COMPLETE RIEMANNIAN 4-MANIFOLDS WITH UNIFORMLY POSITIVE SCALAR CURVATURE

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Abstract. We obtain topological obstructions to the existence of complete Riemannian with uniformly positive scalar curvature on certain (non-compact) 4-manifolds. In particular, such a metric on the interior of a compact contractible 4-manifold uniquely distinguishes the standard 4-ball up to diffeomorphism among Mazur manifolds and up to homeomorphism in general.

We additionally show there exist uncountably many exotic $\mathbb{R}^4$'s that do not admit such a metric and that any (non-compact) tame 4-manifold has a smooth structure that does not admit such a metric.

1. Introduction

This article is concerned with the relationship between the *scalar curvature* of a Riemannian manifold and the topology of the underlying smooth manifold. Scalar curvature (defined in (A.1)) $R \in C^\infty(M)$ is the weakest curvature invariant of a Riemannian manifold (M, g) but the existence of positive scalar curvature can place strong topological constraints on the underling topology. We survey some known results along these lines in Section 2 and Appendix A.

One easily sees that for $n \geq 3$, the standard $\mathbb{R}^n$ admits a complete metric with uniformly positive scalar curvature $R \geq 1$ by capping off a half-cylinder $(-\infty, 0] \times S^{n-1}$ with a hemisphere S^n_+ (and then smoothing). A natural question is whether or not the existence of such a metric *distinguishes* the standard $\mathbb{R}^n$ from other contractible manifolds. Indeed, for 3-manifolds this is true: work of Chang–Weinberger–Yu [CWY10, Theorem 1] shows that $\mathbb{R}^3$ is the only contractible M^3 that admits a complete Riemannian metric with uniformly positive scalar curvature (cf. Remark 3.3). See also [BBMM21, Don23, Wan23].

In this article we study this problem for 4-manifolds. We are able to completely resolve the problem in the case that M is assumed to be a the interior of a *Mazur manifold*, namely M is the interior of a compact contractible smooth 4-manifold with boundary W admitting a (smooth) handle decomposition with one 0-handle, one 1-handle, and one 2-handle.

Theorem 1.1. *Suppose that M is the interior of a Mazur manifold W. If M admits a complete Riemannian metric with scalar curvature $R \geq 1$ then W must be diffeomorphic to the 4-ball B^4.*

(Of course it follows from this that M is diffeomorphic to the standard $\mathbb{R}^4$.) The main ingredient in the proof of Theorem 1.1 is a complete characterization of the homeomorphism type of contractible tame 4-manifolds admitting a complete metric of uniformly positive scalar curvature as follows:

Theorem 1.2. *Suppose that M is the interior of a compact contractible smooth 4-manifold with boundary W. If M admits a complete Riemannian metric with scalar curvature $R \geq 1$ then W must be homeomorphic to the 4-ball B^4.*

We note that Chang–Weinberger–Yu [CWY10] previously proved (using different methods) that certain Mazur manifolds cannot admit complete metrics of uniformly positive scalar curvature. See also [BW99, CWY17, CWY20, WXY23].

Our methods also show that uniformly positive scalar curvature can detect some exotic smooth structures. While it is not known whether exotic 4-balls can exist, there exists uncountably many non-diffeomorphic $\mathbb{R}^4$'s [DMF92, Gom93]. We show that many do not admit a complete metric with uniformly positive scalar curvature:

Theorem 1.3. *There exist uncountably many exotic $\mathbb{R}^4$ which do not admit a complete Riemannian metric with uniformly positive scalar curvature $R \geq 1$.*

In fact, for general tame 4-manifolds, we show that it's always possible to change the smooth structure to prevent the existence of such a metric:

Theorem 1.4. *Suppose W is a smooth 4-manifold with boundary. Then the interior of W admits a smooth structure that does not admit a complete Riemannian metric with uniformly positive scalar curvature $R \geq 1$.*

Motivated by our results, it's natural to ask if the standard $\mathbb{R}^4$ is the unique contractible smooth 4-manifold that admits a complete metric of uniformly positive scalar curvature.

Remark 1.5. The methods here are insensitive to the scalar curvature of on a compact set (cf. Remark 3.2). For example, Theorem 1.4 shows that for some smooth structure on the once punctured $K3$ manifold does not admit a complete metric of uniformly positive scalar curvature. However, we have been unable to determine whether or not the standard smooth structure has such a metric. (The standard $K3$ manifold does not admit a metric of positive scalar curvature by spin-theoretic obstructions [Lic63].)

In a related direction, we can show that the end Floer homology as defined by Gadgil [Gad09] (see Section 5) can obstruct the existence of complete metrics of uniformly positive scalar curvature.

Theorem 1.6. *Let X be a non-compact smooth 4-manifold such that the end Floer homology $HE(X)$ is non-trivial for at least one end, then X does not admit a complete metric of scalar curvature $R \geq 1$.*

1.1. **On $R > 0$.** We note that the condition that M admits a complete metric of positive scalar curvature $R > 0$ (without the strict positive assumption) is much less restrictive. For example, the product of S^1 with a positively curved metric on $\mathbb{R}^2$ (e.g. a paraboloid) yields such a metric on the genus-one handlebody (which does not admit a complete metric of uniformly positive curvature by e.g. combining Corollary 3.4 with Proposition 2.1 below).

We note that even in 3-dimensions, the classification of manifolds that admit a complete metric of positive curvature is a well-known open problem (cf. [Yau82, # 27]). Some progress in the contractible case has been achieved (e.g. genus one contractible 3-manifolds such as the Whitehead manifold does not admit such a metric [Wan19, Wan24]) but it's still unknown if $\mathbb{R}^3$ is the unique contractible 3-manifold admitting such a metric. It's also unknown whether or not the genus-two (or higher) handlebody admits such a metric. We additionally refer to [CL24, CLSZ21, Che24, CRZ23, CCZ24, Lot24, Zhu24] for other results in this direction.

1.2. **Outline of techniques.** The main idea of this article is to combine Gromov's μ-bubble technique (cf. Section 3) with techniques from 3- and 4-manifold topology. The μ-bubble technique generalizes and localizes the Schoen–Yau stable minimal hypersurface method, and is particularly effective in the setting of non-compact manifolds. The Schoen–Yau minimal surface method was previously combined with some gauge theoretic methods in [Lin19, LRS23, KT20] to study whether or not certain (rational) homology $S^1 \times S^3$ admit metrics of positive scalar curvature.

The proofs of Theorems 1.1 and 1.2 can be roughly described as follows. The μ-bubble technique in conjunction with the classification of positive scalar curvature 3-manifolds places strong constraints (cf. Corollary 3.4) on the topology of ∂W if W is a smooth compact 4-manifold whose interior admits a complete Riemannian metric of uniformly positive scalar curvature. On the other hand, if W is assumed to be contractible then ∂W must also be an integral homology sphere. These facts can be combined (cf. Corollary 2.3) to prove that ∂W is diffeomorphic to either S^3 or $(\#_{\ell=1}^L P)\#(\#_{m=1}^M -P)$ for P the Poincaré homology sphere (and $-P$ an oppositely oriented copy). At this point, the assertions follow from deep results in topology.

The proofs of Theorems 1.3, 1.4, and 1.6...

1.3. **Organization.** In Section 2 we recall several classification results concerning positive scalar curvature that we will use later. Section 3 contains a discussion of the μ-bubble method. The end Floer homology obstruction is discussed in Section 5. Then in Section 6 we construct various examples of exotic smooth structures that do not admit complete metrics with uniformly positive scalar curvature. Finally, in Appendix A we survey some general classification results of positive scalar curvature and in Appendix B we discuss perturbed Heegaard Floer homology.

1.4. **Acknowledgements.** We are grateful to Ian Agol, Simone Cecchini, Bob Gompf, Hokuto Konno, Ciprian Manoulescu, and Ian Zemke for interesting conversations related to this work. O.C. was supported by a Terman Fellowship and an NSF grant (DMS-2304432).

2. Overview of PSC classification results

In this section we review some results concerning the topology of (closed) manifolds admitting metrics with positive scalar curvature (PSC). A classification of smooth closed (compact, no boundary) oriented Riemannian manifolds (M^n, g) with positive scalar curvature is available when $n = 2, 3$:

- When $n = 2$, M is diffeomorphic to S^2.
- When $n = 3$, M is diffeomorphc to $(\#_{j=1}^J S^3/\Gamma_j)\#(\#_{k=1}^K S^2 \times S^1)$ [Per03] (see also [GL83, SY79b, SY79a]).

In fact, we will need the following (well-known) mapping version of these results. It would be possible to state these results in the non-orientable case as well (with appropriate modifications) but we will not need this in the sequel.

Proposition 2.1. *Suppose that X^2 is a closed connected oriented surface and (M^2, g) is a closed oriented surface with a Riemannian metric of positive scalar curvature. If there is a map of non-zero degree $M \to X$ then X is diffeomorphic to S^2.*

Proof. Since $R = 2K$ in 2-dimensions, where K is the Gaussian curvature, the Gauss–Bonnet theorem implies that $\chi(M) > 0$ and thus M is diffeomorphic to S^2.

The only X with $S^2 \to X$ non-zero degree is S^2 (any other X has $\pi_2(X) = 0$ by considering the universal cover, so the map would have to be null-homologous). □

Proposition 2.2. *Suppose that X^3 is a closed connected oriented 3-manifold and (M^3, g) is a closed oriented 3-manifold with a Riemannian metric of positive scalar curvature. If there is a map of non-zero degree $M \to X$ then X is diffeomorphic to S^3 or else a connected sum of the form $(\#_{j=1}^J S^3/\Gamma_j)\#(\#_{k=1}^K S^2 \times S^1)$.*

Proof. This follows from [GL83, SY79a, Per03] as explained in [Ago17]. We sketch an alternative approach. By Perelman's proof of the Poincaré conjecture and the Kneser–Milnor prime decomposition, if X is not of the asserted form, then $X = X'\#K$ with K a closed aspherical (having contractible universal cover) 3-manifold. As such, by composing with the map crushing X' to a point, it suffices to rule out the case that that M admits a non-zero degree map $f : M \to K$ for K a closed oriented aspherical 3-manifold.

