big|_{\Sigma_0}(t),$$

where $\lambda_1\big|_{\Sigma_0}(t) = \frac{1}{T-t}$, estimate (3.9) tells us that the eigenvalue $\lambda_2\big|_{\Sigma_0}(t)$ must blow up like

$$(4.2) \qquad \lambda_2\big|_{\Sigma_0}(t) = \frac{1-\sqrt{2}}{T-t} + O\big((T-t)^{\alpha-1}\big),$$

for some $\alpha > 0$. Moreover, we can compute $\lambda_2\big|_{\Sigma_0}(t)$ directly from (2.7) in the coordinate w and find

$$(4.3) \qquad \lambda_2\big|_{\Sigma_0}(t) = -\frac{f_{wt}(0,t)}{f_w(0,t)}.$$

Integrating (4.3) and using (4.2), we find that

$$(4.4) \qquad f_w(0,t) \sim (T-t)^{1-\sqrt{2}} = e^{(\sqrt{2}-1)\tau},$$

and this gives (4.1) by Remark 3.12. Theorem 1.1 is then proved.

5. The cone of metrics with non-negative Ricci curvature

In this section we prove Corollary 1.3 by constructing a metric on M with strictly positive Ricci curvature and belonging to the class $\mathcal{C}$. We recall that by (2.10) one has for the eigenvalues of Ricci and r near $+\infty$ that

$$\lambda_1 = \frac{3}{b_0} + O(e^{-r}),$$

$$\lambda_2 = \frac{1}{b_0} + \frac{2b_2}{b_1^2} + O(e^{-r}).$$

Let φ_{KC} be the potential for the Cao–Koiso soliton, which has positive Ricci curvature everywhere. The metric φ^{KC} is not the metric we are looking for only because $b_0 = 3a_0$. In fact, one can check explicitly that the Cao–Koiso metric is above the barrier as required in Definition 3.6. Thus we can perturb φ^{KC} by a small amount near $r = +\infty$, to obtain a metric potential φ with $b_0 > 3a_0$. Since the perturbation is only made near $r = +\infty$, where the above expansion for the eigenvalues holds, φ will still have strictly positive Ricci curvature everywhere, and also belong to the class $\mathcal{C}$.

Remark 5.1. The construction above provides explicit examples of solutions demonstrating the linear instability of the Cao-Koiso soliton that was proved by Hall–Murphy [17].

A. Line bundles over $\mathbb{CP}^1$

On the complex projective space $\mathbb{CP}^1$ with projective coordinates $[z_1 : z_2]$, let

$$\varphi_1 : \mathcal{U}_1 = \{[z_1 : z_2] \in \mathbb{CP}^1 : z_1 \neq 0\} \longrightarrow \mathbb{C}$$

and

$$\varphi_2 : \mathcal{U}_2 = \{[z_1 : z_2] \in \mathbb{CP}^1 : z_2 \neq 0\} \longrightarrow \mathbb{C}$$

denote its usual charts given by $\varphi_1([z_1 : z_2]) = z_2/z_1$ and $\varphi_2([z_1 : z_2]) = z_1/z_2$.

We consider two topologically distinct line bundles over $\mathbb{CP}^1$ denoted by L and M described as follows. The bundle L has the complex line $\mathbb{C}$ as fibers:

$$L = [(\mathcal{U}_1 \times \mathbb{C}) \sqcup (\mathcal{U}_2 \times \mathbb{C})] \text{ quotient by } \sim$$

where $\mathcal{U}_1 \times \mathbb{C} \ni ([z_1 : z_2]; \xi) \sim ([y_1 : y_2], \eta) \in \mathcal{U}_2 \times \mathbb{C}$ if, and only if,

$$[z_1 : z_2] = [y_1 : y_2] \quad \text{and} \quad \eta = \left(\frac{y_2}{z_1}\right) \xi.$$

