Cohomology of Harmonic Forms on Riemannian Manifolds With Boundary

Sylvain Cappell, Dennis DeTurck, Herman Gluck, and Edward Y. Miller

To Julius Shaneson on the occasion of his 60th birthday

1. Introduction

The main result of this article is the following.

Theorem 1. *Let M be a compact, connected, oriented, smooth Riemannian n-dimensional manifold with non-empty boundary. Then the cohomology of the complex* $(\mathrm{Harm}^*(M), d)$ *of harmonic forms on M is given by the direct sum:*

$$H^p(\mathrm{Harm}^*(M), d) \cong H^p(M; \mathbb{R}) + H^{p-1}(M; \mathbb{R})$$

for $p = 0, 1, \ldots, n$.

Let M be a smooth compact n-manifold, and $\Omega^*(M)$ the space of smooth differential forms on M. The classical theorem of de Rham [1931] asserts that the cohomology of the complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \longrightarrow 0,$$

where d is exterior differentiation, is isomorphic to the cohomology of M with real coefficients. In other words,

$$\frac{\ker d : \Omega^p(M) \to \Omega^{p+1}(M)}{\mathrm{im}\, d : \Omega^{p-1}(M) \to \Omega^p(M)} \cong H^p(M; \mathbb{R}).$$

If M is oriented, a Riemannian metric on M gives rise to an L^2 inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

on $\Omega^*(M)$, where $*$ denotes the Hodge star operator, and to the co-differential

$$\delta = (-1)^{n(p+1)+1} {*d*} \colon \Omega^p(M) \to \Omega^{p-1}(M),$$

which on a closed manifold is the L^2-adjoint of the exterior differential d. As usual, one defines the Laplacian by

$$\Delta = d\delta + \delta d \colon \Omega^p(M) \to \Omega^p(M),$$

and *harmonic* differential forms ω as those satisfying $\Delta\omega = 0$.

The exterior differential d, since it commutes with Δ, preserves harmonicity of forms, and hence $(\mathrm{Harm}^*(M), d)$ is a subcomplex of the de Rham complex $(\Omega^*(M), d)$. It is therefore natural to compute the cohomology of this complex, which we call the *harmonic cohomology* of M.

When M is a closed manifold, a form ω is harmonic if and only if it is both closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$). In this case, all of the maps in the complex $(\mathrm{Harm}^*(M), d)$ are zero, and so

$$H^p(\mathrm{Harm}^*(M), d) = \mathrm{Harm}^p(M) \cong H^p(M; \mathbb{R}),$$

according to the classical theorem of Hodge [1933].

By contrast, when M is connected and has non-empty boundary, it is possible for a p-form to be harmonic without being both closed and co-closed. Some of these, which are exact (that is, in the image of d), although not the exterior derivatives of *harmonic* $p-1$-forms, represent the "echo" of the ordinary $p-1$-dimensional cohomology within the p-dimensional harmonic cohomology reported in Theorem 1.

Example: Let M be the annulus $a^2 \leq x^2 + y^2 \leq b^2$ in the xy-plane $\mathbb{R}^2$. Then the 2-form $\omega = -\frac{1}{2}\log(x^2 + y^2)\, dx \wedge dy$ is harmonic because $\log(x^2 + y^2)$ is a harmonic function on the annulus. Even though ω is exact, it is not the exterior derivative of any harmonic 1-form. Hence it represents a nonzero element of the 2-dimensional harmonic cohomology $H^2(\mathrm{Harm}^*(M), d)$. Also, ω is not co-closed, in fact

$$\delta\omega = \varphi = (-y\, dx + x\, dy)/(x^2 + y^2),$$

which represents a generator of the 1-dimensional cohomology $H^1(M; \mathbb{R}) \cong \mathbb{R}$. The equation $\delta\omega = \varphi$ is the signal that the 2-form ω is the echo of the 1-form φ, as we will see in Lemma 3 below.