This contradicts known results about positive scalar curvature and aspherical manifolds as described in [Cho21, Proposition 7.23] (cf. [CL24]). We sketch the proof for completeness. One may lift to $\tilde{f} : \hat{M} \to \tilde{K}$ proper of non-zero degree, where $\tilde{K}$ is the universal cover and $\hat{M}$ is an appropriate cover (cf. [CLL23, Lemma 18] or [Cho21, Lemma 7.22]). We can then find linked embedding of $\mathbb{R}$ and S^1 into $\tilde{K}$ at a distance $L \gg 1$ apart (this uses $\tilde{K}$ contractible). On the other hand, since $(\hat{M}, \hat{g})$ has uniformly positive scalar curvature (being a cover of a closed Riemannian manifold of positive scalar curvature) the preimage of the S^1 (perturbing f slightly) can be filled by a minimal disk of bounded in-radius by diameter bounds for stable minimal hypersurfaces in positive scalar curvature [SY83]. After pushing this filling disk back to $\tilde{K}$, any point in the intersection with the linked $\mathbb{R}$ must have bounded distance to the S^1. This contradicts the fact that we can take the distance $L \gg 1$ arbitrarily large. □

The following corollary will be used in the sequel. Although it follows from known results, we could not find this exact statement in the literature and expect it to be useful in other settings. We begin by recalling the binary icosahedral group $I^* = \langle s, t (st)^2 = s^3 = t^5 \rangle$. Letting s, t act on $\mathbb{R}^4$ as quaternions $s = \frac{1}{2}(1 + i + j + k), t = \frac{1}{2}(\varphi + \varphi^{-1}i + j)$ (where $\varphi = \frac{\sqrt{5}-1}{2}$) one may check that $I^* \subset SO(4)$ acts on S^3 without fixed points. Thus $P = S^3/I^*$ is a spherical spaceform, so admits a metric of constant positive sectional (and thus scalar curvature). Writing $-P$ for P with the opposite orientation, clearly $-P$ admits positive scalar curvature as well. Since positive scalar curvarture is preserved under connected sum (in dimensions ≥ 3) we see that $(\#_{\ell=1}^L P)\#(\#_{m=1}^M - P)$ admits a metric of positive scalar curvature for any $L, M \geq 0$. On the other hand, it's well known that $H_*(P; \mathbb{Z}) = H_*(S^3; \mathbb{Z})$, i.e. P is an oriented integer homology sphere (this follows from the fact that I^* is a perfect group, combined with Poincaré duality).

Since the connected sum of integral homology spheres is again an integral homology sphere (this follows from Mayer–Vietoris) we conclude that $(\#_{\ell=1}^L P)\#(\#_{m=1}^M - P)$ is an integral homology sphere that admits a metric of positive scalar curvature.

Corollary 2.3. *Suppose that (M^3, g) is a closed oriented 3-manifold with a Riemannian metric of positive scalar curvature. Suppose also that M is an integral homology sphere,$H_*(M; \mathbb{Z}) = H_*(S^3; \mathbb{Z})$. Then M is diffeomorphic to either S^3 or the connected sum of Poincaré homology spheres $(\#_{\ell=1}^L P)\#(\#_{m=1}^M - P)$.*

Proof. By the classification of PSC 3-manifolds (see Proposition 2.2) we conclude that M is diffeomorphic to S^3 or

$$(\#_{j=1}^J S^3/\Gamma_j)\#(\#_{k=1}^K S^2 \times S^1).$$

Since M is oriented, we can use Mayer–Vietoris to conclude that

$$H_i((\#_{j=1}^J S^3/\Gamma_j)\#(\#_{k=1}^K S^2 \times S^1);\mathbb{Z}) = (\oplus_{j=1}^j H_i(S^3/\Gamma_j);\mathbb{Z}) \oplus (\oplus_{k=1}^K H_i(S^2 \times S^1;\mathbb{Z})$$

for $i = 1, 2$. Since $H_1(S^2 \times S_1;\mathbb{Z}) = \mathbb{Z}$ we see that $K = 0$. Finally, one may consider the groups Γ_j in the first summand and check that the only one with trivial abelianization is I^* (and the trivial group). See [Ker69, Theorem 2]. This proves the assertion. □

In dimensions ≥ 4, a complete classification of closed manifolds admitting positive scalar curvature is not known. Several partial classification results have been obtained. We survey some of the techniques and results in Appendix A.

We briefly discuss the following result recently obtained by Räde [R2̈3, Proposition 2.17] (for $n \geq 6$) using surgery theory (see also [GL80, Sto92, CRZ23]) since it will be referenced in the sequel:

Proposition 2.4. *For $n \in \{3, 4\} \cup \{6, 7, \dots\}$, suppose that Y is a closed connected oriented smooth manifold and $\Sigma \subset Y \times \mathbb{R}$ is a closed embedded separating hypersurface. If Σ admits positive scalar curvature, then so does Y.*

The projection map $Y \times \mathbb{R} \to Y$ restricts to Σ to yield a degree 1 map $\Sigma \to Y$. Thus, Proposition 2.4 is a consequence of Propositions 2.1 and 2.2 when $n = 3, 4$. For the results in this article concerned with 4-manifolds, we'll only need this version of Proposition 2.4 (as opposed to the high dimensional surgery result from [R2̈3]).

Remark 2.5. We emphasize that there is a counterexample to Proposition 2.4 with $n = 5$, see [Ros07, Remark 1.25].

3. Exhaustions via μ-bubbles

The following result is due to Gromov. After describing some consequences, we will explain the proof for completeness.

Proposition 3.1 ([Gro23, §3.7.2])**.** *For $3 \leq n \leq 7$ suppose that (M^n, g) is a complete Riemannian manifold with scalar curvature $R \geq 1$. Then there's an exhaustion $\Omega_1 \subset \Omega_2 \subset \Omega_3 \dots$ with $M = \cup_{i=1}^\infty \Omega_i$ where the Ω_i are compact with smooth boundaries $\partial\Omega_i$. The $(n-1)$-manifolds $\partial\Omega_i$ admit metrics of positive scalar curvature.*

Remark 3.2. As will be clear from the proof, the same fact would hold with the weaker requirement that $R \geq 1$ outside of a compact set.

The dimensional restriction $n \leq 7$ is due to the potential presence of singularities in area-minimizing hypersurfaces in 8-dimensional (and higher) manifolds. One may hope that with technical improvements in the study of generic regylarity of (generalized) area minimizing hypersurfaces, one might be able to remove this condition (cf. [HS85, CMS23]).

Remark 3.3. We note that Proposition 3.1 (and the fact that S^2 is the only oriented closed 2-manifold admitting positive scalar curvature) implies that if (M^3, g) is a complete oriented Riemannian manifold with uniformly positive scalar curvature, then it admits an exhaustion by Ω_i with $\partial\Omega_i$ the disjoint union of S^2's. For example, this implies that $\mathbb{R}^3$ is the only contractible 3-manifold that admits such a metric, as originally proven in [CWY10].

We have the following important consequences of Proposition 3.1.

Corollary 3.4. *For $n \in \{3,4,6,7\}$, suppose that W is a compact smooth n-manifold with boundary and that some component of ∂W does not admit a Riemannian metric with positive scalar curvature. Then the interior of W does not admit a complete Riemannian metric of with uniformly positive scalar curvature $R \geq 1$.*

Remark 3.5. The restrictions on the dimension is enforced by Propositions 2.4 and 3.1. (See Remark 2.5.)

Proof of Corollary 3.4. Let M denote the interior of W. The collar neighborhood theorem yields a neighborhood of infinity U diffeomorphic to $\partial W \times \mathbb{R}$. For i sufficiently large, the exhaustion obtained in Proposition 3.1 will have $\partial \Omega_i$ a separating hypersurface in U. The assertion follows by combining Propositions 2.4 and 3.1. ☐

Corollary 3.6. *Suppose (M^4, g) is complete smooth oriented with scalar curvature $R \geq 1$, then there's an exhaustion $\Omega_1 \subset \Omega_2 \subset \dots$ so that each component of $\partial\Omega_i$ is a PSC 3-manifold, namely of the form $(\#_{j=1}^J S^3/\Gamma_j)\#(\#_{k=1}^K S^2 \times S^1)$.*

Proof. Combine Propositions 3.1 and 2.2. ☐

3.1. **μ-bubbles and the proof of Proposition 3.1.** The basic method goes back to work of Schoen–Yau who used studied the second variation of area at area minimizing hypersurfaces [SY79a, SY79b]. They observed that if Σ^{n-1}, $n \geq 3$, is a 2-sided stable minimal hypersurface inside of a Riemannian manifold (M^n, g) with positive scalar curvature then a suitable conformal deformation of the induced metric $g|_\Sigma$ yields a metric of positive scalar curvature on Σ. This gives a topological obstruction to positive scalar curvature when paired with an appropriate existence result for minimal hypersurfaces.

A careful examination of Schoen-Yau argument reveals that it relies mostly on the second variation of area and not on minimality per se. Following [CL24], we describe next an idea of Gromov [Gro23] in which one gives up minimality by considering instead modified area functional. The key point is that the minimization of this functions can be *localized* and the second variation of the functional can still obstruct positivity of scalar curvature in certain cases.

For $3 \leq n \leq 7$, suppose (M^n, g) is a Riemannian manifold with non-empty boundary ∂M. Assume $\partial M = \partial_- M \sqcup \partial_+ M$ is labeling of the boundary components such that each of them is non-empty and let h be a smooth function on $\mathring{M}$ such that

$$h \to +\infty \text{ on } \partial_- M, \quad h \to -\infty \text{ on } \partial_+ M.$$

Let Ω_0 be a Caccioppoli set with smooth boundary $\partial\Omega_0 \subset \mathring{M}$. For all Caccioppoli sets Ω in M with $\Omega \triangle \Omega_0 \Subset \mathring{M}$, we consider the μ-bubble functional:

$$\mathcal{A}(\Omega) = |\partial^* \Omega| - \int_M (\chi_\Omega - \chi_{\Omega_0}) h \, d vol. \tag{3.1}$$

A minimizer of $\mathcal{A}$ is called a μ-bubble. As the next proposition shows, they are easier to construct than stable minimal hypersurfaces (see Proposition 12 of [CL24] for a proof).