The manifold M has fibers $\mathbb{C} \cup \{\infty\}$:

$$M = \left[(\mathcal{U}_1 \times \mathbb{CP}^1) \sqcup (\mathcal{U}_2 \times \mathbb{CP}^1)\right] / \sim$$

where $\mathcal{U}_1 \times \mathbb{CP}^1 \ni ([z_1 : z_2]; \xi) \sim ([y_1 : y_2], \eta) \in \mathcal{U}_2 \times \mathbb{CP}^1$ if, and only if,

$$[z_1 : z_2] = [y_1 : y_2] \quad \text{and} \quad \eta = \left(\frac{y_2}{z_1}\right) \xi.$$

For our geometric purposes, we think of L and M in the following alternative manner. On M, consider the global sections $\Sigma_0 = \{[z_1 : z_2]; 0\}$ and $\Sigma_\infty = \{[z_1 : z_2]; \infty\}$ and define a map $\Psi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \widehat{M}$, where $\widehat{M} = M \backslash (\Sigma_0 \cup \Sigma_\infty)$, given as

$$\Psi : (z_1, z_2) \mapsto ([z_1 : z_2]; z_\alpha)$$

if $z_\alpha \neq 0$. Because $([z_1 : z_2]; z_\alpha) \sim ([z_1 : z_2]; z_\beta)$ whenever $z_\alpha \neq 0$ and $z_\beta \neq 0$, Ψ is well defined. Moreover, one can check that Ψ is biholomorphism. We then think of M as $\mathbb{C}^2 \backslash \{0\}$ with one $\mathbb{CP}^1$ glued at 0 (the section Σ_0) and another at ∞ (the section Σ_∞) and of $L = \widehat{M} \cup \Sigma_0$ as $\mathbb{C}^2 \backslash \{0\}$ with a $\mathbb{CP}^1$ glued at 0.

B. Comparison principle

The comparison principle used in Section 3 for equation (3.4) is similar in spirit to [1, Lemma 3]. For the reader's convenience, we outline the proof in what follows.

Since the evolving function y in Section 3 is non-negative for all times, any point of contact between y and one of the barriers must be a point where the barrier is non-negative. Moreover, one can check that the barriers used have spatial second derivative bounded in time. We can then reduce our analysis to the following:

Proposition B.1. *Let $y^-(\phi, \tau)$ and $y^+(\phi, \tau)$ be non-negative sub- and super-solutions, respectively, of $(\partial_\tau - \mathcal{E})[\cdot]$ on the interval $[1, (b_0 - 3a_0)e^\tau + 3]$. Suppose that either y^+ or y^- satisfy the uniform bound $y_{\phi\phi} < C$ for some constant $C < \infty$ on a compact space-time set $[1, (b_0 - 3a_0)e^{\overline{\tau}} + 3] \times [0, \overline{\tau}]$. Moreover, assume that*

(i) *$y^+(\phi, 0) > y^-(\phi, 0)$ in $(1, b_0 - 3a_0 + 3)$,*

(ii) *$y^+(1, \tau) \geq y^-(1, \tau)$ and $y^+((b_0 - 3a_0)e^\tau + 3, \tau) \geq y^-((b_0 - 3a_0)e^\tau + 3, \tau)$, for any $\tau \in [0, \overline{\tau}]$.*

Then, one must have $y^+(\phi, \tau) \geq y^-(\phi, \tau)$ in $(1, (b_0 - 3a_0)e^{\overline{\tau}} + 3) \times [0, \overline{\tau}]$.

Proof. Suppose first that $y^+_{\phi\phi} < C$. For some $\lambda > 0$ to be chosen later and any $\alpha > 0$, define

$$\text{(B.1)} \qquad\qquad w = e^{-\lambda\tau}(y^+ - y^-) + \alpha.$$

Then $w > 0$ on the parabolic boundary of our evolution. We will prove w is also positive in the interior of our domain and the lemma will follow by letting $\alpha \searrow 0$.

