Metrics with prescribed Ricci curvature of constant rank. I. The integrable case

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Introduction

In this paper, we give local existence results in the real-analytic category for the non-linear partial differential equation arising from the problem of finding Riemannian metrics with prescribed Ricci curvature. Let R be a given symmetric 2-form on a real-analytic manifold X of dimension $n \geq 3$; we seek (locally) a Riemannian metric g whose Ricci curvature $\mathrm{Ric}(g)$ is equal to R, that is, we wish to solve the equation

$$\mathrm{Ric}(g) = R \tag{1}$$

for the metric g. When R is non-degenerate, a local solution always exists both in the C^∞-case and in the real-analytic category (see [3]). Here we are interested in the case when R is degenerate, but still has constant rank. We also impose an additional condition on R, namely, that the kernel K of the morphism $R^\flat$ from the tangent bundle T of X to the cotangent bundle T^* of X determined by R be an integrable sub-bundle of T. Our main result asserts that, if R is real-analytic and if the 1-jet of R at a point $x \in X$ satisfies certain appropriate conditions, the equation (1) admits local real-analytic solutions in a neighborhood of x.

Our work on metrics with degenerate Ricci tensors began with a study of symmetric 2-forms R for which the kernel K has rank one. In this case, K is always an integrable sub-bundle of T. We discovered a simple sufficient condition on the 1-jet of R (see Theorem 3 below) that guarantees the local existence of a solution of equation (1).

Left-invariant metrics on certain three-dimensional non-unimodular Lie groups provide simple examples of Riemannian metrics which are not product metrics and whose Ricci tensors have corank one. Let α be a positive real number, and let $\mathfrak{g}$ be the Lie algebra, endowed with a positive-definite inner product, which possesses an orthonormal basis $\{e_1, e_2, e_3\}$

[1] The first author was supported in part by NSF Grant DMS 92-03362 and the second author was supported in part by NSF Grant DMS 92-03974.

such that

$$[e_1, e_2] = \frac{2+\alpha^2}{1+\alpha^2}\, e_2 + \frac{(2+\alpha^2)\alpha}{1+\alpha^2}\, e_3,$$
$$[e_1, e_3] = \frac{-\alpha^3}{1+\alpha^2}\, e_2 + \frac{\alpha^2}{1+\alpha^2}\, e_3$$

and $[e_2, e_3] = 0$. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$ and let g be the left-invariant Riemannian metric on G determined by the inner product on $\mathfrak{g}$. According to [9, §6], the Ricci tensor of g is diagonal with respect to the basis of left-invariant vector fields on G obtained from the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$, and its eigenvalues are $-(4+2\alpha^2)/(1+\alpha^2)$, -2 and 0.

If the rank of the sub-bundle K is greater than one, then K may or may not be integrable. Each case seems to require a different approach; our original proof of Theorem 3 provided substantial insight into the problem when K is integrable and led to the results of this paper. Our analysis of the non-integrable case will appear in a subsequent publication.

The first non-trivial examples of symmetric 2-forms R whose kernels are of rank > 1 occur in dimension three; these forms R have rank one. In general, if a symmetric 2-form R on X has rank one, it can be locally written in the form $R = \pm\omega \otimes \omega$, where ω is a nowhere-vanishing one-form on X. Suppose that the kernel K is integrable; this assumption is equivalent to the equality $\omega \wedge d\omega = 0$. We may then choose a local coordinate system $x = (x^1, \ldots, x^n)$ and a function f on an open subset U of X such that $\omega = f(x)dx^n$; thus we have $R = \pm f(x)^2 dx^n \otimes dx^n$. If g is a solution of equation (1) on U, from the Bianchi identity for the Ricci curvature of g, we obtain the relations

$$g^{nn}\frac{\partial(f^2)}{\partial x^k} = 0$$

on U, for $1 \leq k \leq n-1$. These equalities express the vanishing of the traces, with respect to g, of the quadratic forms Q^k defined below. Since g^{nn} can never be zero for the Riemannian metric g, we conclude that $d\omega$ must be zero in order for a solution of (1) to exist. This remark leads to examples of non-existence in the case where ω is integrable, but not closed. Indeed, the one-form $\omega = (1 + x^1)dx^n$ is nowhere-vanishing on a neighborhood V of the origin in $\mathbb{R}^n$ and is integrable, and the 2-form $d\omega$ does vanish at any point of V; thus the equation (1) corresponding to the symmetric 2-form $R = \omega \otimes \omega$ does not admit a solution in any neighbood of any point of V. On the other hand, explicit solutions of equation (1) can be given in the case where ω is closed (see [1, Chapter 5]). Notice that, when $d\omega = 0$, the Ricci tensor is a product; however, when $n = 3$, since no metric on a manifold of dimension one or two has a Ricci tensor of rank one, a metric g which satisfies equation (1) cannot be a product metric.

Left-invariant metrics on $SU(2)$, $SL(2,\mathbb{R})$ and the group $E(1,1)$ of rigid motions of Minkowski 2-space provide us with examples of three-dimensional Riemannian manifolds whose Ricci curvatures have kernels of rank two which are not integrable (see [9, §4]).

Any analysis of the problem of prescribing Ricci curvature must take into account a necessary condition, which R must satisfy in order that equation (1) have local solutions. As indicated above, this condition arises from the Bianchi identity for the Ricci curvature, and only occurs when R is degenerate. In general, it leads to examples of degenerate symmetric 2-forms on $\mathbb{R}^n$ which are not Ricci tensors of any Riemannian metric near the origin (Theorem 2.1). To a point x of X and a vector ξ of K_x, we associate a quadratic form Q_ξ on the tangent space T_x of X at x, which depends only on R and ξ (Lemma 2.1). The obstruction to the local solvability of the equation (1) can be formulated as follows: At every point of X, the trace of each of these forms Q_ξ, with respect to a solution g of (1), must vanish. Our first positive result states that, if all these forms vanish at every point of X, the equation (1) admits local solutions in the real-analytic category (Theorem 6.1); this includes the case when R is non-degenerate.

We also assume the space of quadratic forms $\mathcal{Q}_x = \{Q_\xi\}$, with $\xi \in K_x$, associated to a point $x \in X$ has constant dimension m, independent of the point x. For $x \in X$, the null-space of the pencil $\mathcal{Q}_x$ of quadratic forms contains K_x, and so its dimension $\nu(x)$ is $\geq m$.