Proposition 3.7. *There exists a smooth minimizer Ω for $\mathcal{A}$ such that $\Omega \triangle \Omega_0$ is compactly contained in the interior of M.*

The first and second variations of the μ-bubble functional are as follows. If Ω_t is a smooth 1-parameter family of regions with $\Omega_0 = \Omega$ and normal speed φ at $t = 0$, then

$$\frac{d}{dt}\mathcal{A}(\Omega_t) = \int_{\partial\Omega_t} (H - h)\varphi$$

where H is the mean curvature of $\partial\Omega_t$. In particular, a μ-bubble must satisfy $H = h$ along $\partial\Omega = \Sigma$.

Remark 3.8. The first variation can be used to formally explain why a minimizer should exist: $\partial_- M$ has mean curvature $H_{\partial_- M} - h = -\infty$ and thus acts as a strict barrier for a minimizing sequence for $\mathcal{A}(\cdot)$. Similar considerations hold for $\partial_+ M$, and thus a minimizing sequence Ω_j has $\Omega_j \Delta \Omega_0$ compactly contained in the interior of M. Thus, standard arguments in geometric measure theory allow one to pass to a weak limit to find Ω minimizing $\mathcal{A}(\cdot)$ so that $\Omega\Delta\Omega_0$ compactly contained in the interior of M. Since $\mathcal{A}(\cdot)$ is a perturbation of the area functional (at small scales in the interior of M), standard regularity theory implies that $\partial\Omega = \Sigma$ is a smooth compact submanifold contained in the interior of M. (When $n \geq 8$ there could exist singularities along $\partial\Omega$, so we cannot directly apply this method without further modification.)

Assuming that $\partial\Omega$ satisfies $H = h$ along Σ, the second variation is

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}(\Omega_t) = \int_\Sigma |\nabla_\Sigma \varphi|^2 - \frac{1}{2}(R_M - R_\Sigma + |A|^2 + h^2 + 2\langle \nabla_M h, \nu\rangle)\varphi^2 \tag{3.2}$$

See e.g. Lemma 14 [CL24] (one must take $u = 1$ and rearrange the terms slightly). In the above formula $\mathring{A}$ is the trace-free part of the second fundamental form of Σ (this will not matter since we'll discard it using $|\mathring{A}|^2 \geq 0$ in the sequel), R_M is the scalar curvature of M and R_Σ is the scalar curvature of the induced metric on Σ.

The minimizer Ω obtained above will satisfy the stability inequality: $\frac{d^2}{dt^2}|_{t=0}\mathcal{A}(\Omega_t) \geq 0$ for any $\varphi \in C^\infty(\Sigma)$.

Proposition 3.9. *For $3 \leq n \leq 7$, suppose (M^n, g) is a Riemannian manifold with non-empty boundary and scalar curvature $R \geq \Lambda > 0$. Assume $\partial M = \partial_- M \sqcup \partial_+ M$ is a labeling of the boundary components such that each of them is non-empty. There exists a constant $D = D(\Lambda) > 0$ such that if the distance $d(\partial_- M, \partial_+ M) > D$, then in the interior of M there must be a there exists a smoothly embedded closed 2-sided hypersurface Σ^{n-1} which itself admits a metric with positive scalar curvature.*

The estimate for $D(\Lambda)$ given below could be strengthened to give a sharp/explicit estimate, but we will not bother with this here since it will not matter in the sequel. See e.g. [Gro96, Gro23, Cho21] and references therein.

Proof. It suffices to assume there is $\rho : M \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ smooth with $\mathrm{Lip}(\rho) \leq \frac{\pi}{D}$, $\rho^{-1}(\pm\frac{\pi}{2}) = \partial_\pm M$. Indeed, one may smooth out (and rescale) the distance function from $\partial_- M$ to obtain such ρ with $\partial_- M = \rho^{-1}(-\frac{\pi}{2})$ and $\frac{\pi}{2}$ a regular value.

Then we can define M to be $\rho^{-1}([-\frac{\pi}{2}, \frac{\pi}{2}])$. Then define $h(x) = -\frac{\pi}{D}\tan(\rho(x))$ on the interior of M. Note that

$$h^2 + 2\langle \nabla_M h, \nu\rangle \geq h^2 - 2|\nabla_M h| \geq \frac{\pi^2}{D^2}\left(\tan(\rho(x))^2 - \sec(\rho(x))^2\right) = -\frac{\pi^2}{D^2}.$$

Using this in the stability inequality (3.2) (and using $R_M \geq \Lambda$, and discarding the $|\mathring{A}|^2$ term) we obtain

$$\int_\Sigma |\nabla_\Sigma \varphi|^2 + \frac{1}{2} R_\Sigma \varphi^2 - \frac{1}{2}\left(\Lambda - \frac{\pi^2}{D^2}\right)\varphi^2 \geq 0.$$

As long as $D \geq D(\Lambda) > 0$ is sufficiently large, the term $\frac{1}{2}\left(\Lambda - \frac{\pi^2}{D^2}\right)$ will be uniformly positive. To be definite we can take $D(\Lambda)^2 = \frac{2\pi}{\Lambda}$ so this term will be $\geq \frac{\Lambda}{4}$.

Thus

$$\int_\Sigma |\nabla_\Sigma \varphi|^2 + \frac{1}{2} R_\Sigma \varphi^2 \geq \frac{\Lambda}{4}\int_\Sigma \varphi^2. \tag{3.3}$$

When $n = 3$, $\dim \Sigma = 2$, so $R_\Sigma = 2K_\Sigma$ is the Gaussian curvature and we can take $\varphi = 1$ on any connected component $\Sigma' \subset \Sigma$ to find

$$2\pi\chi(\Sigma') = \int_{\Sigma'} K_\Sigma > 0,$$

so Σ' is a sphere (or possibly projective plane if M is nonorientable) and thus admits positive scalar curvature.

When $n \geq 4$ we observe that

$$\frac{2(m-2)}{m-3} \geq 1$$

so using $|\nabla_\Sigma \varphi|^2 \geq 0$ we find that

$$\int_\Sigma \frac{2(m-2)}{m-3}|\nabla_\Sigma \varphi|^2 + \frac{1}{2} R_\Sigma \varphi^2 \geq \frac{\Lambda}{4}\int_\Sigma \varphi^2,$$

i.e. the operator

$$L_\Sigma := -\frac{4(m-2)}{m-3}\Delta_\Sigma + R_\Sigma$$

is a positive operator (e.g. all eigenvalues are positive). Letting $u > 0$ denote the first eigenfunction of L_Σ (so $L_\Sigma u = \lambda u > 0$), a computation (see p. 9 in [SY79a]) implies that $\tilde{g}_\Sigma = u^{\frac{4}{n-3}} g_\Sigma$ has scalar curvature

$$\tilde{R} = u^{\frac{m+1}{m-3}} L u > 0.$$

This completes the proof. □

We can now prove the exhaustion result.

Proof of Proposition 3.1. Fix a compact region M_1' with smooth boundary so that for some $p \in M_1'$, $d(p, \partial M_1') > D + \varepsilon$ for $D = D(\Lambda) > 0$ as in Proposition 3.9, and $\varepsilon > 0$ small enough so that $B_\varepsilon(p)$ has smooth boundary. Let $M_1 = \overline{M_1' \setminus B_\varepsilon(p)}$ and set $\partial_- M_1 = \partial B_\varepsilon(p)$ and $\partial_+ M_1 = \partial M_1'$. Proposition 3.9 produces $\Omega \subset M_0$ so that $\partial\Omega = \partial B_\varepsilon(p) \cup \partial'\Omega$ and each component of $\partial'\Omega$ admits positive scalar curvature. Set $\Omega_1 = \Omega \cup B_\varepsilon(p)$. Assume that $\Omega_1 \subset \dots \Omega_{i-1}$ have been chosen as in the statement of the theorem. Choose a smooth compact region $\Omega_{i-1} \subset M_i'$ with $d(\partial M_i', \Omega_{i-1}) > D$. As before, applying Proposition 3.9 to $M_i := \overline{M_i' \setminus \Omega_{i-1}}$ yields $\Omega \subset M_i$ whose boundary components in the interior of M_i admit positive scalar curvature. Thus, setting $\Omega_i := \Omega \cup \Omega_{i-1}$ completes the inductive step. □

4. Classifying uniformly PSC tame contractible 4-manifolds

We recall the following well-known fact which follows from Lefschetz duality:

Lemma 4.1. *If W is a compact contractible topological 4-manifold then ∂W is an integral homology sphere.*

The next result collects several deep results in topology together to obstruct connect sums of the Poincaré homology sphere P from bounding a *smooth* contractible 4-manifold. (Note that by Friedman's work [Fre82], any integral homology sphere bounds a compact contractible topological 4-manifold.)

Proposition 4.2. *For $L, M \geq 0$ suppose that W is a compact contractible smooth 4-manifold with ∂W diffeomorphic to $(\#_{\ell=1}^{L} P)\#(\#_{m=1}^{M} - P)$. Then $L = M = 0$ ∂W is diffeomorphic to S^3, and W is homeomorphic to the 4-ball.*

Proof. Since ∂W is the boundary of a contractible smooth 4-manifold, it follows that the Heegard Floer d-invariant[1] satisfies $d(\partial W) = 0$ by [OS03, Theorem 1.2]. On the other hand, by [OS03, Theorem 4.3 and Proposition 6.3] we find that

$$d((\#_{\ell=1}^{L} P)\#(\#_{m=1}^{M} - P)) = (L - M)d(P).$$

and $d(P) \neq 0$ by [OS03, §8.1]. Thus $L = M$ so we see that either ∂W is S^3 or $\#_{\ell=1}^{L}(P\# - P)$. The latter case cannot occur by the periodic end theorem of Taubes [Tau87] which implies that if Y is a homology 3-sphere that bounds a negative definite 4-manifold with non-diagonal intersection form (e.g. $Y = \#_{\ell=1}^{L} P$ bounds the boundary connect sum of the E_8-plumbing of spheres) then $Y\# - Y$ cannot bound a contractible 4-manifold.