Remarks: 1. The co-differential δ also commutes with the Laplacian Δ, and therefore $(\mathrm{Harm}^*(M), \delta)$ is a subcomplex of $(\Omega^*(M), \delta)$. We can apply the Hodge star operator to the isomorphism given by Theorem 1 and replace $n - p$ by p to obtain

$$H^p(\mathrm{Harm}^*(M), \delta) \cong H^p(M, \partial M; \mathbb{R}) + H^{p+1}(M, \partial M; \mathbb{R}),$$

where $H^p(M, \partial M; \mathbb{R})$ is the cohomology of M relative to its boundary. In this case, the homological "echo" is shifted *down* by one unit.

2. Theorem 1, Remark 1 and their proofs can be readily generalized to harmonic forms with coefficients in a flat bundle with metric.

3. It would be interesting to understand to what extent these results have analogues for harmonic forms on smoothly stratified manifolds with singularities. In this regard, we look to the work of Cheeger [1980], and Mazzeo and Melrose [1999]. It would also be interesting to consider analogous questions for the $\overline{\partial}$ operator, cf. Epstein [2005] for

some results on solutions of $\overline{\partial}$ equations with modified $\overline{\partial}$-Neumann conditions along the boundary.

4. In the proof of Theorem 1 given here, we never make use of the structure of the space of all harmonic forms on M, but focus only on the closed ones. Theorem 3.25 on pp 48–49 of Parsley [2004] gives the structure of all harmonic vector fields on a compact Riemannian 3-manifold with boundary; the analogous result holds for differential p-forms on compact Riemannian n-manifolds with boundary.

2. The Hodge Decomposition Theorem

Conventions and definitions. In what follows, the reference to the manifold M is understood, and so we omit it and write Ω^p for the space of smooth differential p-forms on M. We will write C^p and cC^p for the spaces of closed and co-closed p-forms on M, and E^p and cE^p for the spaces of exact and co-exact (that is, in the image of δ) p-forms on M. We juxtapose letters to indicate intersections of spaces, so CcC^p is the subspace of p-forms which are both closed and co-closed (these were called *harmonic fields* by Kodaira [1949]). Similarly, $EcC^p = E^p \cap cC^p \subset CcC^p$ and $CcE^p = C^p \cap cE^p \subset CcC^p$. Finally, we use the symbol $+$ between spaces to indicate a direct sum, and reserve $\oplus$ for an orthogonal direct sum.

To prepare for the proof of Theorem 1, we consider boundary conditions on differential forms and the related Hodge decompositions of Ω^* on manifolds with boundary. Along the boundary of M, any smooth differential p-form ω has a natural decomposition into tangential and normal components. For $x \in \partial M$, we write

$$\omega(x) = \omega_{\tan}(x) + \omega_{\text{norm}}(x),$$

where $\omega_{\tan}(x)$ agrees with $\omega(x)$ when evaluated on a p-tuple of vectors, all of which are tangent to ∂M, but is zero if any one of the vectors is orthogonal to ∂M. We then define $\omega_{\text{norm}}(x)$ by the above equation. We have that $\omega_{\tan}(x) = 0$ if and only if the restriction $(\omega|_{\partial M})(x) = 0$.

Let Ω^p_N be the space of smooth p-forms on M that satisfy *Neumann boundary conditions* at every point of ∂M,

$$\Omega^p_N = \{\omega \in \Omega^p \mid \omega_{\text{norm}} = 0\},$$

and similarly let Ω^p_D be the space of smooth p-forms on M that satisfy *Dirichlet boundary conditions* at every point of ∂M,

$$\Omega^p_D = \{\omega \in \Omega^p \mid \omega_{\tan} = 0\}.$$

We define $cE^p_N = \delta(\Omega^{p+1}_N)$ and $E^p_D = d(\Omega^{p-1}_D)$, and emphasize that the boundary conditions are applied *before* we take co-differentials and differentials.