Let $(x^1, \dots, x^n)$ be a local coordinate system on a neighborhood U of $x \in X$ such that $\{\partial/\partial x^1, \dots, \partial/\partial x^r\}$ is a frame for the integrable subbundle K of T. In terms of this coordinate system, we have $R_{ij} = 0$, for $1 \leq i \leq n$ and $1 \leq j \leq r$; the quadratic form Q^k corresponding to the section $\partial/\partial x^k$ of K over U, with $1 \leq k \leq r$, is given by

$$Q^k_{ij} = -\frac{1}{2}\frac{\partial R_{ij}}{\partial x^k},$$

for $1 \leq i, j \leq n$. According to our assumption, at every point x of U, the vector space of quadratic forms on T_x generated by $\{Q^1(x), \dots, Q^r(x)\}$ is m-dimensional.

If $m = 0$, then on the open set U the tensor R depends only on the variables $x^{r+1}, \dots, x^n$. Thus, R respects the local product structure induced on X by the leaves of K. In this case, if $n - r \geq 3$, or if $n - r = 2$ and R is semi-definite, then the results of [3] can be used to prove the local existence of a product metric which satisfies equation (1). However, as indicated by the corank-one examples on Lie groups given above, there may be other solutions of (1) which do not respect the local product structure of X.

Our main result can be stated as follows:

THEOREM 1. *Let R be a real-analytic symmetric 2-form on a real-analytic manifold X of dimension $n \geq 3$. Suppose that the kernel K of $R^\flat$ is an integrable sub-bundle of T and that, for all $x \in X$, the space $\mathcal{Q}_x$ of quadratic forms $\{Q_\xi\}$, with $\xi \in K_x$, has constant dimension m. Let $x_0 \in X$ and suppose that there exists a Riemannian metric g_0 on X such that the trace (with respect to g_0) of the form Q_ξ vanishes, for all $\xi \in K_{x_0}$. Suppose that the following conditions do not hold:*
(i) *we have $m = 2$ and $n = 4$;*
(ii) *we have $m = 2$, $n = 5$ and $\nu(x_0) = 3$;*
(iii) *we have $m = 3$ and $n = 6$.*
Then there exists a real-analytic Riemannian metric g solution of the equation (1) on a neighborhood of x_0.

In the case where m is equal to the rank of the bundle K, Theorem 1 admits the following simplified reformulation:

THEOREM 2. *Let $n \geq 3$ and $0 \leq m \leq n$ be given integers. Let $R = (R_{ij})$ be a real-analytic symmetric 2-form on $\mathbb{R}^n$ defined in a neighborhood U of the origin satisfying $R_{ij} = 0$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, and*

$$R_{ij}(x) = R^0_{ij} + \sum_{k=1}^{m} Q^k_{ij} x^k + \sum_{l=m+1}^{n} R^l_{ij} x^l + O(|x|^2),$$

for all $m+1 \leq i, j \leq n$ and $x \in U$, where

$$R^0 = (R^0_{ij})_{m+1 \leq i,j \leq n}, \quad R^l = (R^l_{ij})_{m+1 \leq i,j \leq n}, \quad Q^k = (Q^k_{ij})_{m+1 \leq i,j \leq n},$$

with $m+1 \leq l \leq n$ and $1 \leq k \leq m$, are $(n-m) \times (n-m)$ symmetric real matrices which possess the following three properties:
(i) *the matrix R^0 is invertible;*
(ii) *the matrices $Q^1, \dots, Q^m$ are linearly independent;*
(iii) *the matrices $Q^1, \dots, Q^m$ are traceless.*
If ν is the dimension of the common null-space (in $\mathbb{R}^{n-m}$) of the matrices $Q^1, \dots, Q^m$, suppose that the following conditions do not hold:
(iv) *we have $m = 2$ and $n = 4$;*
(v) *we have $m = 2$, $n = 5$ and $\nu = 1$;*

(vi) *we have* $m = 3$ *and* $n = 6$.
Then there exists a real-analytic Riemannian metric g *on a neighborhood of the origin in* $\mathbb{R}^n$ *which is a solution of the equation* (1).

We remark that, in the above theorem, conditions (i) and (ii) imply that, over a neighborhood of the origin, the kernel K of $R^\flat$ is the sub-bundle generated by $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$ and that the m quadratic forms

$$\left\{ \frac{\partial R_{ij}}{\partial x^1}(x), \ldots, \frac{\partial R_{ij}}{\partial x^m}(x) \right\}$$

on T_x are linearly independent for all x in a neighborhood of the origin. Condition (iii) says that the traces of these quadratic forms, with respect to the Euclidean metric $g_{0,ij} = \delta_{ij}$, vanish at the origin. Thus Theorem 2 follows directly from Theorem 1.

Let R be a symmetric 2-form on $\mathbb{R}^n$ of the form described in Theorem 2. Suppose that condition (i) of the theorem holds, but that either condition (ii) or condition (iii) is violated. Then there may be no solution of equation (1) in any neighborhood of the origin, as shown by the following examples. Let $p \geq 1$ be a given integer. The symmetric 2-form

$$R = dx^2 \otimes dx^2 + (1 + (x^1)^p)\, dx^3 \otimes dx^3$$

on $\mathbb{R}^3$ is of the form considered in Theorem 2, with $m = 1$, and satisfies condition (i). We see that

$$Q^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

when $p = 1$, and that Q^1 vanishes when $p \geq 2$. Thus if $p = 1$, condition (ii) of the theorem holds, but Q^1 is definite, and so condition (iii) is violated. If $p \geq 2$, condition (ii) does not hold, but (iii) is true. In either case, there does not exist a solution of the equation (1) on any neighborhood of the origin. Indeed, by considering the x^1-derivative of order $p - 1$ of the first component of the Bianchi identity, we see that a solution g of (1) must satisfy the relation $g^{33} = 0$ at the origin, which is impossible for any Riemannian metric on $\mathbb{R}^3$.

In the special case when K is one-dimensional, the obstruction to local solvability described above and Theorem 1 lead to a condition, which is essentially necessary and sufficient, given by the following result:

Theorem 3. *Let R be a real-analytic symmetric 2-form on a real-analytic manifold X of dimension $n \geq 3$. Suppose that the kernel K of $R^\flat$ is a sub-bundle of T of rank 1. Let $x \in X$ and $\xi \in K_x$. If Q_ξ is non-zero, there exists a Riemannian metric g solution of the equation (1) on a neighborhood of x if and only if Q_ξ is not semi-definite.*