Thus, ∂W is diffeomorphic to S^3. The work of Friedman [Fre82] implies that W is homeomorphic to the 4-ball. This finishes the proof. □

We now prove that a complete metric with uniformly positive scalar curvature distinguishes the ball up to homeomorphism among tame contractible 4-manifold.

Proof of Theorem 1.2. Consider W a compact contractible smooth 4-manifold and denote by M the interior. Assume that M admits a complete Riemannian metric of with uniformly positive scalar curvature $R \geq 1$. By Corollary 3.4, ∂W admits positive scalar curvature. Furthermore, by Lemma 4.1, ∂W is an integral homology sphere. Thus by Corollary 2.3, we see that ∂W is either S^3 or $(\#_{\ell=1}^{L} P)\#(\#_{m=1}^{M} - P)$. The assertion thus follows from Proposition 4.2. □

Using this, we can now consider the case of W a Mazur manifold.

Proof of Theorem 1.1. Assume that W is a a compact contractible smooth 4-manifold admitting a smooth handle decomposition with one 0-handle, one 1-handle and one 2-handle. Assume that the interior M admits a complete Riemannian metric with uniformly positive scalar curvature $R \geq 1$. By Theorem 1.2 (proven above) we see that W is homeomorphic to the 4-ball.

We now observe that the 2-handle is attached along a knot on the boundary of the 1-handle which is $S^1 \times S^2$. Consequently, we have obtained S^3 by performing surgery along a knot K in $S^1 \times S^2$. Hence, Gabai's property R theorem [Gab87] implies that K is smoothly isotopic to the S^1 factor of $S^1 \times S^2$. In the 4-dimensional handle picture, the attaching sphere of the 2-handle intersects the belt sphere of

[1] Note that a homology 3-sphere has a unique spin^c structure.

the 1-handle geometrically once, allowing us to cancel the 1- and 2-handle. Thus, W must be diffeomorphic to a 4-ball. This completes the proof. □

5. A general obstruction via end Floer homology

In order to obstruct the existence of complete positive scalar curvature metric in open 4-manifolds we will first recall the construction of end Floer homology.

Definition 5.1. Let X be a smooth open 4-manifolds. An asymptotic spin^c structure $\mathfrak{s}$ on X is a spin^c structure defined on $X \setminus K$ for some compact subset $K \subset X$. Two such spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ defined on $X \setminus K_1$ and $X \setminus K_2$ are said to be equal at infinity if there exists a compact set K' which contains both K_1 and K_2 and $\mathfrak{s}_1|_{X\setminus K_1} = \mathfrak{s}_2|_{X\setminus K_2}$.

Definition 5.2. Let X be a smooth 4-manifold. We call $X_1 \subset X_2 \subset \cdots$ be an *exhaustion* if it satisfies the following properties:

(i) $\cup_i X_i = X$,
(ii) X_i is a smooth compact 4-manifold with boundary Y_i for all i.

Moreover, an exhaustion is called *admissible* if the map induced by the inclusion $H^1(X_{i+1} \setminus X_i) \to H^1(Y_{i+1})$ is surjective for all i.

Remark 5.3. If $W = W_1 \cup_Y W_2$ be a cobordism such that the induced map by inclusion $H^1(W_1) \to H^1(Y)$ is surjective then given $\mathfrak{s}_i \in spin^c(W_i)$ for $i = 1, 2$, there exists an unique $spin^c$ structure $\mathfrak{s}$ on W which restricts to $\mathfrak{s}_i$ on W_i.

Remark 5.4. Let W be a cobordism from Y_1 to Y_2 which is obtained from $Y_1 \times [0, 1]$ in one of the following ways:

- Attach a $2-$handle along a knot $K \in Y_1 \times \{1\}$ which represents a primitive, non-torsion element in $H_1(Y_1)$ or,
- W is a rational-homology cobordism, i.e. $H_k(W, Y_i; \mathbb{Q}) = 0$ for all k and $i = 0, 1$ or,
- Attach a $1-$handle along $Y_1 \times \{1\}$,

then the induced map by the inclusion $H^1(W) \to H^1(Y_2)$ is surjective.

Consider a smooth open 4-manifold X with an asymptotic spin^c structure $\mathfrak{s}$. Given an admissible exhaustion $X_1 \subset X_2 \subset \cdots$ of X, we can define the end Floer homology $HE(X, \mathfrak{s})$ as the direct limit of the reduced Floer homology groups $HF^+_{red}(Y_i, \mathfrak{s}|_{Y_i})$, where the morphism is induced by the cobordisms $X_{i+1} \setminus \mathring{X}_i$. To be more precise, suppose ω is a $2-$form on $X \setminus K$ for some compact set $K \subset X$.Now, consider the admissible cobordism $W_{ij} = X_j \setminus \mathring{X}_i$ from Y_i to Y_j. Then we have the ω-twisted induced map $\underline{F}^+_{W_{ij};\omega} : \underline{HF}^+_{\mathrm{red}}(Y_i, \omega|_{Y_i}) \to \underline{HF}^+_{\mathrm{red}}(Y_j, \omega|_{Y_j})$ (see Appendix B). The *end Floer homology* $\underline{HE}(X, \mathfrak{s}, \omega)$ is defined as the direct limit of $\underline{F}^+_{W_{ij};\omega}$. Note that if Y is a rational homology sphere, then the Heegaard Floer homology with twisted coefficients is equivalent to the untwisted version. For further details on twisted Heegaard Floer theory, refer to [OS04a].

Theorem 5.5 (Gadgil [Gad09])**.** *Let X be a smooth open 4-manifold and $\mathfrak{s}$ be an admissible spin^c structure on X. Then, $\underline{HE}(X, \mathfrak{s}; \omega)$ does not depend on the choices of admissible exhaustion.*

We call a 3-manifold Y an $L-$space if $\underline{HF}^+_{\text{red}}(Y, M) = 0$ holds for every $\mathbb{Z}[H^1(Y, \mathbb{Z})]$ module M. Notably, any 3-manifold that permits a positive scalar curvature metric also is an $L-$space [OS05]*Proposition 2.3. With this in mind, we have the following vanishing result of end Floer homology.

Theorem 5.6. *Let X be a smooth open 4-manifold which admits an exhaustion $\Omega_1 \subset \Omega_2 \subset \dots$ so that each component of $\partial\Omega_i$ is an $L-$space. Then $\underline{HE}(X, \mathfrak{s}; \omega) = 0$ for all choice of asymptotic* spin^c *structures $\mathfrak{s}$ and 2-forms ω.*

Proof. Suppose W is a cobordism between Y_1 and Y_2, with Y' smoothly embedded in W, serving as an $L-$space that separates the two boundaries Y_1 and Y_2. Then, observe that the induced twisted cobordism map $\underline{F}^+_{W;\omega} : \underline{HF}^+_{\text{red}}(Y_1, \omega|_{Y_1}) \to \underline{HF}^+_{\text{red}}(Y_2, \omega|_{Y_2})$ factors through $\underline{HF}^+_{\text{red}}(Y', \omega|_{Y'}) = 0$. Consequently, the image of $\underline{F}^+_{W;\omega}$ is 0.

Now consider an admissible exhaustion $X_0 \subset X_1 \subset X_2 \subset \cdots$ for X. As X admits an exhaustion by $\Omega_1 \subset \Omega_2 \subset \dots$, where each component of $\partial\Omega_i$ is an $L-$space, we can refine the admissible exhaustion to $X_{i_1} \subset X_{i_2} \subset \cdots$ of X_i's such that in this refined exhaustion, for each cobordism $X_{i_j} \setminus \mathring{X}_{i_{j-1}}$, $\partial\Omega_l$ embeds smoothly in a boundary-separating manner for some l. Hence, the induced cobordism map is trivial, yielding $\underline{HE}(X, \mathfrak{s}; \omega) = 0$. □

Now we are ready to prove Theorem 1.4 which says that a non-compact 4-manifold with non-trivial end Floer homology cannot admit a complete positive scalar metric $R \geq 1$.

Proof of Theorem 1.6. If X admits a complete positive scalar curvature metric $R \geq 1$, then by Proposition 3.1, it admits an exhaustion $\{\Omega_i\}$ such that $\partial\Omega_i$ admits a positive scalar curvature metric, and hence an L-space. Thus by Theorem 5.6, the end Floer homology satisfies $\underline{HE}(X, \mathfrak{s}; \omega) = 0$ for all choice of asymptotic spin^c structures $\mathfrak{s}$ and 2-forms ω, which is a contradiction. □

6. Exotic smooth structures with no uniformly PSC metric

In this section, we will use contact and symplectic geometry to produce many examples of smooth 4-manifolds that do not admit complete scalar curvature $R \geq 1$ metric.

6.1. **The idea of the construction.** Begin with a disjoint collection of smooth disks D in B^4 whose boundary $\partial D \subset \partial B^4 = S^3$ gives a link. By removing a tubular neighborhood of D from B^4 we obtain a 4-manifold B' whose boundary is the 3-manifold which is obtained from S^3 by doing 0-surgery on the link ∂D. If we were to attach 2 handles on B' along the meridians of D, we will recover B^4. However, we will instead glue (Stein) Casson handles [Cas86, Gom98] to these meridians of D. The resultant interior will be homeomorphic to $\mathbb{R}^4$ (since Casson handles are homeomorphic to open 2-handles [Fre82]), but it may not be diffeomorphic to $\mathbb{R}^4$. In Figure 2 we describe a ribbon disk complement B' which is obtained from deleting the standard disk D bounded by Pretzel knot $P(-3, -3, 3)$ and the dashed circle in Figure 2 is the meridian of D where we will attach various Stein Casson handle to construct our desired exotic $\mathbb{R}^4$ which has no complete positive scalar curvature metric ≥ 1.