Assuming the contrary, there must be an interior point ϕ_0 and a first time τ_0 such that $w(\phi_0, \tau_0) = 0$. Then $w_\tau(\phi_0, \tau_0) \leq 0$, and at (ϕ_0, τ_0):

$$y^+ = y^- - \alpha e^{\lambda\tau_0}, \quad y^+_\phi = y^-_\phi, \quad y^+_{\phi\phi} \geq y^-_{\phi\phi}.$$

Thus, at that point, we have $0 \geq e^{\lambda\tau}w_\tau$ and

$$
\begin{aligned}
e^{\lambda\tau}w_\tau &= y^+_\tau - y^-_\tau - \lambda(y^+ - y^-) \\
&= y^-(y^+_{\phi\phi} - y^-_{\phi\phi}) + (y^- - y^+)\left[\lambda - y^+_{\phi\phi} - 1 + \frac{y^+ + y^-}{\phi^2}\right].
\end{aligned}
$$

But if we use the uniform bound $y^+_{\phi\phi} < C$, we have

$$0 \geq w_\tau > \alpha e^{\lambda\tau}\left[\lambda - C - 1 + \frac{y^+ + y^-}{\phi^2}\right],$$

and, since

$$\frac{y^+ + y^-}{\phi^2} \geq 0,$$

this is a contradiction for any $\lambda > C + 1$. The result then follows in the case $y^+_{\phi\phi} < C$.

To prove the lemma in the case that the subsolution y^- satisfies the uniform bound

$$y^-_{\phi\phi} < C,$$

one uses the fact that at a first interior zero (ϕ_0, τ_0) of w one has

$$e^{\lambda\tau}w_\tau = y^+(y^+_{\phi\phi} - y^-_{\phi\phi}) + (y^- - y^+)\left[\lambda - y^-_{\phi\phi} - 1 + \frac{y^+ + y^-}{\phi^2}\right]. \qquad \square$$

References

[1] *S. Angenent, M. C. Caputo* and *D. Knopf,* Minimally invasive surgery for Ricci flow singularities, J. reine angew. Math., to appear.

[2] *S. Angenent, J. Isenberg* and *D. Knopf,* Formal matched asymptotics for degenerate Ricci flow neckpinches, Nonlinearity **24** (2011), 2265–2280.

[3] *S. Angenent* and *D. Knopf,* An example of neckpinching for Ricci flow on S^{n+1}, Math. Res. Lett. **11** (2004), no. 4, 493–518.

[4] *S. Angenent* and *D. Knopf,* Precise asymptotics of the Ricci flow neckpinch, Comm. Anal. Geom. **15** (2007), no. 4, 773–844.

[5] *C. Böhm* and *B. Wilking,* Manifolds with positive curvature operators are space forms, Ann. of Math. (2) **167** (2008), no. 3, 1079–1097.

[6] *S. Brendle* and *R. Schoen,* Manifolds with 1/4-pinched curvature are space forms, J. Amer. Math. Soc. **22** (2009), no. 1, 287–307.

[7] *E. Calabi*, Extremal Kähler metrics, in: Seminar on differential geometry, Ann. of Math. Stud. **102**, Princeton University Press, Princeton (1982), 259–290.

[8] *H.-D. Cao*, Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, Invent. Math. **81** (1985), no. 2, 359–372.

[9] *H.-D. Cao*, Existence of gradient Kähler–Ricci solitons, in: Elliptic and parabolic methods in geometry (Minneapolis 1994), A K Peters, Wellesley (1996), 1–16.

[10] *H.-D. Cao*, Recent progress on Ricci solitons, in: Recent advances in geometric analysis, Adv. Lect. Math. (ALM) **11**, International Press, Somerville (2010), 1–38.

[11] *B.-L. Chen*, Strong uniqueness of the Ricci flow, J. Differential Geom. **82** (2009) no. 2, 363–382.