We now present an outline of the proof of Theorem 1. We denote by $\bigotimes^k T^*$ and $S^l T^*$ the k-th tensor power and the l-th symmetric product of the cotangent bundle T^*. If g is a Riemannian metric on X, we denote by $\langle\ ,\ \rangle_g$ is the inner product on $\bigotimes^k T^*$ determined by g; then $\operatorname{Tr}_g h = \langle h, g\rangle_g$ is the trace, with respect to g, of an element h of S^2T^*. We denote by $j_k(s)$ the jet of order k of a section s of a vector bundle. A Riemannian metric g which is a solution of (1) must necessarily satisfy an equation

$$\operatorname{Bian}(g, R) = 0 \tag{2}$$

obtained from the Bianchi identity for the Ricci curvature; here $\operatorname{Bian}(g, R)$ is a section of T^* over X. If R is degenerate, we choose a fixed complement T' of K in T which allows us to split this equation into two pieces. The first one is a first-order equation which we modify by a method similar to the one used in [3] when R is non-degenerate (see also [2, Chapter IX]). This transformed equation can written in the form

$$g^\flat S \operatorname{Bian}(g, R) = 0, \tag{3}$$

where $g^\flat : T \to T^*$ is the isomorphism determined by the metric g and $S : T^* \to T$ is a morphism of vector bundles depending only on R and T' whose image is equal to T'; when R is non-degenerate, S is the isomorphism equal to inverse of $R^\flat$. The second equation obtained from (2) imposes a condition of order zero on the metric g and expresses the vanishing of the traces, with respect to g, of all the quadratic forms Q_ξ. To a metric g, we associate the section $\chi(g)$ of K^* determined by

$$\langle \xi, \chi(g)\rangle = \operatorname{Tr}_g Q_\xi,$$

for $\xi \in K$. We then attempt to apply the Cartan-Kähler theorem to the system

$$\operatorname{Ric}(g) = R, \qquad j_1(g^\flat S \operatorname{Bian}(g, R)) = 0, \qquad j_2(\chi(g)) = 0 \tag{4}$$

of partial differential equations of order 2 for the metric g. If the quadratic forms $\{Q_\xi\}$ do not all vanish at all points of X, this approach fails. Indeed, by prolonging these equations, we find that a set of m additional identities must be taken into consideration. These scalar-valued identities give totally new obstructions to the local solvability of (1). This is quite different from the case of a non-degenerate R, when the equation (2) is the only further equation which needs to be considered. We then study the new system consisting of (4) and these new identities using the theory of overdetermined partial differential equations of [7] or [2, Chapter IX]. We shall use the notation and terminology introduced there.

The strategy of our proof will be to construct a formal power series solution of this system, which has the special property that one of its finite jets is strongly prolongable (i.e. every prolongation of this jet is in turn prolongable). Then a result of Malgrange [8] asserts that these equations admit a convergent series solution, and thus yields a solution of our original equation (1). We do not apply the Cartan-Kähler theorem, but show directly the existence of formal solutions, when appropriate conditions are imposed on R.

If g is a Riemannian metric on X, we consider the sequence of vector bundles

$$(5)\quad S^2T^* \otimes S^2T^* \xrightarrow{\sigma_g} S^2T^* \oplus (T^* \otimes g^\flat(T')) \oplus (S^2T^* \otimes K^*) \xrightarrow{\nu_g} K^* \to 0,$$

where the morphism of vector bundles σ_g is the symbol of the system (4) associated to g, and where ν_g is the morphism of vector bundles determined by

$$\langle \xi, \nu_g(u \oplus v \oplus w)\rangle = \langle u + v, Q_\xi\rangle_g - \tfrac{1}{2}\mathrm{Tr}_g\langle \xi, w\rangle,$$

for $u \in S^2T^*$, $v \in T^* \otimes g^\flat(T')$, $w \in S^2T^* \otimes K^*$ and $\xi \in K$. We first observe that the sequence (5) is a complex, because K is integrable (Lemma 4.2). For simplicity, we now assume that the morphism $\iota : K \to S^2T^*$ sending ξ into Q_ξ is injective and also suppose that $m \geq 1$. If $\chi(g)$ vanishes at $x \in X$, then we prove that the sequence (5) is exact at x (Lemma 4.4). Given the complexity of the morphism σ_g, it is remarkable that the cokernel morphism ν_g can be expressed in such a simple and natural manner.

The non-surjectivity of the morphism σ_g leads us to our additional identities. Let x be a point of X and g_0 be a given Riemannian metric on X; since we are interested in the local solvability of the equation (1), we may suppose that $X = \mathbb{R}^n$ with a system of linear coordinates $(x^1, \ldots, x^n)$ such that $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$ is a frame for K over X. We set $\xi_j = \partial/\partial x^j$, for $1 \leq j \leq n$. Without loss of generality, we may also suppose that $g_{0,ij}(x) = \delta_{ij}$ and that $\{\xi_1(x), \ldots, \xi_q(x)\}$ is a basis for the null-space of the pencil $\mathcal{Q}_x$ of quadratic forms, where $q = \nu(x) \geq m$. We choose T'

to be the integrable complement of K in T generated by the vector fields $\{\xi_{m+1}, \ldots, \xi_n\}$ on X; thus T'_x is the orthogonal complement of K_x in T_x. We consider the flat connection ∇ in T for which all the vector fields ξ_j are horizontal. This connection preserves K and so, if g is a Riemannian metric on X, we may consider the section

$$P(g) = \nu_g\big((\mathrm{Ric}(g) - R) \oplus (\nabla g^\flat SB_g R) \oplus \nabla^2\chi(g)\big)$$

of K^*. A Riemannian metric g, which is a solution of (1), must also satisfy the equation $P(g) = 0$, which, in light of the complex (5), is in fact of first order. However, as P arises from the lower order terms of the quasi-linear Ricci operator, $P(g)$ is an expression which is quadratic in the first derivatives of g.

We now examine the new system N'_2 of order 2 consisting of the equations (4) together with the second-order equation $j_1(P(g)) = 0$ for a Riemannian metric g. The crucial step in proving the existence of formal solutions of this system at x consists in constructing a 1-jet $j_1(g)(x)$ of a Riemannian metric g at x, which is a solution of the first-order system

$$g^\flat S\,\mathrm{Bian}(g, R) = 0, \qquad j_1(\chi(g)) = 0, \qquad P(g) = 0 \tag{6}$$

determined by our new system, and such that the symbol of the system N'_2 is involutive at $j_1(g)(x)$.