6.2. **Contact and Symplectic Topology.** Let ξ denote a contact structure on an oriented 3-manifold. A knot K contained in (Y,ξ) is termed *Legendrian* if the tangent space T_pK lies within ξ_p for every $p \in K$. In a contact manifold (Y,ξ), a Legendrian knot K possesses a standard neighborhood N and a framing fr_ξ determined by the contact planes. When K is null-homologous, the framing fr_ξ relative to the Seifert framing represents the *Thurston–Bennequin* number of K, denoted by $tb(K)$. Performing $fr_\xi - 1$-surgery on K, i.e. by removing N and attaching a solid torus to realize the desired surgery, results in a unique extension of $\xi|_{Y-N}$ over the surgery torus, maintaining tightness on the surgery torus. The resulting contact manifold is termed obtained from (Y,ξ) by *Legendrian surgery* on K. Furthermore, for a knot K in (S^3,ξ_{std}), the *maximal Thurston–Bennequin number* is defined as the maximal value among all Thurston–Bennequin numbers for all Legendrian representations of K.

A *symplectic cobordism* from the contact manifold (Y_-,ξ_-) to (Y_+,ξ_+) constitutes a compact symplectic manifold (W,ω) with boundary $-Y_- \cup Y_+$, where Y_- is a *concave* boundary component and Y_+ is *convex*. This implies the existence of a vector field v near ∂W pointing transversally inwards at Y_- and transversally outwards at Y_+, satisfying $\mathcal{L}_v\omega = \omega$ and $\iota_v\omega|Y_\pm$ representing a contact form of $\xi_\pm$. If Y_- is empty, (W,ω) is referred to as a *symplectic filling.*

Our approach predominantly follows a technique for constructing symplectic cobordisms known as *Stein handle attachment* [Eli90, Wei91]. This involves attaching 1- or 2-handles to the convex end of a symplectic cobordism to obtain a new symplectic cobordism with the modified convex end as follows: for a 1-handle attachment, the convex boundary undergoes a connected sum, potentially internal. On the other hand, a 2-handle is attached along a Legendrian knot L with framing one less than the contact framing, leading to a Legendrian surgery on the convex boundary.

Theorem 6.1. *For a contact 3-manifold $(Y,\xi = \ker\theta)$, consider W as a segment of its symplectization, i.e., $(W = [0,1]\times Y, \omega = d(e^t\theta))$. Let L be a Legendrian knot in (Y,ξ), with Y regarded as $Y\times 1$. If W' is obtained from W by attaching a 2-handle along L with framing one less than the contact framing, then the upper boundary (Y',ξ') remains a convex boundary. Furthermore, if the 2-handle is attached to a symplectic filling of (Y,ξ), then the resulting manifold would be a strong symplectic filling of (Y',ξ').*

Eliashberg established the theorem for Stein fillings [Eli90], for strong fillings by Weinstein [Wei91], and was initially formulated for weak fillings by Etnyre and Honda [EH02].

Definition 6.2. A Stein surface (W,ω) with a concave boundary (Y,ξ) is called an admissible Stein surface if it admits an admissible exhaustion $\{W_i\}$ where $W_0 = Y\times[0,1]$, and W_j is obtained by attaching a Stein handle of index 1 or 2 on the convex boundary $(\partial W_{j-1},\xi_{j-1})$.

Theorem 6.3. *Let (Y,ξ) be a contact 3-manifold such that reduced contact invariant $c^+_{\text{red}}(\xi;\omega) \neq 0$ in $\underline{HF}^+_{\text{red}}(-Y,\omega)$. If (W,ω) is an admissible Stein surface with a concave boundary (Y,ξ), then for any 4-manifold X with boundary Y, the interior of $X\cup_Y W$ doesn't admit a complete positive scalar curvature metric $R \geq 1$.*

Proof of Theorem 6.3. We start with an admissible Stein exhaustion $W_0 \subset W_1 \subset \cdots$ of W. Let $\mathfrak{s}$ represent the asymptotic *spin*^{c} structure on W corresponding to the symplectic 2-form ω. By construction, $\mathfrak{s}$ restricts to a unique *spin*^{c}

structure $\mathfrak{s}_i$ on W_i, which coincides with the Stein cobordism structure on W_i. Denoted by (Y_i, ξ_i) the convex boundary of the Stein cobordism W_i, we have a cobordism map $\underline{F}^+_{-W_i,\mathfrak{s}_i;\omega} : \underline{HF}^+_{red}(-Y_i, \omega|_{-Y_i}) \to \underline{HF}^+_{red}(-Y, \omega|_{-Y})$ such that $F^+_{-W_i,\mathfrak{s}_i;\omega}(c^+_{red}(\xi_i;\omega)) = c^+_{red}(\xi;\omega) \neq 0$ [OS04a] where $-W_i$ is consider the upside-down cobordism from $-Y_i$ to $-Y$.

Now if the interior of $X \cup_Y W$ admits a complete positive scalar curvature metric $R \geq 1$, then Corollary 3.6 implies that there exists an exhaustion $\Omega_1 \subset \Omega_2 \subset \dots$ of $X \cup_Y W$ so that each component of $\partial\Omega_i$ is of the form $(\#_j S^3/\Gamma_j)\#(\#_j S^2 \times S^1)$. Moreover, we have $X_i = X \cup_Y W_i$ an exhaustion of $X \cup_Y W$. So by compactness argument, there will exist an integer k and k' such that $X_1 \subset \Omega_{k'} \subset X_k$. Thus the map $F^+_{-W_k,\mathfrak{s}_k;\omega} : \underline{HF}^+_{red}(-Y_k, \omega|_{-Y_k}) \to \underline{HF}^+_{red}(-Y, \omega|_{-Y})$ factors through $HF^+_{\text{red}}(-\partial\Omega_{k'}, \omega|_{\partial\Omega_{k'}}) = 0$. But $F^+_{-W_k,\mathfrak{s}_k;\omega}(c^+_{red}(\xi_k;\omega)) = c^+_{red}(\xi;\omega) \neq 0$, which is a contradiction. □

6.3. **Constructing an admissible Stein Casson Handle.** Now, we will elaborate on the process of constructing a Stein-admissible Casson handle, derived from Gompf's method outlined in [Gom98]*Section 3, Proof of Theorem 3.1. Consider H, a 4-manifold comprising one 0-handle, one 1-handle, and one 2-handle, denoted as $H = h^0 \cup h^1_0 \cup h^2_0$. Eliashberg demonstrated [Eli90] that if the Thurston-Bennequin number of the attaching 2-handle exceeds the smooth framing of its attaching sphere, the manifold admits a Stein structure. However, this inequality might not always hold. Bearing this in mind, we can construct the handle structure of H as follows: Let $X_1 = H_1 \cup h^1_1 \cup \cdots \cup h^1_k$ for some $k > 0$, where H_1 represents the 1-skeleton of H, to which a few new 1-handles are appended. Then, attach a 2-handle h^2_1 to X_1, with its attaching sphere passing over h^1_0 identical to that of h^2_0 but in addition to that this will twist the remaining newly attached 1-handles h^1_k, as illustrated in Figure 1, with the smooth framing of h^2_1 matching that of h^2_0. This process increases the Thurston–Bennequin number of the attaching sphere of h^2_1 by $2k$ (bottom picture of Figure 1).

By selecting an appropriate k, we ensure X_1 becomes Stein. This modification of the original 2-handle h^2_0 into the new attaching 2-handle h^2_1 is referred to as a kinky-handle. Subsequently, attaching k new 0-framed 2-handles along the meridian of the dotted 1-handles h^1_j for $j > 0$ results in a 4-manifold diffeomorphic to H. However, the attaching spheres of these 2-handles will have Thurston–Bennequin numbers equal to 0, aligning with the smooth framing. To establish a Stein structure, we can repeat the above process for each pair of handles by introducing new kinky 2-handles such that the interior admits the Stein structure. This construction can be iterated, adding a third layer of handles onto the second layer, and so forth, generating a manifold X with infinitely many handles, inherently possessing a Stein interior. Note that in this infinite iteration process cancel all the 1-handles with some 2-handles and thus it is simply-connected at infinity and there is no new homology can arise. Additionally, each layer of handle attachments satisfies the conditions of Remark 5.4. The work of Casson [Cas86] and Freedman [Fre82] demonstrated that the aforementioned handle attachment method gives rise to Casson handles, and the resultant 4-manifold X is homeomorphic to the interior of H.

6.4. **Proofs.** We can now construct uncountably many smooth structures on $\mathbb{R}^4$ that don't admit complete metrics with uniformly positive scalar curvature.

