Homework Set 6

Due: Never

1. Cheng’s Theorem (Rigidity in Bonnet-Myers) If \((M^n, g)\) has \(Rc \geq (n - 1)k > 0\) and \(\text{diam}(M, g) = \frac{\pi}{\sqrt{k}}\), then \((M^n, g)\) is isometric to the round sphere \(S^n_k\).

2. Show that if \((M^n, g)\) has \(\text{sec} \geq k > 0\) and \(\text{vol}(M, g) > \frac{1}{2}\text{vol}(S^n_k)\), then \(\text{diam}(M, g) > \frac{1}{2}\text{diam}(S^n_k)\) and hence \(M\) is homeomorphic to a sphere.

3. Show that if \((M^n, g)\) has \(Rc \geq (n - 1)k > 0\) and \(\text{vol}(M, g) > \frac{1}{2}\text{vol}(S^n_k)\), then \(M\) is simply connected.

4. Show that if \((M^n, g)\) has \(\text{sec} \geq k > 0\) and \(\text{diam}(M, g) > \frac{\pi}{\sqrt{k}}\), then given any point \(p \in M\), there exists a unique point \(q \in M\) with \(d(p, q) = \text{diam}(M, g)\).

5. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold that is isometric to Euclidean space outside some compact subset \(K \subset M\), i.e., \(M \setminus K\) is isometric to \(\mathbb{R}^n \setminus C\) for some compact set \(C \subset \mathbb{R}^n\). If \(Rc(g) \geq 0\), show that \(M\) isometric to \(\mathbb{R}^n\).

6. Is the converse to the Bishop-Gromov Inequality true? In other words, if, for a complete \(n\)-dimensional Riemannian manifold \(M\), say simply connected, there is \(k \in \mathbb{R}\) such that: defining \(V_k(R)\) to be the size of a ball of radius \(R\) in the standard space \(S^n_k\), for any \(p \in M\) and \(R \geq 0\) we have that

\[
\text{vol}_M(B_R(p)) \leq V_k(R)
\]

then is it true that on \(M\), \(Rc \geq (n - 1)k\)?