In §6, we consider the symbol morphism $\sigma(P)_g : T^* \otimes S^2T^* \to K^*$ of the operator P and a section $\omega(g)$ of $K^* \otimes K^*$ associated to a Riemannian metric g on X. We denote by $\sigma_{g,l}$ and $\sigma_l(P)_g : S^{l+1}T^* \otimes S^2T^* \to S^lT^* \otimes K^*$ the l-th prolongations of the morphisms σ_g and $\sigma(P)_g$, respectively, and by g'_{l+2} the kernel of the morphism $\sigma_{g,l}$. For $0 \leq k \leq n$, we denote by T^*_k the subspace of T^*_x consisting of all $\alpha \in T^*_x$ satisfying $\langle \xi_j, \alpha\rangle = 0$, for $k+1 \leq j \leq n$, and we consider the l-th symmetric product $S^lT^*_k$ of T^*_k as a subspace of $S^lT^*_x$; for $l \geq 0$, we then obtain mappings

$$\sigma_l(P)_g : S^{l+1}T^*_k \otimes S^2T^*_x \to S^lT^*_k \otimes K^*_x.$$

If g is a Riemannian metric on X satisfying $g(x) = g_0(x)$, we give expressions in terms of the coordinate system $(x^1, \ldots, x^n)$ for $P(g)(x)$ and for the symbol morphism $\sigma(P)_g$ at x. For $1 \leq s \leq m$, if Q^s is the section Q_{ξ_s} of S^2T^* over X, we verify that

$$P(g)(\xi_s)(x) = \sum_{j=1}^{m} \omega^s_j \frac{\partial g_{jj}}{\partial x^j}(x) + \phi_s(g), \tag{7}$$

where

$$\omega(g)(\xi_j,\xi_s)(x) = \omega_j^s = \sum_{i,k=m+1}^{n} Q_{ik}^s(x)\Big(\frac{\partial g_{kj}}{\partial x^i} - \frac{1}{4}\frac{\partial g_{ik}}{\partial x^j}\Big)(x),$$

with $1 \leq j \leq m$, and where $\phi_s(g)$ is an expression which does not involve the derivatives

$$\frac{\partial g_{kk}}{\partial x^k}(x)$$

of g, with $1 \leq k \leq m$.

We now suppose that $\mathrm{Tr}_{g_0} Q_\xi = 0$, for all $\xi \in K_x$. We modify appropriately our coordinate system in such a way that the new system retains all the properties of the original one and that the bundle T' and the connection ∇ do not change. Then replacing the original system by this new one, we construct a Riemannian metric g on X satisfying the following properties:

(i) we have $g_{ij}(x) = \delta_{ij}$ and

$$\mathrm{Tr}_g Q_\xi = 0, \qquad g_{|T'_x} = g_{0|T'_x}, \qquad g(T'_x, K_x) = 0,$$

for all $\xi \in K_x$;

(ii) the 1-jet $j_1(g)(x)$ verifies the equations

$$g^\flat S\,\mathrm{Bian}(g,R)(x) = 0, \qquad j_1(\chi(g))(x) = 0;$$

(iii) $\omega(g)(x)$ is an element of $(K^* \otimes K^*)_x$ of maximal rank;

(iv) the mappings

$$\sigma_1(P)_g : g'_{2,x} \cap (S^2T_k^* \otimes S^2T_x^*) \to T_k^* \otimes K_x^*$$

are surjective for $1 \leq k \leq n$;

(v) the mapping

$$\sigma_2(P)_g : g'_{3,x} \cap (S^3T_q^* \otimes S^2T_x^*) \to S^2T_q^* \otimes K_x^*$$

is surjective.

The affine structure of the jet bundles allows us to parametrize the affine space $\mathcal{A}_x$ of 1-jets $j_1(\tilde{g})(x)$ of Riemannian metrics $\tilde{g}$ on X satisfying $\tilde{g}(x) = g(x)$ by the vector space $(T^* \otimes S^2T^*)_x$; in fact, if $\tilde{g}$ is such a metric on X, there is a unique element u of $(T^* \otimes S^2T^*)_x$ such that

$$j_1(\tilde{g})(x) = j_1(g)(x) + u. \tag{8}$$

We suppose the conditions (i)–(iii) of Theorem 1 do not hold. The main difficulty now resides in modifying the metric g in such a way that the 1-jet $j_1(g)(x)$ is a solution of (6) and that the kernel of the morphism $\sigma_g \oplus \sigma_1(P)_g$ is involutive at x. It suffices to find a Riemannian metric $\tilde{g}$ on X, with $\tilde{g}(x) = g(x)$, such that the properties (ii)–(v) still hold for $\tilde{g}$, that the mapping

$$\sigma_2(P)_{\tilde{g}} : g'_{3,x} \to (S^2T^* \otimes K^*)_x \tag{9}$$

is surjective, and that the 1-jet $j_1(\tilde{g})(x)$ is a solution of (6).

According to equation (7) and (iii), we see that there exists an element u of $(T^* \otimes S^2T^*)_x$ of the form

$$u = \sum_{j=1}^{m} u_j \, dx^j \otimes (dx^j)^2 \tag{10}$$

such that any Riemannian metric $\tilde{g}$ on X satisfying (8) is a solution of the equations (6) and possesses the properties (iv) and (v). However the mapping (9) may not be surjective.

In order to remedy this situation, we introduce a non-zero subspace V of $(T^* \otimes S^2T^*)_x$ and families of elements $\{w_\alpha\}_{\alpha \in A}$ of $g'_{3,x}$ and orthonormal bases $\{B_\alpha\}_{\alpha \in A}$ of T'_x (with respect to g) and matrix-valued functions $\{M_\gamma\}_{\gamma \in \Gamma}$ on the space $\mathcal{A}_x$ depending only on $\mathcal{Q}_x$ and $g(x)$ which have the following properties. For $\gamma \in \Gamma$, the function M_γ takes its values in the space of real $d_\gamma \times d_\gamma$ matrices, where the integer d_γ is equal either to m or $2m$. If $\tilde{g}$ is a Riemannian metric on X satisfying $\tilde{g}(x) = g(x)$, for $\gamma \in \Gamma$ each entry $M_\gamma(\tilde{g})_{ij}$, with $1 \leq i, j \leq d_\gamma$, of the matrix $M_\gamma(\tilde{g}) = M_\gamma(j_1(\tilde{g})(x))$ is equal to a fixed coefficient (independent of $\tilde{g}$) in the expansion of the element $\sigma_2(P)_{\tilde{g}} w_\alpha$ of $(S^2T^* \otimes K)_x$ in terms of the orthonormal basis $\{\xi_1(x), \ldots, \xi_m(x), B_\alpha\}$ of T_x, for an appropriately chosen element $\alpha \in A$ depending only on γ, i and j. If u is an element of $(T^* \otimes S^2T^*)_x$ and $\tilde{g}$ is a Riemannian metric on X satisfying (8), we set

$$M_\gamma(g, u) = M_\gamma(\tilde{g}) - M_\gamma(g).$$

These objects have the following properties: if u is an element of $(T^* \otimes S^2T^*)_x$ of the form (10), we have

$$M_\gamma(g, u) = 0, \tag{11}$$

for all $\gamma \in \Gamma$; on the other hand if u is an element of V, assertions (ii)–(v) still hold for a Riemannian metric $\tilde{g}$ on X satisfying (8) and we have

$$M_\gamma(g, tu) = tM_\gamma(g, u), \tag{12}$$

for all $t \in \mathbb{R}$ and $\gamma \in \Gamma$. If $m \geq 2$, the subspace V is in fact independent of $\mathcal{Q}_x$.