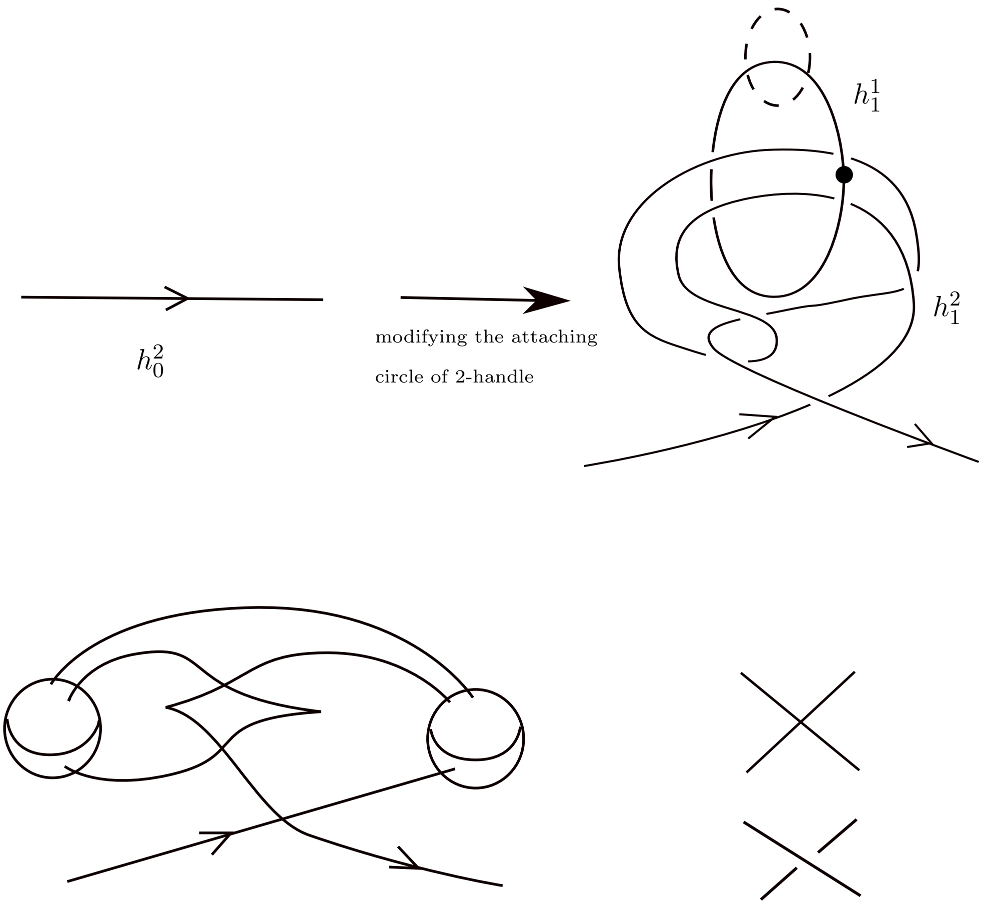

FIGURE 1. On the top figure we are showing how to modify the attaching 2-handle h_0^2 by passing over the new 1-handle h_1^1 that increases the Thurston–Bennequin number as shown in the bottom left figure where we can see a Legendrian representation of the new 2-handle h_1^2 (where each crossing are as indicated in the right bottom picture).

Proof of Theorem 1.3. Let K denote the non-trivial Pretzel slice knot $P(-3,-3,3)$ in $S^3 = \partial B^4$ and $Y = S^3_0(K)$ be the 3-manifold resulting from 0-surgery on K. If μ represents the meridian of K in S^3, it is notable that $[\mu]$ normally generates $\pi_1(Y)$. Now let us examine the cobordism W obtained by attaching a 0-framed 2-handle on $Y \times [0,1]$ along $\mu \subset Y \times \{1\}$. Then W is a cobordism from Y to S^3. Gabai proved that [Gab87] the constructed 3-manifold Y admits a taut foliation. Since Y can be equipped with a taut foliation, Eliashberg–Thurston showed that [ET22] a perturbation induces a contact structure ξ on Y and moreover there exists a symplectic structure ω on $Y \times I$ such that both boundary component are convex. Now by capping off one convex boundary component by a symplectic cap as constructed in [EMM22] we can construct a symplectic filling of (Y, ξ) with $b^+ > 0$. And thus by [OS04a]*Theorem 4.2 we know that (Y, ξ) be a contact 3-manifold such that reduced contact invariant $c^+_{\text{red}}(\xi;\omega) \neq 0$ in $\underline{HF}^+_{\text{red}}(-Y,\omega)$.

Let D be a slice disk bounded by K in B^4. Consider the $4-$manifold $B' = B^4 \setminus \nu(D)$ in Figure 2 , with Y as boundary.

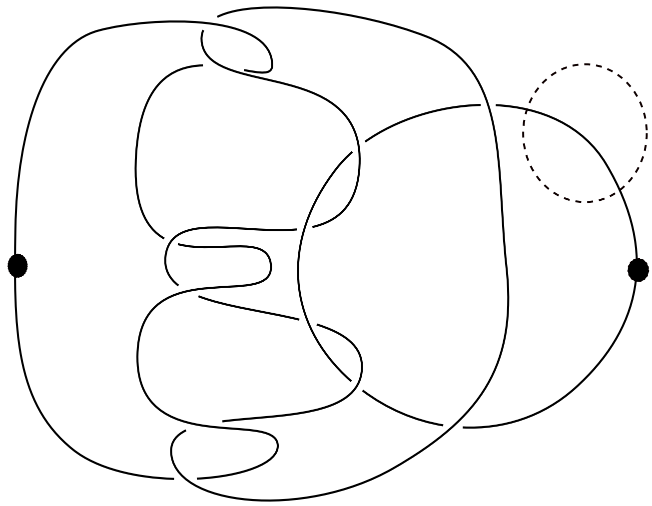

FIGURE 2. Kirby picture of B' where the dashed circle denotes the meridian of the disk

Now let us focus on the open 4-manifold $R = B' \cup W_0$ where $W_0 = W \setminus S^3$. Gompf showed in [Gom98]*Theorem 3.4 that by replacing the open 2-handle of W_0 with Stein Casson handles (attaching them along the dashed circle in Figure 2), one can construct uncountable many non-diffeomorphic 4-manifolds $R_c = B' \cup CH_c$, indexed by Cantor set (compare with [DMF92, BG96]). These R_c's are seen to have trivial homology in all dimensions. Furthermore, $\pi_1(R_c) = 1$, and R_c is simply-connected at infinity. According to Freedman [Fre82], R_c is thus homeomorphic to $\mathbb{R}^4$. Since CH_c is a Stein Casson handle, by the result of Eliashberg, it has a handle decomposition $\{h_c^i\}$ with 1- and 2-handles. Moreover, we can arrange the handles h_c^i (as explained in Section 6.3) to satisfy Remark 5.4. Therefore we have an admissible cobordism $\{X_c^i\}$ of R_c, where $X_c^0 = B'$, and X_c^i obtained from X_c^{i-1} by attaching a Stein 1- or 2-handle h_c^i along ∂X_c^{i-1}. Let ω be the symplectic 2-form of CH_c obtained by the Stein structure. Hence, by Theorem 6.3, R_c doesn't admit a complete positive scalar curvature metric $R \geq 1$. □

Similarly we can show that if W is any smooth 4-manifold with boundary then the interior of W admits a smooth structure with no complete Riemannian metric with uniformly positive scalar curvature.

Proof of Theorem 1.4. Let us consider the following decomposition of W as $W \cup h_1 \cup h_2$, where the 1-handle h_1 is attached along ∂W and the 2-handle h_2 attached along a attaching circle that goes over the co-core of the 1-handle h_1 geometrically once. Thus h_1 and h_2 smoothly cancel each other.

Now we will use a similar strategy of the proof of Theorem 1.3 above and replace h_2 by a Stein Casson handle, then we can change the smooth structure of the interior of the 4-manifold W. However, now the new smooth structure decomposes as in Theorem 6.3 and thus it will not admit a complete positive scale curvature metric $R \geq 1$. □

APPENDIX A. OVERVIEW OF SCALAR CURVATURE

This appendix contains a brief survey of the topological study of scalar curvature. Many other more comprehensive surveys exist, including [Ros07, Gro23, Cho21].

The scalar curvature of a Riemannian manifold (M^n, g) is a function $R : M \to \mathbb{R}$ such that for every point $p \in M$, the volume of balls has the following infinitesimal expansion:

$$\mathrm{Vol}_M(B_\varepsilon(p)) = \mathrm{Vol}_{\mathbb{R}^n}(B_\varepsilon(0))\left(1 - \frac{R(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^2)\right). \quad \text{(A.1)}$$

This is the weakest curvature invariant one can define on a manifold and it is well-known to be very flexible in some ways but rigid in others. On any smooth manifold, any function that is negative at some point can be prescribed as the scalar curvature of some metric (c.f. [Ros07, Theorem 0.1]) but, remarkably, there are global obstructions a manifold must satisfy to carry a metric with positive scalar curvature.

A.1. **Obstructions.** All known obstructions arise from one of three methods. The first, discovered by Lichnerowicz [Lic63], employs the Atiyah–Singer index theorem for the Dirac operator. He demonstrated that any closed spin manifold with a non-vanishing $\hat{A}$-genus cannot support a metric of positive scalar curvature. This, for example, implies that the $K3$ surface cannot accommodate such a metric. A refinement by Hitchin [Hit74] further rules out the existence of positive scalar curvature metrics on certain exotic spheres of dimensions $8k+1, 8k+2$ for all $k \geq 1$.

The next obstruction, due to Schoen–Yau [SY79a], uses the second variation of the area of minimal hypersurfaces. They discovered it while investigating the positive mass conjecture in general relativity [SY79c]. Their method works for manifolds of dimension $3 \leq n \leq 7$. It obstructs positive scalar curvature on manifolds satisfying a certain condition on their integral cohomology ring, a class that includes closed manifolds admitting non-zero degree maps to a torus.

Finally, in dimension 4, Witten [Wit94] demonstrated that closed manifolds M with positive scalar curvature and a second Betti number $b_2^+(M) > 1$ must have vanishing Seiberg-Witten invariants. However, as shown by Taubes [Tau94], this is not possible if such manifolds admit a symplectic structure, thereby obstructing the existence of positive scalar curvature metrics on a significant number of closed smooth 4-manifolds. Furthermore, for 4-manifolds with periodic ends, various generalizations of Seiberg-Witten theory also obstruct the existence of positive scalar curvature metrics [KT20, Lin19, LRS23].

A.1.1. *μ-bubbles.* Recently, Gromov introduced [Gro23] a new idea that generalizes the minimal hypersurface technique to other situations (cf. Section 3) by localizing the minimizer via distance function. Using this extension, Chodosh-Li proved that closed aspherical n-manifold does not admit metric with positive scalar curvature for $n = 4, 5$. The $n = 5$ was independently obtained by Gromov [Gro20]. Other obstructions using incompressible surfaces were obtained by [CLSZ21, CRZ23]. We also note that similar localizations have been obtained for spinors by various authors [Zei22, Zei20, GXY23, Cec20, CZ24].

