DEFORMING METRICS IN THE DIRECTION OF THEIR RICCI TENSORS

(Improved version)

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In [4], R. Hamilton has proved that if a compact manifold M of dimension three admits a C^∞ Riemannian metric g_0 with positive Ricci curvature, then it also admits a metric $\overline{g}$ with constant positive sectional curvature, and is thus a quotient of the sphere S^3. In fact, he shows that the original metric can be deformed into the constant-curvature metric by requiring that, for $t \geq 0$, $x \in M$ and $g = g(t,x)$,

$$\frac{\partial g}{\partial t} = \frac{2}{3} r_t g - 2\,\mathrm{Ric}(g), \qquad g(0,x) = g_0(x), \tag{1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of g on M at time t, and r_t is the average scalar curvature of the metric $g_t = g(t,x)$ over M, i.e.,

$$r_t = \frac{1}{\mathrm{Vol}_{g_t}(M)} \int_M \mathrm{Scal}(g_t)\, dV_{g_t}.$$

Hamilton's proof has two parts. In the first part, he proves local-in-time existence for the initial-value problem (IVP) (1), which is equivalent to proving local existence for the IVP

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g), \qquad g(0,x) = g_0(x) \tag{2}$$

(see [2, §3]). This part of the proof is valid for all dimensions $n \geq 3$. In the second part, which is specific to three dimensions, he proves that, as t approaches ∞, $g(t,x)$ approaches $\overline{g}(x)$ and that the Ricci curvature of g remains positive throughout the deformation.

To do the first (local) part of the proof, Hamilton uses a deep and powerful theorem from analysis: the Nash-Moser implicit-function theorem. (Some special technique is required because the IVP (2) is almost, but not strictly parabolic.) The purpose of this note is to prove local-in-time existence for (2) without recourse to the Nash-Moser theorem. In fact, our only analytic tools will be the "classical" existence and uniqueness theorems for initial-value problems for quasilinear parabolic systems and for systems of ordinary differential equations. The author gratefully acknowledges Philippe Delanoe, who organized a seminar at the Mathematical Sciences Research Institute in Berkeley in 1983, where this proof was discovered and presented.

Theorem. *For some $\varepsilon > 0$, the initial value problem*

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g), \qquad g(0,x) = g_0(x) \tag{2}$$

has a unique solution for $x \in M$ and $t \in [0, \varepsilon)$.

The idea of our proof is simple: we show that (2) is equivalent to an IVP for a parabolic system, modulo the action of the diffeomorphism group of M. In other words, we replace (2) by a parabolic IVP, and produce solutions $\tilde{g}_t$ of the new system. Then, we find a one-parameter family of diffeomorphisms ϕ_t of M having the property that the family of metrics $g_t = \phi_t^*(\tilde{g}_t)$ is a solution of (2). Conceptually, this proof is like the proof of local existence of metrics with prescribed Ricci curvature given in Chapter 5 of [1], and thus replaces the more unwieldy computational version given in [3].

We use the notation of [1] and [2]. In particular, $\mathrm{Ric}(g)$ denotes the Ricci tensor of the metric g, and with respect to the metric g (that will be clear from context), we define for any symmetric tensor $T \in S^2T^*(M)$,

$$\mathrm{tr}(T) = g^{kl}T_{kl}, \qquad G(T)_{ij} = T_{ij} - \frac{1}{2}(\mathrm{tr}(T))g_{ij},$$

$$\delta(T)_i = -g^{jk}T_{ij|k}$$

Note that δ maps S^2T^* to T^*, and so its L^2 adjoint δ^* maps T^* to S^2T^* as follows: for $v \in T^*$,

$$\delta^*(v)_{ij} = \frac{1}{2}(v_{i|j} + v_{j|i}).$$

The IVP (2) is not parabolic because the right-hand side $-2\,\mathrm{Ric}(g)$ is not an elliptic operator. The linearization of the Ricci operator is

$$\mathrm{Ric}'(g)h = \frac{1}{2}\Delta_L h - \delta^*(\delta G(h))$$

where Δ_L is the Lichnerowicz Laplacian, and the other term in the linearization is such that symmetric squares of one-forms are in the kernel of the symbol of the entire operator. This is demonstrated in [2], where it is also shown that, for *any* fixed invertible symmetric tensor field $T \in S^2T^*$, it is the case that the expression:

$$\delta^*(T^{-1}\delta G(T))$$

considered as a second-order quasilinear operator *on the metric* g, has precisely the same symbol as the second term in the symbol of the Ricci operator. Therefore the operator

$$Q(g) = \mathrm{Ric}(g) - \delta^*(T^{-1}\delta G(T))$$

has the same symbol as the Laplacian, and so is elliptic. Therefore the IVP

$$\frac{\partial g}{\partial t} = -2Q(g), \qquad g(0) = g_0 \tag{3}$$

is a parabolic IVP (once a tensor T has been fixed — a reasonable choice for T would be to let $T = g_0$). Because it is a quasilinear parabolic IVP, (3) has a solution for small time by the standard parabolc existence theorems.

To show how to get solutions of (2) from those of (3), we need the following two lemmas:

LEMMA 1. *Let* $v(y,t)$, *(*$y \in M$, $t \in {}^+$*) be a time-varying vector field on* M. *Then for small* t, *there exists a unique family of diffeomorphisms* $\phi_t \colon M \to M$ *such that*

$$\frac{\partial \phi_t(x)}{\partial t} = v(\phi_t(x), t)$$

for all $x \in M$ *and with* ϕ_0 *equal to the identity diffeomorphism.*

Proof. The standard proof when v does not depend on t still applies, via the existence and uniqueness theorem for ordinary differential equations, see e.g., [5].

LEMMA 2. *Let* $g_{ij}(y,t)$ *(*$y \in M$, $t \in {}^+$*) be a time-varying Riemannian metric on* M, *and* ϕ_t *be the family of diffeomorphisms from Lemma 1. Then:*

$$\frac{\partial \phi_t^*(g)}{\partial t}(x) = \phi_t^*\left(\frac{\partial g}{\partial t}(\phi_t(x))\right) + 2\phi_t^*\left(\delta^*(v^\flat(\phi_t(x)))\right)$$

where the δ^* *and* $\flat$ *(map from vector fields to one-forms) operations are those of* $g(y,t)$.

Proof. In local coordinates,

$$\phi_t^*(g)_{ij} = \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j} g_{\alpha\beta}(\phi_t(x), t)$$

therefore

$$\begin{aligned}\frac{\partial \phi_t^*(g)}{\partial t} &= \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j}\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial v^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j} g_{\alpha\beta} + \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial v^\beta}{\partial x^j} g_{\alpha\beta} + \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j}\frac{\partial g_{\alpha\beta}}{\partial \phi^k} v^k \\ &= \phi_t^*\left(\frac{\partial g}{\partial t}\right) + \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j}\left[\frac{\partial v^\gamma}{\partial \phi^\alpha} g_{\gamma\beta} + \frac{\partial v^\gamma}{\partial \phi^\beta} g_{\gamma\alpha} + \frac{\partial g_{\alpha\beta}}{\partial \phi^\gamma} v^\gamma\right] \\ &= \phi_t^*\left(\frac{\partial g}{\partial t}\right) + 2\phi_t^*(\delta^*(v^\flat)).\end{aligned}$$

Proof of the Theorem. To get solutions of (2) from those of (3), let g be the solution of (3), and let v be the vector field associated via g to the one-form

$$v^\flat = -T^{-1}(\delta G(T))$$

obtained using T and g. Finally, let ϕ_t be the family of diffeomorphisms obtained by integrating v using Lemma 1. Then:

$$\begin{aligned}\frac{\partial \phi_t^*(g)}{\partial t} &= \phi_t^*\left(\frac{\partial g}{\partial t}\right) + 2\phi_t^*(\delta^*(v^\flat)) \\ &= \phi_t^*(-2Q(g)) + 2\phi_t^*\left(\delta^*(-T^{-1}\delta G(T))\right) \\ &= \phi_t^*\left(-2(\mathrm{Ric}(g) - \delta^*(T^{-1}\delta G(T)))\right) + 2\phi_t^*\left(\delta^*(-T^{-1}\delta G(T))\right) \\ &= -2\,\mathrm{Ric}(\phi_t^*(g)) - 2\phi_t^*\left(\delta^*(-T^{-1}\delta G(T))\right) + 2\phi_t^*\left(\delta^*(-T^{-1}\delta G(T))\right) \\ &= -2\,\mathrm{Ric}(\phi_t^*(g))\end{aligned}$$

Thus $\phi_t^*(g)$ satisfies the initial-value problem (2), which was to be shown.

References

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