In §5, we prove a sequence of algebraic lemmas concerning pencils of quadratic forms; the results of this section can be read independently of the rest of the paper. In §7, we use these lemmas to deduce the existence of a finite subset Γ' of Γ and an element u_0 of V such that:

(a) if $\tilde{g}$ is a Riemannian metric on X satisfying $\tilde{g}(x) = g(x)$, and if all the matrices $M_\gamma(\tilde{g})$, with $\gamma \in \Gamma'$, are invertible, then the mapping

$$g'_{3,x} \to (S^2T^*_x/S^2T^*_q) \otimes K^*_x$$

induced by $\sigma_2(P)_{\tilde{g}}$ is surjective;

(b) $\det M_\gamma(g, u_0)$ is non-zero, for all $\gamma \in \Gamma'$.

The fact that some of these determinants $\det M_\gamma(g, u)$, with $d_\gamma = 2m$, are squares for certain elements $u \in V$ is a crucial point which arises in the verification of (b). Preliminary computations, which we carried out using *Maple*, lead us to these particular determinants and their special form.

If g_t are Riemannian metrics on X satisfying

$$j_1(g_t)(x) = j_1(g)(x) + tu_0$$

for $t \in \mathbb{R}$, then from the relations (12) we infer that

$$\det M_\gamma(g_t) = \det(M_\gamma(g) + tM_\gamma(g, u_0))$$

is a polynomial in t of degree d_γ whose leading coefficient is equal to $\det M_\gamma(g, u_0)$, for all $\gamma \in \Gamma$. Thus according to (b), there exists a real number t_0 such that $\det M_\gamma(g_{t_0})$ is non-zero, for all $\gamma \in \Gamma'$. Since g_{t_0} possesses properties (ii)–(v), we know that there exists an element v of $T^* \otimes S^2T^*$ of the form (10) such that the 1-jet $j_1(g_{t_0})(x) + v$ is a solution of the equations (6). A Riemannian metric $\tilde{g}$ on X, satisfying the relation

$$j_1(\tilde{g})(x) = j_1(g_{t_0})(x) + v,$$

also possesses properties (iv) and (v). According to (11), the matrix $M_\gamma(\tilde{g})$ is invertible, for all $\gamma \in \Gamma'$, and so, by (a) and property (v) for $\tilde{g}$, the mapping (9) is surjective. Then from property (iv), we infer that the kernel of the morphism $\sigma_g \oplus \sigma_1(P)_{\tilde{g}}$ is involutive at x. We deduce the existence of a metric g' whose 2-jet is a solution of the equation N_2' and which is strongly prolongable. This completes our outline of the proof of Theorem 1, when m is equal to the rank of K and is ≥ 1.

We wish to point out that, when $1 \leq m \leq 3$, special difficulties arise. The cases $m = 2$ or 3 require additional lemmas given in §5, which are then exploited in §7. The case $m = 1$ is presented separately in §7 and §8.

1. The Bianchi operator

Let X be a manifold of dimension $n \geq 3$. We denote by T and T^* its tangent and cotangent bundles. By $\bigotimes^m E$, $\bigwedge^k E$ and $S^l E$, we shall mean the m-th tensor power, k-th exterior product and the l-th symmetric product of a vector bundle E over X, respectively. We shall identify $S^m T^*$ with a sub-bundle of $\bigotimes^m T^*$ by means of the imbedding considered in Chapter IX, §2 of [2]. In particular, if $\alpha, \beta \in T^*$, the symmetric product $\alpha \cdot \beta$ is identified with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If E is a fibered manifold over X, we denote by $J_k(E)$ the fibered manifold of k-jets of sections of E, and by $\pi_k : J_{k+l}(E) \to J_k(E)$ the natural projection. If s is a section of E over a neighborhood of $x \in X$, then $j_k(s)(x)$ is the k-jet of s at x. We identify $J_0(E)$ with E. If E is a vector bundle over X, we denote by $\mathcal{E}$ the sheaf of sections of E over X and we recall that there is a monomorphism of vector bundles

$$\varepsilon : S^k T^* \otimes E \to J_k(E);$$

if s is a section of E over a neighborhood of $x \in X$ satisfying $j_{k-1}(s)(x) = 0$, then $j_k(s)(x)$ is equal to the image under ε of a unique element $\varepsilon^{-1} j_k(s)(x)$ of $S^k T^* \otimes E$.

If $x \in X$ and $\{t_1, \ldots, t_n\}$ is a basis of T_x, for $0 \leq k \leq n$ we denote by $T^*_{\{t_1,\ldots,t_k\}}$ the subspace of T^*_x consisting of all $\alpha \in T^*_x$ satisfying $\langle t_j, \alpha \rangle = 0$, for $1 \leq j \leq k$. If $k = 0$, then $T^*_{\{t_1,\ldots,t_k\}} = T^*_x$. We shall consider the m-th symmetric product $S^m T^*_{\{t_1,\ldots,t_k\}}$ of $T^*_{\{t_1,\ldots,t_k\}}$ as a subspace of $S^m T^*_x$. If E and F are vector bundles over X, we denote by

$$(\varphi)_{+l} : S^{m+l} T^* \otimes E \to S^l T^* \otimes F$$

the l-th prolongation of a morphism

$$\varphi : S^m T^* \otimes E \to F$$

of vector bundles. We then obtain by restriction a mapping

$$(\varphi)_{+l} : S^{m+l} T^*_{\{t_1,\ldots,t_k\}} \otimes E_x \to S^l T^*_{\{t_1,\ldots,t_k\}} \otimes F_x.$$

A linear differential operator $D : \mathcal{E} \to \mathcal{F}$ of order k determines a morphism of vector bundles $p(D) : J_k(E) \to F$ such that $p(D)j_k(s) = Ds$, for all $s \in \mathcal{E}$. The symbol

$$\sigma(D) : S^kT^* \otimes E \to F$$

of D is the unique morphism of vector bundles such that $\sigma(D) \circ \varepsilon = p(D)$; its l-th prolongation is denoted by $\sigma_l(D)$.

If h is a section of S^2T^* over X, let

$$h^\flat : T \to T^*$$

be the morphism of vector bundles determined by

$$h(\xi, \eta) = \langle \eta, h^\flat(\xi) \rangle,$$

for $\xi, \eta \in T$; if h is non-degenerate, that is, if $h^\flat$ is an isomorphism, we denote by

$$h^\sharp : T^* \to T$$

the inverse of $h^\flat$.