A.2. **(Partial) classification results.** In $n = 3$ dimensions, the minimal hypersurface technique can be used to impose strong restrictions on the fundamental group of closed manifolds with positive scalar curvature [SY79b]. After Perelman solved the Poincaré conjecture [Per03], a complete classification of closed 3-manifolds with positive scalar curvature up to diffeomorphism could be obtained (cf. Proposition 2.2).

In dimensions $n \geq 5$, the problem is also understood for simply connected manifolds: Gromov-Lawson [GL80] showed that every closed simply connected non-spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric, and Stolz [Sto92] showed that spin manifolds can only admit such metrics if their α-index invariant vanishes.

Also using μ-bubbles, Chodosh–Li–Liokumovich [CLL23] showed that if M is a closed manifold of dimension $n = 4$ (resp. $n = 5$) with $\pi_2(M) = 0$ (resp. $\pi_2(M) = \pi_3(M) = 0$) that admits a metric of positive scalar curvature, then a finite cover $\widehat{M}$ of M is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$. In a different vein, Bamler–Li–Mantoulidis showed that if M is a closed smooth 4-manifold that admits a metric of positive scalar curvature then it can be obtained by performing 0- and 1-surgeries on a disjoint union of PSC orbifolds with first Betti number $b_1 = 0$ [BLM23].

Appendix B. Perturbed Heegaard Floer homology

Recall that the Novikov ring over $\mathbb{F}_2$ is a set of formal series

$$\Lambda = \left\{ \sum_{x \in \mathbb{R}} n_x z^x : n_x \in \mathbb{F}_2 \right\}$$

where the set

$$\{x \in (-\infty, c] : n_x \neq 0\}$$

is finite for every $c \in \mathbb{R}$. One may easily check that this is a field under the obvious operations.

Let Y be a closed 3–manifold equipped with a closed 2-form $\omega \in \Omega^2(Y)$. Then there exists an action of a group ring $\mathbb{F}_2[H^1(Y)] \cong \mathbb{F}_2[H_2(Y)]$ on Λ, induced by ω, which is defined as follows. For $a \in H_2(Y)$,

$$e^a \cdot z^x = z^{x + \int_a \omega}.$$

The naturality of perturbed Heegaard Floer homology is conveniently described by projective transitive systems, which were first introduced by Baldwin and Sivek in [BS15].

Recall that Heegaard Floer homology of a 3-manifold Y with a spin^c structure is a of $\mathbb{F}_2[U]$-modules $HF^\circ(Y, \mathfrak{s})$ for $\circ \in \{\infty, +, -, \wedge\}$, which fit into a long exact sequence

$$\cdots \xrightarrow{\tau} HF^-(Y, \mathfrak{s}) \to HF^\infty(Y, \mathfrak{s}) \to HF^+(Y, \mathfrak{s}) \xrightarrow{\tau} HF^-(Y, \mathfrak{s}) \to \cdots$$

In [OS04b], Ozsváth–Szabó introduced Heegaard Floer homology perturbed by a second real cohomology class, which is more thoroughly discussed in [JM08]. For completeness we will discuss the construction now- Let $\omega \in \Omega^2(Y)$ be a closed 2-from on Y and $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ an $\mathfrak{s}$-admissible pointed Heegaard diagram of Y. Denote the two handlebodies determined by $\mathcal{H}$ by H_α and H_β respectively and let D_α and D_β be sets of compressing disks of H_α and H_β, respectively, such that D_α intersects Σ along α and D_β intersects Σ along β. Note that $\phi \in \pi_2(x, y)$ determines a 2–chain $\mathcal{D}(\phi)$ on Σ with boundary a union the loops in $\alpha \cup \beta$. One may cone these loops in the compressing disks D_α and D_β to obtain a 2–chain $\widetilde{D}(\phi)$. Now we define

$$A_\omega(\phi) := \int_{\widetilde{D}(\phi)} \omega.$$

Consider a chain complex $CF^\infty(\mathcal{H}, \mathfrak{s}; \omega)$ which is a free Λ-module generated by $U^i x$ for $x \in T_\alpha \cap T_\beta$, such that the spin^c structure associated to x is $\mathfrak{s}$, i.e. $\mathfrak{s}(x) = \mathfrak{s}$. The differential is defined as follows.

$$\partial^\infty(U^i x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot z^{A_\omega(\phi)} \cdot U^{i - n_w(\phi)} y \pmod 2,$$

where $n_w(\phi)$ is the algebraic intersection number between $\phi \in \pi_2(x, y)$ and $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$ and $\mathcal{M}(\phi)$ is the space of J-holomorphic disks in the homotopy class ϕ for $J \in \mathcal{J}$, where $\mathcal{J}$ is a generic family of almost complex structures on $\mathrm{Sym}^g(\Sigma)$. For $\circ \in \{+, -, \wedge\}$, ∂° is induced from ∂^∞ in the usual way and $(\partial^\circ)^2 = 0$ as in the original Heegaard Floer homology. Now we define

$$HF^\circ(Y, \mathfrak{s}; \omega) := H_*(CF^\circ(\mathcal{H}, \mathfrak{s}; \omega), \partial^\circ).$$

The homology $HF^\circ(Y, \mathfrak{s}; \omega)$ depends on the choice of $\mathcal{H}$ and J, but Juhász and Zemke in [JZ23] proved the well defineness.

We have the following functoriality in Heegaard Floer homology groups induced by cobordisms.

Theorem B.1 (Ozsváth–Szabó [OS04a, Section 3.1])**.** *Let W be a cobordism from Y_1 to Y_2. Suppose ω is a closed 2-form on W and $\mathfrak{s} \in \mathrm{spin}^c(W)$ is a spin^c structure on W. Then the cobordism map*

$$F^\circ_{W,\mathfrak{s};\omega} \colon HF^\circ(Y_1, \mathfrak{s}|_{Y_1}; \omega|_{Y_1}) \to HF^\circ(Y_2, \mathfrak{s}|_{Y_2}; \omega|_{Y_2})$$

is well-defined up to overall multiplication by z^x for $x \in \mathbb{R}$.

Consider $\omega \in \Omega^2(W, \partial W)$ which is a closed 2-form on W compactly supported in the interior of W. We will say such an ω is a 2-form on $(W, \partial W)$. Let W be a cobordism from Y_0 to Y_1 equipped with a closed 2-form ω and $\mathfrak{S} \subseteq \mathrm{spin}^c(W)$ a subset of spin^c structures on W such that each $\mathfrak{s} \in \mathfrak{S}$ has the same restriction to ∂W.

If $\circ \in \{\infty, -\}$, we further assume that there exists only finitely many $\mathfrak{s} \in \mathfrak{S}$ such that $F^\circ_{W,\mathfrak{s};\omega}$ is non-vanishing. Then, there exists a cobordism map

$$F^\circ_{W,\mathfrak{S};\omega} : HF^\circ(Y_1; \omega|_{Y_1}) \to HF^\circ(Y_2; \omega|_{Y_2}),$$

which is also well-defined up to overall multiplication by z^x for $x \in \mathbb{R}$. Although addition is not well-defined in projective systems, we may find representatives of $F^\circ_{W,\mathfrak{S};\omega}$ for $\mathfrak{s} \in \mathfrak{S}$ so that

$$F^\circ_{W,\mathfrak{S};\omega} \doteq \sum_{\mathfrak{s} \in \mathfrak{S}} F^\circ_{W,\mathfrak{s};\omega}.$$

If $\omega = \{\omega_1, \ldots, \omega_n\}$ is an n-tuple of closed 2-forms on a 3–manifold Y, we can define a $\Lambda_n[U]$-module $HF^\circ(Y, \mathfrak{s}; \omega)$ as above, where Λ_n is the n-variable Novikov ring over $\mathbb{F}_2$. All the theorems and lemmas in this section hold for this version.

Let $a = (a_1, \ldots, a_n)$ be an n-tuple of integers. We will use the notation

$$z^a := z_1^{a_1} \cdots z_n^{a_n}.$$

Lemma B.2 (Juhász–Zemke [JZ23, Lemma 3.4])**.** *Let W be a cobordism from Y_1 to Y_2 and $\omega = \{\omega_1, \ldots, \omega_n\}$ be an n-tuple of closed 2-forms on $(W, \partial W)$. Suppose $\mathfrak{S} \subseteq \mathrm{spin}^c(W)$ is a subset of spin^c structures on W. If $\circ \in \{-, \infty\}$, we further* *assume that there are only finitely many* $\mathfrak{s} \in \mathfrak{S}$ *where* $F^{\circ}_{W,\mathfrak{s}} \neq 0$*. Fix an arbitrary* spin^c *structure* $\mathfrak{s}_0 \in \mathrm{spin}^c(W)$*. Then,*

$$F^{\circ}_{W,\mathfrak{S};\omega} \doteq \sum_{s \in \mathfrak{S}} z^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup[\omega],[W,\partial W]\rangle} \cdot F^{\circ}_{W,\mathfrak{s}},$$

where $i_* : H^2(W;\mathbb{Z}) \to H^2(W;\mathbb{R})$ *is induced by the inclusion* $i : \mathbb{Z} \to \mathbb{R}$*.*

References

[Ago17] Ian Agol, *For a 3-manifold* Y*, when does* $Y \times S^1$ *admits a Riemannian metric with positive scalar curvature?*, MathOverflow, 2017, https://mathoverflow.net/q/215872 (version: 2017-06-08).