As noted above, on a closed manifold, $CcC^p(M)$ and $\mathrm{Harm}^p(M)$ coincide, but in the presence of a boundary, there are more harmonic forms than fields. We apply the boundary conditions to $CcC^p(M)$ as follows:

$$CcC^p_N = \{\omega \in \Omega^p \mid d\omega = 0,\ \delta\omega = 0,\ \omega_{\text{norm}} = 0\}$$
$$CcC^p_D = \{\omega \in \Omega^p \mid d\omega = 0,\ \delta\omega = 0,\ \omega_{\text{tan}} = 0\}.$$

Hodge Decomposition Theorem[1] . *Let M be a compact, connected, oriented, smooth Riemannian n-manifold, with or without boundary. Then we have the orthogonal direct sum*

$$\Omega^p = cE^p_N \oplus CcC^p \oplus E^p_D. \tag{1}$$

Furthermore,

$$CcC^p = CcC^p_N \oplus EcC^p = CcE^p \oplus CcC^p_D.$$

When the manifold M is closed, the boundary conditions are vacuous, and we get the original Hodge decomposition, $\Omega^p = cE^p \oplus CcC^p \oplus E^p$. In this case, $C^p = CcC^p \oplus E^p$, and thus CcC^p is the orthogonal complement of the exact p-forms within the closed ones, so $CcC^p \cong H^p(M;\mathbb{R})$. Likewise, $cC^p = cE^p \oplus CcC^p$, and so CcC^p is simultaneously the orthogonal complement of the co-exact p-forms within the co-closed ones.

When the boundary of M is non-empty, the space C^p of closed p-forms decomposes as

$$C^p = CcC^p \oplus E^p_D = CcC^p_N \oplus EcC^p \oplus E^p_D = CcC^p_N \oplus E^p.$$

Thus, CcC^p_N is the orthogonal complement of the exact p-forms within the closed ones, so $CcC^p_N \cong H^p(M;\mathbb{R})$. Similarly, the space cC^p of co-closed p-forms decomposes as

$$cC^p = cE^p_N \oplus CcC^p = cE^p_N \oplus CcE^p \oplus CcC^p_D = cE^p \oplus CcC^p_D.$$

Thus, CcC^p_D is the orthogonal complement of the co-exact p-forms within the co-closed ones, so $CcC^p_D \cong H^p(M, \partial M;\mathbb{R})$.

All the decompositions given above are canonical, once the Riemannian metric on M is specified.

[1] The Hodge Decomposition Theorem arose historically with increasing generality in the papers and books of de Rham [1931], Hodge [1933], Weyl [1940], Hodge [1941], Tucker [1941], Weyl [1943], Bidal and de Rham [1946], Kodaira [1949], Duff [1952], Duff and Spencer [1952], de Rham [1955], Friedrichs [1955], Conner [1955] and Morrey [1956].

3. The image of the Laplacian

If M is a closed, oriented Riemannian n-manifold, the Hodge Decomposition Theorem tells us that $\Omega^p = cE^p \oplus CcC^p \oplus E^p$. The Laplacian Δ acting on p-forms is self-adjoint, and its image $\Delta(\Omega^p)$ is the orthogonal complement $cE^p \oplus E^p$ of its kernel CcC^p. Thus $\Omega^p = CcC^p \oplus \Delta(\Omega^p)$.

By contrast, when the boundary of the manifold is non-empty, we have

Lemma 1. *Let M be a compact, connected, oriented, smooth Riemannian n-manifold with non-empty boundary. Then the Laplacian on forms, $\Delta\colon \Omega^p \to \Omega^p$, is surjective.*

Proof. Equation (1) in the Hodge Decomposition Theorem asserts that $\Omega^p = cE^p_N \oplus CcC^p \oplus E^p_D$, and we will compute the image of the Laplacian on each summand.

On cE^p_N, we have $\Delta = \delta d$. Since $C^p = CcC^p \oplus E^p_D$, the exterior derivative d must take cE^p_N isomorphically to $E^{p+1} = EcC^{p+1} \oplus E^{p+1}_D$. Applying the co-differential δ to this, we see that δ kills EcC^{p+1} and takes E^{p+1}_D isomorphically to cE^p. Thus

$$\Delta(cE^p_N) = cE^p = cE^p_N \oplus CcE^p.$$

Likewise,

$$\Delta(E^p_D) = E^p = EcC^p \oplus E^p_D.$$

And naturally, $\Delta(CcC^p) = 0$.