Let $S^2_+T^*$ be the open fibered submanifold of S^2T^* consisting of the positive-definite symmetric 2-forms on X. A section g of $S^2_+T^*$ is a Riemannian metric on X, whose Levi-Civita connection and Ricci curvature we denote by ∇^g and $\mathrm{Ric}(g)$, respectively.

Let g be a Riemannian metric on X. We consider the trace mappings

$$\mathrm{Tr}_g : S^kT^* \to S^{k-2}T^*, \qquad \mathrm{Tr}^1_g : T^* \otimes S^2T^* \to T^* \tag{1.1}$$

and the morphism of vector bundles

$$\tau_g : S^2T^* \otimes S^2T^* \to S^2T^*$$

defined by

$$\begin{aligned}
(\mathrm{Tr}_g\, u)(\xi_1, \ldots, \xi_{k-2}) &= \sum_{i=1}^n u(t_i, t_i, \xi_1, \ldots, \xi_{k-2}), \\
(\mathrm{Tr}^1_g\, v)(\xi) &= \sum_{i=1}^n v(t_i, \xi, t_i), \\
(\tau_g w)(\xi, \eta) &= \tfrac{1}{2} \sum_{i=1}^n \{w(t_i, \xi, t_i, \eta) + w(t_i, \eta, t_i, \xi) \\
&\qquad\qquad - w(t_i, t_i, \xi, \eta) - w(\xi, \eta, t_i, t_i)\},
\end{aligned}$$

for $u \in S^kT^*_x$, $v \in (T^* \otimes S^2T^*)_x$, $w \in (S^2T^* \otimes S^2T^*)_x$ and $\xi_1, \dots, \xi_{k-2}, \xi, \eta \in T_x$, with $x \in X$, where $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. We denote by $S^k_0T^*$ the kernel of the first mapping of (1.1); we have $S^1_0T^* = T^*$. If $x \in X$ and $\{t_1, \dots, t_n\}$ is a basis of T_x, we set

$$S^m_0 T^*_{\{t_1,\dots,t_k\}} = S^m_0 T^*_x \cap S^m T^*_{\{t_1,\dots,t_k\}},$$

for $0 \leq k \leq n$. We also consider the inner product $\langle\ ,\ \rangle_g$ on $\bigotimes^m T^*$ determined by the metric g.

Let

$$\lambda : T^* \otimes S^2T^* \to S^2T^* \otimes T^*$$

be the isomorphism of vector bundles defined by

$$\lambda(v)(\xi, \eta, \zeta) = \tfrac{1}{2}(v(\xi, \eta, \zeta) + v(\eta, \zeta, \xi) - v(\zeta, \xi, \eta)),$$

for $v \in T^* \otimes S^2T^*$, $\xi, \eta, \zeta \in T$. The Bianchi operator

$$B_g : S^2\mathcal{T}^* \to \mathcal{T}^*$$

is defined by

$$B_g h = (\mathrm{Tr}^1_g - \tfrac{1}{2}\mathrm{id} \otimes \mathrm{Tr}_g)\nabla^g h = (\mathrm{Tr}_g \otimes \mathrm{id})\lambda\nabla^g h, \tag{1.2}$$

for $h \in S^2\mathcal{T}^*$, and the symbol

$$\sigma(B_g) : T^* \otimes S^2T^* \to T^*$$

of this first-order differential operator is equal to $\mathrm{Tr}^1_g - \frac{1}{2}\mathrm{id} \otimes \mathrm{Tr}_g$ or to $(\mathrm{Tr}_g \otimes \mathrm{id})\lambda$. The Ricci curvature of g satisfies the Bianchi identity

$$B_g \mathrm{Ric}(g) = 0. \tag{1.3}$$

Let $x \in X$ and α be a non-zero element of T^*_x. Let V_1 be the subspace $\alpha \cdot T^*_x$ of $S^2T^*_x$ and let V be the subspace of $S^2T^*_x$ generated by g and V_1. Let $\{t_1, \dots, t_n\}$ be an orthonormal basis of T_x such that $\langle t_j, \alpha\rangle = 0$, for $1 \leq j \leq n-1$. We denote by $\{\alpha_1, \dots, \alpha_n\}$ the dual basis of T^*_x; then α is a multiple of α_n.

Lemma 1.1. *The mappings*

$$\sigma(B_g) : T_x^* \otimes V_1 \to T_x^*, \qquad \sigma_1(B_g) : S_0^2 T^*_{\{t_1,\dots,t_k\}} \otimes V \to T^*_{\{t_1,\dots,t_k\}} \otimes T_x^*$$

are surjective for $0 \leq k \leq n-2$.

Proof: For $\beta \in T_x^*$, we have

$$\sigma(B_g)(\alpha_n \otimes \alpha_n \cdot \beta) = \beta.$$

If $1 \leq j \leq n-1$ and $\beta \in T_x^*$ is orthogonal to α_j and α_n, then

$$\sigma_1(B_g)(\alpha_j \cdot \alpha_n \otimes \beta \cdot \alpha_n) = \alpha_j \otimes \beta.$$

If $\beta \in T_x^*$ is orthogonal to α_{n-1} and α_n, then

$$\frac{1}{2}\sigma_1(B_g)((\alpha_n^2 - \alpha_{n-1}^2) \otimes \beta \cdot \alpha_n) = \alpha_n \otimes \beta.$$

For $1 \leq j \leq n-2$, we have

$$\frac{1}{2}\sigma_1(B_g)((\alpha_j^2 - \alpha_{n-1}^2) \otimes \alpha_j \cdot \alpha_n) = \alpha_j \otimes \alpha_n.$$

If $1 \leq j \leq n-1$, we have

$$\begin{aligned}
\sigma_1(B_g)(\alpha_j \cdot \alpha_n \otimes \alpha_j \cdot \alpha_n) &= \alpha_j \otimes \alpha_j + \alpha_n \otimes \alpha_n, \\
\sigma_1(B_g)((\alpha_j^2 - \alpha_n^2) \otimes \alpha_j \cdot \alpha_n) &= 2\alpha_j \otimes \alpha_n - \alpha_n \otimes \alpha_j, \\
\sigma_1(B_g)(\alpha_{n-1} \cdot \alpha_n \otimes g_0) &= \Big(1 - \frac{n}{2}\Big)(\alpha_{n-1} \otimes \alpha_n + \alpha_n \otimes \alpha_{n-1}), \\
\sigma_1(B_g)((\alpha_{n-1}^2 - \alpha_n^2) \otimes g_0) &= (2-n)(\alpha_{n-1} \otimes \alpha_{n-1} - \alpha_n \otimes \alpha_n).
\end{aligned}$$

The lemma follows from all these formulas.