[12] *X. Chen and S. Sun*, Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics, preprint 2011, `http://arxiv.org/abs/1004.2012`.

[13] *J. Enders*, *R. Müller* and *P. M. Topping*, On Type-I singularities in Ricci flow, Comm. Anal. Geom. **18** (2011), no. 5, 905–922

[14] *M. Feldman*, *T. Ilmanen* and *I. Knopf*, Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons, J. Differential Geom. **65** (2003), no. 2, 169–209.

[15] *F. T.-H. Fong*, Kähler–Ricci flow on projective bundles over Kähler–Einstein manifolds, Trans. Amer. Math. Soc., to appear.

[16] *H.-L. Gu* and *X.-P. Zhu*, The existence of type II singularities for the Ricci flow on S^{n+1}, Comm. Anal. Geom. **16** (2008), no. 3, 467–494.

[17] *S. J. Hall* and *T. Murphy*, On the linear stability of Kähler–Ricci solitons, Proc. Amer. Math. Soc. **139** (2011), no. 9, 3327–3337.

[18] *R. S. Hamilton*, Three-manifolds with positive Ricci curvature, J. Differential Geom. **17** (1982), no. 2, 255–306.

[19] *R. S. Hamilton*, The formation of singularities in the Ricci flow, in: Surveys in differential geometry, vol. II (Cambridge 1993), International Press, Cambridge (1995), 7–136.

[20] *T. Ivey*, Ricci solitons on compact three-manifolds, Differential Geom. Appl. **3** (1993), no. 4, 301–307.

[21] *N. Koiso*, On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics, in: Recent topics in differential and analytic geometry, Adv. Stud. Pure Math. **18**, Academic Press, Boston (1990), 327–337.

[22] *G. Lieberman*, Second order parabolic differential equations, rev. ed., World Scientific, River Edge 2005.

[23] *D. Máximo*, Non-negative Ricci curvature on closed manifolds under Ricci flow, Proc. Amer. Math. Soc. **139** (2011), no. 2, 675–685.

[24] *G. Perelman*, The entropy formula for the Ricci flow and its geometric applications, preprint 2002, `http://arxiv.org/abs/math/0211159`. arxiv:0211159.

[25] *G. Perelman*, Ricci flow with surgery on three-manifolds, preprint 2003, `http://arxiv.org/abs/math/0303109`.

[26] *N. Šešum*, Curvature tensor under the Ricci flow, Amer. J. Math. **127** (2005), no. 6, 1315–1324.

[27] *J. Song*, Some type I solutions of Ricci flow with rotational symmetry, preprint 2012, `http://arxiv.org/abs/1203.2688`.

[28] *J. Song* and *G. Tian*, The Kähler–Ricci flow through singularities, preprint 2009, `http://arxiv.org/abs/0909.4898`.

[29] *J. Song* and *B. Weinkove*, The Kähler–Ricci flow on Hirzebruch surfaces, J. reine angew. Math. **659** (2011), 141–168

[30] *J. Song* and *B. Weinkove*, Contracting exceptional divisors by the Kähler–Ricci flow, preprint 2011, `http://arxiv.org/abs/1003.0718`.

[31] *J. Song* and *B. Weinkove*, Contracting exceptional divisors by the Kähler–Ricci flow II, preprint 2011, `http://arxiv.org/abs/1102.1759`.

[32] *J. Song* and *B. Weinkove*, Lecture notes on the Kähler–Ricci flow, preprint 2011.

[33] *G. Tian* and *Z. Zhang*, On the Kähler–Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192.

[34] *Z. Zhang*, Ricci lower bound for Kähler–Ricci flow, preprint 2011, `http://arxiv.org/abs/1110.5954`.

———————————

Davi Máximo, Department of Mathematics, University of Texas at Austin, TX 78712, USA
e-mail: maximo@math.utexas.edu