[BBMM21] Laurent Bessières, Gérard Besson, Sylvain Maillot, and Fernando C. Marques, *Deforming 3-manifolds of bounded geometry and uniformly positive scalar curvature*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 1, 153–184. MR 4186465

[BG96] Žarko Bižaca and Robert E. Gompf, *Elliptic surfaces and some simple exotic* $\mathbf{R}^4$*'s*, J. Differential Geom. **43** (1996), no. 3, 458–504. MR 1412675

[BLM23] Richard H. Bamler, Chao Li, and Christos Mantoulidis, *Decomposing 4-manifolds with positive scalar curvature*, Adv. Math. **430** (2023), Paper No. 109231, 17. MR 4621960

[BS15] John A. Baldwin and Steven Sivek, *Naturality in sutured monopole and instanton homology*, J. Differential Geom. **100** (2015), no. 3, 395–480. MR 3352794

[BW99] Jonathan Block and Shmuel Weinberger, *Arithmetic manifolds of positive scalar curvature*, J. Differential Geom. **52** (1999), no. 2, 375–406. MR 1758300

[Cas86] Andrew J. Casson, *Three lectures on new-infinite constructions in 4-dimensional manifolds*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, With an appendix by L. Siebenmann, pp. 201–244. MR 900253

[CCZ24] Shuli Chen, Jianchun Chu, and Jintian Zhu, *Positive scalar curvature metrics and aspherical summands*, available at https://arxiv.org/abs/2312.04698 (2024).

[Cec20] Simone Cecchini, *A long neck principle for Riemannian spin manifolds with positive scalar curvature*, Geom. Funct. Anal. **30** (2020), no. 5, 1183–1223. MR 4181824

[Che24] Shuli Chen, *A generalization of the Geroch conjecture with arbitrary ends*, Math. Ann. **389** (2024), no. 1, 489–513. MR 4735953

[Cho21] Otis Chodosh, *Lecture notes "Stable minimal surfaces and positive scalar curvature"*, https://web.stanford.edu/~ochodosh/Math258-min-surf.pdf (2021).

[CL24] Otis Chodosh and Chao Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, Ann. of Math. (2) **199** (2024), no. 2, 707–740. MR 4713021

[CLL23] Otis Chodosh, Chao Li, and Yevgeny Liokumovich, *Classifying sufficiently connected PSC manifolds in 4 and 5 dimensions*, Geom. Topol. **27** (2023), no. 4, 1635–1655. MR 4602422

[CLSZ21] Jie Chen, Peng Liu, Yuguang Shi, and Jintian Zhu, *Incompressible hypersurface, positive scalar curvature and positive mass theorem*, available at https://arxiv.org/abs/2112.14442 (2021).

[CMS23] Otis Chodosh, Christos Mantoulidis, and Felix Schulze, *Generic regularity for minimizing hypersurfaces in dimensions 9 and 10*, available at https://arxiv.org/abs/2302.02253 (2023).

[CRZ23] Simone Cecchini, Daniel Räde, and Rudolf Zeidler, *Nonnegative scalar curvature on manifolds with at least two ends*, J. Topol. **16** (2023), no. 3, 855–876. MR 4611408

[CWY10] Stanley Chang, Shmuel Weinberger, and Guoliang Yu, *Taming 3-manifolds using scalar curvature*, Geom. Dedicata **148** (2010), 3–14. MR 2721617

[CWY17] ———, *Contractible manifolds with exotic positive scalar curvature behavior*, Manifolds and K-theory, Contemp. Math., vol. 682, Amer. Math. Soc., Providence, RI, 2017, pp. 51–64. MR 3603893

[CWY20] ———, *Positive scalar curvature and a new index theory for noncompact manifolds*, J. Geom. Phys. **149** (2020), 103575, 22. MR 4045309

[CZ24] Simone Cecchini and Rudolf Zeidler, *Scalar and mean curvature comparison via the Dirac operator*, Geom. Topol. **28** (2024), no. 3, 1167–1212. MR 4746412

[DMF92] Stefano De Michelis and Michael H. Freedman, *Uncountably many exotic* $\mathbf{R}^4$*'s in standard 4-space*, J. Differential Geom. **35** (1992), no. 1, 219–254. MR 1152230

[Don23] Conghan Dong, *Three-manifolds with bounded curvature and uniformly positive scalar curvature*, J. Geom. Anal. **33** (2023), no. 6, Paper No. 169, 11. MR 4567570

[EH02] John B. Etnyre and Ko Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. **148** (2002), no. 3, 609–626. MR 1908061

[Eli90] Yakov Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. **1** (1990), no. 1, 29–46. MR 1044658

[EMM22] John B. Etnyre, Hyunki Min, and Anubhav Mukherjee, *On 3-manifolds that are boundaries of exotic 4-manifolds*, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4307–4332. MR 4419060

[ET22] Yakov M. Eliashberg and William P. Thurston, *Confoliations*, Collected works of William P. Thurston with commentary. Vol. I. Foliations, surfaces and differential geometry, Amer. Math. Soc., Providence, RI, 2022, Reprint of [1483314], pp. 281–351. MR 4554446

[Fre82] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), no. 3, 357–453. MR 679066

[Gab87] David Gabai, *Foliations and the topology of* 3*-manifolds. III*, J. Differential Geom. **26** (1987), no. 3, 479–536. MR 910018

[Gad09] Siddhartha Gadgil, *Open manifolds, Ozsvath-Szabo invariants and exotic R^4's*, Expo. Math. (2009).

[GL80] Mikhael Gromov and H. Blaine Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), no. 3, 423–434. MR 577131

[GL83] ———, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196. MR 720933

[Gom93] Robert E. Gompf, *An exotic menagerie*, J. Differential Geom. **37** (1993), no. 1, 199–223. MR 1198606

[Gom98] Robert E. Gompf, *Handlebody construction of stein surfaces*, Annals of Mathematics **148** (1998), no. 2, 619–693.

[Gro96] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993), Progr. Math., vol. 132, Birkhäuser Boston, Boston, MA, 1996, pp. 1–213. MR 1389019

[Gro20] Misha Gromov, *No metrics with positive scalar curvatures on aspherical 5-manifolds.*

[Gro23] Misha Gromov, *Four lectures on scalar curvature*, Perspectives in scalar curvature. Vol. 1, World Sci. Publ., Hackensack, NJ, 2023, pp. 1–514. MR 4577903

[GXY23] Hao Guo, Zhizhang Xie, and Guoliang Yu, *Quantitative K-theory, positive scalar curvature, and bandwidth*, Perspectives in scalar curvature. Vol. 2, World Sci. Publ., Hackensack, NJ, 2023, pp. 763–798. MR 4577930

[Hit74] Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. MR 358873

[HS85] Robert Hardt and Leon Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129. MR 809969

[JM08] Stanislav Jabuka and Thomas E. Mark, *Product formulae for Ozsváth-Szabó 4-manifold invariants*, Geom. Topol. **12** (2008), no. 3, 1557–1651. MR 2421135

[JZ23] András Juhász and Ian Zemke, *Concordance surgery and the Ozsváth-Szabó 4-manifold invariant*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 3, 995–1044. MR 4577958

[Ker69] Michel A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. **144** (1969), 67–72. MR 253347

[KT20] Hokuto Konno and Masaki Taniguchi, *Positive scalar curvature and 10/8-type inequalities on 4-manifolds with periodic ends*, Invent. Math. **222** (2020), no. 3, 833–880. MR 4169052

[Lic63] André Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7–9. MR 156292

[Lin19] Jianfeng Lin, *The Seiberg-Witten equations on end-periodic manifolds and an obstruction to positive scalar curvature metrics*, J. Topol. **12** (2019), no. 2, 328–371. MR 3911569

[Lot24] John Lott, *Some obstructions to positive scalar curvature on a noncompact manifold*, available at https://arxiv.org/abs/2402.13239 (2024).

[LRS23] Jianfeng Lin, Daniel Ruberman, and Nikolai Saveliev, *On the Frøyshov invariant and monopole Lefschetz number*, J. Differential Geom. **123** (2023), no. 3, 523–593. MR 4584860

[OS03] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261. MR 1957829

[OS04a] ———, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334. MR 2023281

[OS04b] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158. MR 2113019

[OS05] ———, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300. MR 2168576

[Per03] Grisha Perelman, *Ricci flow with surgery on three-manifolds*, available at https://arxiv.org/abs/math/0303109 (2003).

[R2̈3] Daniel Räde, *Scalar and mean curvature comparison via μ-bubbles*, Calc. Var. Partial Differential Equations **62** (2023), no. 7, Paper No. 187, 39. MR 4612761

[Ros07] Jonathan Rosenberg, *Manifolds of positive scalar curvature: a progress report*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 259–294. MR 2408269

[Sto92] Stephan Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), no. 3, 511–540. MR 1189863

[SY79a] R. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183. MR 535700

[SY79b] R. Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142. MR 541332

[SY79c] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR 526976

[SY83] Richard Schoen and S. T. Yau, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys. **90** (1983), no. 4, 575–579. MR 719436

[Tau87] Clifford Henry Taubes, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geom. **25** (1987), no. 3, 363–430. MR 882829

[Tau94] ———, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), no. 6, 809–822. MR 1306023

[Wan19] Jian Wang, *Contractible 3-manifolds and positive scalar curvature (I)*, to appear in J. Diff. Geom., available at https://arxiv.org/abs/1901.04605 (2019).

[Wan23] ———, *Topology of 3-manifolds with uniformly positive scalar curvature*, available at https://arxiv.org/abs/2212.14383 (2023).

[Wan24] Jian Wang, *Contractible 3-manifolds and positive scalar curvature (II)*, J. Eur. Math. Soc. (JEMS) **26** (2024), no. 2, 537–572. MR 4705657

[Wei91] Alan Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991), no. 2, 241–251. MR 1114405

[Wit94] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. MR 1306021

[WXY23] Shmuel Weinberger, Zhizhang Xie, and Guoliang Yu, *On gromov's compactness question regarding positive scalar curvature*, 2023.

[Yau82] Shing Tung Yau, *Problem section*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. No. 102, Princeton Univ. Press, Princeton, NJ, 1982, pp. 669–706. MR 645762

[Zei20] Rudolf Zeidler, *Width, largeness and index theory*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 127, 15. MR 4181525

[Zei22] ———, *Band width estimates via the Dirac operator*, J. Differential Geom. **122** (2022), no. 1, 155–183. MR 4507473

[Zhu24] Jintian Zhu, *Calabi-Yau type theorem for complete manifolds with nonnegative scalar curvature*, available at https://arxiv.org/abs/2402.15118 (2024).

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