Referring again to the Hodge decomposition (1), we see that the only way that the Laplacian $\Delta\colon \Omega^p \to \Omega^p$ could fail to be surjective would be for CcE^p and EcC^p to fail to span CcC^p. But from the Hodge Decomposition Theorem, the orthogonal complement of CcE^p in CcC^p is $CcC^p_D \cong H^p(M, \partial M; \mathbb{R})$, and the orthogonal complement of EcC^p in CcC^p is $CcC^p_N \cong H^p(M; \mathbb{R})$. Thus the subspaces in question both have finite codimension in CcC^p, and so the only way they could fail to span CcC^p would be for some non-zero $\omega \in CcC^p$ to be orthogonal to both subspaces. This would force ω to lie in $CcC^p_D \cap CcC^p_N$, telling us that ω is closed, co-closed, and vanishes on the boundary of M. But such a form must be zero, according to the following Lemma, which will complete the proof of Lemma 1.

Lemma 2. *On a connected, oriented, smooth Riemannian n-manifold with non-empty boundary, a smooth differential form which is both closed and co-closed, and which vanishes on the boundary, must be identically zero.*

In order to prove Lemma 2, we will appeal to the "strong unique continuation theorem", orginally due to Aronszajn [1957], Aronszajn, Krzywicki and Szarski [1962], and given by Kazdan [1988] in the following form:

Strong Unique Continutation Theorem. *Let N be a Riemannian manifold with Lipshitz continuous metric, and let ω be a differential form having first derivatives in L^2 that satisfies $\Delta\omega = 0$. If ω has a zero of infinite order at some point in N, then ω is identically zero.*

Proof of Lemma 2. Let M be a connected, oriented, smooth Riemannian n-manifold with non-empty boundary, and ω a smooth differential p-form on M which is closed, co-closed, and vanishes on ∂M. We will show that ω is identically zero. Since the result is local, we can take M to be the upper half-space in $\mathbb{R}^n$, with $\partial M = \mathbb{R}^{n-1}$.

Extend the metric from the upper half-space to all of $\mathbb{R}^n$ by reflection in $\mathbb{R}^{n-1}$. The resulting metric will be Lipschitz continuous. Extend the p-form ω to all of $\mathbb{R}^n$ by making it odd with respect to reflection in $\mathbb{R}^{n-1}$. Because the original ω vanished on $\mathbb{R}^{n-1}$ and was closed and co-closed, the extended ω will be of class C^1 and will be closed and co-closed on all of $\mathbb{R}^n$.

These facts, together with the vanishing of ω on $\mathbb{R}^{n-1}$, are enough to show that the first derivatives of the coefficients of ω vanish along $\mathbb{R}^{n-1}$, even when computed in the normal direction. Repeated differentiation of the equations which express the fact that ω is closed and co-closed, together with the vanishing of ω on $\mathbb{R}^{n-1}$, show that all higher partial derivatives of the coefficients of ω vanish on $\mathbb{R}^{n-1}$. In other words, ω vanishes to infinite order at each point of $\mathbb{R}^{n-1}$.

The Strong Unique Continuation Theorem then implies that ω must be identically zero on all of $\mathbb{R}^n$. Since M was assumed to be connected, ω must be identically zero on all of M. This completes the proof of Lemma 2, and with it, the proof of Lemma 1.

For a different proof of Lemma 1, see Theorem 3.4.10 on page 137 of Schwarz [1995].