Lemma 1.2. *The mapping*

$$\sigma_2(B_g) : S_0^3 T_x^* \otimes V \to (S_0^2 T^* \otimes T^*)_x$$

is surjective.

Proof: We set $W = \sigma_2(B_g)(S_0^3 T_x^* \otimes V)$. If $1 \leq i, j \leq n$ are distinct integers, we consider the element

$$\theta_{ij} = \frac{1}{2}\alpha_i^2 \cdot \alpha_j - \frac{1}{6}\alpha_j^3$$

of $S^3_0 T^*_x$. If $1 \leq i, j, k \leq n$ are distinct integers, we have

$$\sigma_2(B_g)(\theta_{ij} \otimes \alpha_j \cdot \alpha_k) = \frac{1}{2}(\alpha_i^2 - \alpha_j^2) \otimes \alpha_k,$$
$$\sigma_2(B_g)(\theta_{ij} \otimes \alpha_i \cdot \alpha_k) = \alpha_i \cdot \alpha_j \otimes \alpha_k.$$

From these formulas with i, j or k equal to n, we deduce that

$$\alpha_j \cdot \alpha_n \otimes \alpha_i, \qquad \alpha_i \cdot \alpha_j \otimes \alpha_n, \qquad (\alpha_j^2 - \alpha_n^2) \otimes \alpha_i, \qquad (\alpha_i^2 - \alpha_j^2) \otimes \alpha_n$$

are elements of W, whenever $1 \leq i, j < n$ are distinct integers. If $1 \leq i, j, k < n$ are distinct integers, we see that

$$\sigma_2(B_g)(\alpha_j \cdot \alpha_k \cdot \alpha_n \otimes \alpha_i \cdot \alpha_n) = \alpha_j \cdot \alpha_k \otimes \alpha_i,$$
$$\sigma_2(B_g)((\alpha_j^2 - \alpha_k^2) \cdot \alpha_n \otimes \alpha_i \cdot \alpha_n) = (\alpha_j^2 - \alpha_k^2) \otimes \alpha_i.$$

For $1 \leq i < n$, we have

$$\sigma_2(B_g)(\theta_{ni} \otimes \alpha_i \cdot \alpha_n) = \alpha_i \cdot \alpha_n \otimes \alpha_i - \frac{1}{2}(\alpha_i^2 - \alpha_n^2) \otimes \alpha_n,$$
$$\sigma_2(B_g)(\theta_{in} \otimes \alpha_i \cdot \alpha_n) = \alpha_i \cdot \alpha_n \otimes \alpha_n + \frac{1}{2}(\alpha_i^2 - \alpha_n^2) \otimes \alpha_i,$$
$$\sigma_2(B_g)(\theta_{in} \otimes g) = \Big(1 - \frac{n}{2}\Big)\Big(\alpha_i \cdot \alpha_n \otimes \alpha_i + \frac{1}{2}(\alpha_i^2 - \alpha_n^2) \otimes \alpha_n\Big),$$
$$\sigma_2(B_g)(\theta_{ni} \otimes g) = \Big(1 - \frac{n}{2}\Big)\Big(\alpha_i \cdot \alpha_n \otimes \alpha_n - \frac{1}{2}(\alpha_i^2 - \alpha_n^2) \otimes \alpha_i\Big),$$

and we deduce that

$$\alpha_i \cdot \alpha_n \otimes \alpha_i, \qquad \alpha_i \cdot \alpha_n \otimes \alpha_n, \qquad (\alpha_i^2 - \alpha_n^2) \otimes \alpha_n, \qquad (\alpha_i^2 - \alpha_n^2) \otimes \alpha_i$$

belong to W. If $1 \leq i, j < n$ are distinct integers, we have

$$\sigma_2(B_g)(\alpha_i \cdot \alpha_j \cdot \alpha_n \otimes \alpha_i \cdot \alpha_n) = \alpha_i \cdot \alpha_j \otimes \alpha_i + \alpha_j \cdot \alpha_n \otimes \alpha_n,$$
$$\sigma_2(B_g)(\theta_{ij} \otimes \alpha_n^2) = -\frac{1}{2}(\alpha_i^2 - \alpha_j^2) \otimes \alpha_j - \alpha_i \cdot \alpha_j \otimes \alpha_i,$$

and we conclude that $W = (S^2_0 T^* \otimes T^*)_x$.

Let E' be a vector bundle over X and let

$$\Phi : J_k(S^2_+T^*) \to E'$$

be a morphism of fibered manifolds. The symbol

$$\sigma(\Phi) : S^kT^* \otimes S^2T^* \to E'$$

of Φ is the morphism of vector bundles over $J_k(S^2_+T^*)$ defined by

$$\sigma(\Phi)(j_k(g)(x), u) = \frac{d}{dt}\,\Phi(j_k(g + th)(x))_{|t=0},$$

where g is a Riemannian metric on a neighborhood of $x \in X$ and u is an element of $(S^kT^* \otimes S^2T^*)_x$, where h is a section of S^2T^* satisfying $j_{k-1}(h)(x) = 0$ and $\varepsilon^{-1}j_k(h)(x) = u$. For $l \geq 1$, the l-th prolongation

$$p_l(\Phi) : J_{k+l}(S^2_+T^*) \to J_l(E')$$

of Φ, sending $j_{k+l}(g)(x)$ into $j_l(\Phi(j_k(g)))(x)$, where g is a Riemannian metric on a neighborhood of $x \in X$, is quasi-linear in the sense of [2, Chapter IX] and its symbol

$$\sigma_l(\Phi) : S^{k+l}T^* \otimes S^2T^* \to S^lT^* \otimes E'$$

is the morphism of vector bundles over $J_k(S^2_+T^*)$ equal to the l-th prolongation of $\sigma(\Phi)$. In fact, if g is a Riemannian metric on a neighborhood of $x \in X$ and h is a section of S^2T^* satisfying $j_{k+l-1}(h)(x) = 0$, then

$$\begin{aligned} &p_l(\Phi)(j_{k+l}(g + h)(x)) \\ &\qquad = p_l(\Phi)(j_{k+l}(g)(x)) + \sigma_l(\Phi)(j_k(g)(x), \varepsilon^{-1}j_{k+l}(h)(x)). \end{aligned} \tag{1.4}$$

If g is a Riemannian metric on X, we shall denote by

$$\sigma(\Phi)_g : S^kT^* \otimes S^2T^* \to E'$$

the morphism of vector bundles over X sending $u \in (S^kT^* \otimes S^2T^*)_x$, with $x \in X$, into $\sigma(\Phi)(j_k(g)(x), u)$, and by $\sigma_l(\Phi)_g$ the l-th prolongation of $\sigma(\Phi)_g$.