4. Proof of Theorem 1

To prove Theorem 1, we must show that

$$H^p(\mathrm{Harm}^*(M), d) \cong H^p(M; \mathbb{R}) + H^{p-1}(M; \mathbb{R}).$$

By definition, we have

$$H^p(\mathrm{Harm}^*(M), d) = \frac{\mathrm{CHarm}^p}{d(\mathrm{Harm}^{p-1})},$$

where CHarm^p denotes the set $C^p \cap \mathrm{Harm}^p$ of p-forms which are both closed and harmonic. Recalling that CcC_N^p is the orthogonal complement of the exact p-forms within the closed ones, we can write

$$\mathrm{CHarm}^p = CcC_N^p \oplus \mathrm{EHarm}^p,$$

where EHarm^p denotes the space of exact harmonic p-forms. We naturally have $d(\mathrm{Harm}^{p-1}) \subset \mathrm{EHarm}^p$, and thus get a direct-sum decomposition

$$H^p(\mathrm{Harm}^*(M), d) = CcC_N^p + \frac{\mathrm{EHarm}^p}{d(\mathrm{Harm}^{p-1})}.$$

The first term on the right is isomorphic to $H^p(M;\mathbb{R})$. The second term on the right measures the extent to which a harmonic p-form can be exact without actually being the exterior derivative of a harmonic $p-1$-form. This is the term that we claim to be the echo of $H^{p-1}(M;\mathbb{R})$. As suggested by the example in section 1, this isomorphism is provided by the co-differential δ. We demonstrate this in the following lemma, which will complete the proof of Theorem 1.

Lemma 3. *Under the assumptions of Theorem 1, the co-differential* $\delta\colon \Omega^p \to \Omega^{p-1}$ *induces an isomorphism*

$$\overline{\delta}\colon \frac{\mathrm{EHarm}^p}{d(\mathrm{Harm}^{p-1})} \to H^{p-1}(M;\mathbb{R}).$$

That is, the isomorphism $\overline{\delta}$ takes the echo back to its source.

Proof of Lemma 3. We show that the linear map

$$\overline{\delta}\colon \frac{\mathrm{EHarm}^p}{d(\mathrm{Harm}^{p-1})} \to \frac{C^{p-1}}{E^{p-1}} \cong H^{p-1}(M;\mathbb{R})$$

is well-defined by seeing that the numerator of the domain of $\overline{\delta}$ maps to the numerator of its range, and likewise for the denominators. First, if $\varphi \in \mathrm{EHarm}^p$, then φ is an exact, harmonic p-form. Being exact, φ is certainly closed, hence $\Delta\varphi = (\delta d + d\delta)\varphi = d\delta\varphi = 0$. Thus $\delta\varphi$ is a closed $p-1$-form. Second, if $\varphi \in d(\mathrm{Harm}^{p-1})$ is the exterior derivative of a harmonic $p-1$-form β, then $\delta\varphi = \delta d\beta = -d\delta\beta$, showing that $\delta\varphi$ is an exact $p-1$-form. Hence $\overline{\delta}$ is well-defined.

Next, we show that $\overline{\delta}$ is one-to-one. To this end, suppose that $\varphi \in \mathrm{EHarm}^p$ and that $\delta\varphi \in E^{p-1}$. We must show that $\varphi \in d(\mathrm{Harm}^{p-1})$. Since φ is exact, write $\varphi = d\beta$ for $\beta \in \Omega^{p-1}$, and note that the Laplacian of β is exact, since

$$\Delta\beta = \delta d\beta + d\delta\beta = \delta\varphi + d\delta\beta \in E^{p-1}.$$

Thus $\Delta\beta = d\eta$ for some $p-2$-form η. Since the Laplacian on $p-2$-forms is surjective (Lemma 2), we write $\eta = \Delta\sigma$. Then, because $\Delta\beta = d\eta = d\Delta\sigma = \Delta d\sigma$, we have that $\beta - d\sigma$ is harmonic. Finally, writing $\varphi = d(\beta - d\sigma)$ shows that $\varphi \in d(\mathrm{Harm}^{p-1})$, as desired.

Finally, to prove that $\overline{\delta}$ is surjective, given $\alpha \in C^{p-1}$, we must find an exact harmonic form $\varphi \in \mathrm{EHarm}^p$ such that $\delta\varphi - \alpha \in E^{p-1}$. Using the surjectivity of the Laplacian on $p-1$-forms (Lemma 2 again), we write $\alpha = \Delta\beta$, and then let $\varphi = d\beta$. Note that $\Delta\varphi = \Delta d\beta = d\Delta\beta = d\alpha = 0$, since α is closed. Therefore φ is harmonic, and hence lies in EHarm^p. Now,

$$\delta\varphi = \delta d\beta = \Delta\beta - d\delta\beta = \alpha - d\delta\beta,$$

so $\delta\varphi - \alpha = -d\delta\beta$, showing that $\delta\varphi - \alpha$ is exact, as desired.

This completes the proof of Lemma 3, and with it, the proof of Theorem 1.

References

[1931] G. de Rham, *Sur l'analysis situs des variétés à n dimensions*, J. Math. Pures Appl., **10**, 115–200.

[1933] W.V.D. Hodge, *A Dirichlet problem for harmonic functions with applications to analytic varieties*, Proc. London Math. Soc., **36**, 257–303.

[1940] H. Weyl, *The method of orthogonal projection in potential theory*, Duke Math. J., **7**, 411–444.

[1941] W.V.D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, Cambridge.

[1941] A. Tucker, *A boundary-value theorem for harmonic tensors*, Bull. AMS, **47**, 714.

[1943] H. Weyl, *On Hodge's theory of harmonic integrals*, Annals of Math., **44**, 1–6.

[1946] P. Bidal and G. de Rham, *Les formes différentielles harmoniques*, Comment. Math. Helv., **19**, 1–49.

[1949] K. Kodaira, *Harmonic fields in Riemannian manifolds (Generalized potential theory)*, Annals of Math., **50**, 587–665.

[1952] G.F.D. Duff, *Differential forms in manifolds with boundary*, Annals of Math., **56**, 115–127.

[1952] G.F.D. Duff and D. Spencer, *Harmonic tensors on Riemannian manifolds with boundary*, Annals of Math. **56**, 127–156.

[1955] K.O. Friedrichs, *Differential forms on Riemannian manifolds*, Comm. Pure Appl. Math., **8**, 551–590.

[1955] G. de Rham, *Variétés Différentiables*, Hermann, Paris (English edition: *Differentiable Manifolds*, Grund. der Math. Wiss., **266**, Springer-Verlag, Berlin 1984).

[1956] P. Conner, *The Neumann problem for differential forms on Riemannian manifolds*, Memoirs of the AMS **20**, AMS, Providence.

[1956] C.B. Morrey, *A variational method in the theory of harmonic integrals*, II, Amer. J. Math., **78**, 137–170.

[1957] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures et Appl., **36**, 235–249.

[1962] N. Aronszajn, A. Krzywicki and J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Ar. Kat., **4**, 417–453.

[1966] C.B. Morrey, *Multiple Integrals in the Calculus of Variations*, Grund. der Math. Wiss., **130**, Springer-Verlag, Berlin.

[1980] J. Cheeger, *On the Hodge theory of Riemannian Pseudomanifolds*, Proc. Sympos. Pure Math., **36**, (*Geometry of the Laplace operator*), 91–146.

[1988] J. Kazdan, *Unique continuation in geometry*, Comm. Pure Appl. Math., **41**, 667–681.

[1995] G. Schwarz, *Hodge Decomposition — A Method for Solving Boundary Value Problems*, Lecture Notes in Math., **1607**, Springer-Verlag, Berlin.

[1999] R. Mazzeo and R. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries*, Asian J. of Math., **2**, 833–866.

[2004] J. Parsley, *The Biot-Savart operator and electrodynamics on bounded subdomains of the three-sphere*, Ph.D. thesis, University of Pennsylvania.

[2005] C. Epstein, *Subelliptic* $Spin_C$ *Dirac operators, I and II*, to appear.

Sylvain Cappell: Courant Institute, New York University, *cappell@cims.nyu.edu*
Dennis DeTurck: University of Pennsylvania, *deturck@math.upenn.edu*
Herman Gluck: University of Pennsylvania, *gluck@math.upenn.edu*
Edward Y. Miller: Polytechnic University, *emiller@poly.edu*