If $k \geq 1$ and $\Phi = p_0(\Phi)$ is quasi-linear, then $\sigma(\Phi) = \sigma_0(\Phi)$ is a morphism over $J_{k-1}(S^2_+T^*)$ and the relation (1.4) also holds for $l = 0$; in this case, for $l \geq 1$ the l-th prolongation $\sigma_l(\Phi)$ of $\sigma(\Phi)$ is also a morphism over $J_{k-1}(S^2_+T^*)$.

Our strategy for solving analytic differential equations will be to seek a formal power series solution of finite order which is strongly prolongable. Let s' be a given section of E' and set

$$R_{k+l} = \{q \in J_{k+l}(S^2_+T^*)_x \mid p_l(\Phi)q = j_l(s')(x),\, x \in X\}.$$

We say that an element q of $R_{k+l,x}$, with $x \in X$, is strongly prolongable for Φ and s' if, for any element q' of $R_{k+l+m,x}$, with $m \geq 0$, satisfying $\pi_{k+l}q' = q$, there is a jet $q'' \in R_{k+l+m+1,x}$ such that $\pi_{k+l+m}q'' = q'$. If X, E', Φ, s' are real-analytic, a result of Malgrange [8] asserts that, if there is a strongly prolongable element q of $R_{k+l,x}$, with $x \in X$ and $l \geq 0$, the equation

$$\phi(j_k(g)) = s'$$

admits an analytic solution g on a neighborhood of x.

The morphism of fibered manifolds

$$\varphi : J_2(S^2_+T^*) \to S^2T^*,$$

sending the 2-jet $j_2(g)(x)$ at $x \in X$ of a Riemannian metric g on X into $\mathrm{Ric}(g)(x)$, is quasi-linear and its symbol

$$\sigma(\varphi) : S^2T^* \otimes S^2T^* \to S^2T^*$$

over $J_1(S^2_+T^*)$ is determined by the relation

$$\sigma(\varphi)_g = \tau_g.$$

2. Quadratic forms and obstructions

We denote by $\mathcal{L}_\xi u$ the Lie derivative of a section u of $\bigotimes^m T^*$ along a vector field ξ on X. For the remainder of this paper, we consider a fixed section R of S^2T^* over X, and we assume that the morphism of vector bundles

$$R^\flat : T \to T^*$$

has constant rank. We denote by K its kernel and we remark that its image is contained in the annihilator $K^\perp$ of K. It is easily seen that there is a unique morphism of vector bundles

$$\omega : \bigwedge^2 K \to T^*$$

such that

$$\omega(\xi_1, \xi_2) = R^\flat([\xi_1, \xi_2]),$$

for $\xi_1, \xi_2 \in \mathcal{K}$. Clearly the sub-bundle K is integrable if and only if $\omega = 0$. If $\xi \in T$, we consider the element ω_ξ of $\bigwedge^2 K^*$ defined by

$$\omega_\xi(\xi_1, \xi_2) = \langle \xi, \omega(\xi_1, \xi_2) \rangle,$$

for $\xi_1, \xi_2 \in K$. Since the image of ω is contained in $K^\perp$, we see that $\omega_\xi = 0$, for all $\xi \in K$. Let ∇ be an arbitrary torsionless connection in T; we consider the section ∇R of $T^* \otimes S^2 T^*$.

The following lemma associates a section Q of $S^2T^* \otimes K^*$ to the section R of S^2T^*.

Lemma 2.1. *The section*

$$Q = (\lambda(\nabla R))_{|T\otimes T\otimes K}$$

of $S^2T^* \otimes K^*$ is independent of the torsionless connection ∇ and has the following properties:

(i) *If ξ is a section of K over X, we have*

$$Q(\eta, \zeta, \xi) = -\tfrac{1}{2}(\mathcal{L}_\xi R)(\eta, \zeta), \tag{2.1}$$

for all $\eta, \zeta \in T$.

(ii) If $\eta \in T$, $\xi_1, \xi_2, \zeta \in K$, we have

$$Q(\eta, \xi_1, \xi_2) = -\tfrac{1}{2}\omega_\eta(\xi_1, \xi_2), \tag{2.2}$$

$$Q(\eta, \xi_1, \xi_2) + Q(\eta, \xi_2, \xi_1) = 0, \qquad Q(\zeta, \xi_1, \xi_2) = 0. \tag{2.3}$$

(iii) The sub-bundle K of T is integrable if and only if

$$Q(\eta, \xi_1, \xi_2) = 0, \tag{2.4}$$

for all $\xi_1, \xi_2 \in K$, $\eta \in T$.

Proof: If ∇, ∇' are torsionless connections in T, then there is a section L of $S^2T^* \otimes T$ such that

$$\nabla'_\xi \eta - \nabla_\xi \eta = L(\xi, \eta),$$

for $\xi, \eta \in \mathcal{T}$. It is easily verified that

$$(\lambda(\nabla' R - \nabla R))(\zeta_1, \zeta_2, \xi) = -R(L(\zeta_1, \zeta_2), \xi),$$

for $\xi, \zeta_1, \zeta_2 \in T$. If ξ belongs to K, the right-hand side of the above equation vanishes, and so we see that Q is independent of ∇. Since ∇ is torsionless, according to the definition of Q we have

$$Q(\eta, \zeta, \xi) = \tfrac{1}{2}(R([\xi, \eta], \zeta) + R(\eta, [\xi, \zeta]) - \xi \cdot R(\eta, \zeta)) = -\tfrac{1}{2}(\mathcal{L}_\xi R)(\eta, \zeta),$$

for all $\xi \in \mathcal{K}$, $\eta, \zeta \in \mathcal{T}$. We thus obtain (2.1) and see that

$$Q(\eta, \xi_1, \xi_2) = -\tfrac{1}{2} R(\eta, [\xi_1, \xi_2]),$$

for all $\xi_1, \xi_2 \in \mathcal{K}$, $\eta \in \mathcal{T}$. Assertions (ii) and (iii) follow directly from this equality.

If $\xi \in K$, we denote by Q_ξ the element of S^2T^* defined by

$$Q_\xi(\eta, \zeta) = Q(\eta, \zeta, \xi),$$

for $\eta, \zeta \in T$; from (2.3), we deduce that

$$Q_\xi(\xi, \eta) = 0, \tag{2.5}$$

for all $\eta \in T$. If ξ is a section of K over X, by (2.1) we have

$$Q_\xi = -\tfrac{1}{2}\mathcal{L}_\xi R. \tag{2.6}$$

If g is a Riemannian metric on X, using ∇^g to define Q and the relation (1.2), we see that

$$(B_g R)(\xi) = \operatorname{Tr}_g Q_\xi, \tag{2.7}$$

for all $\xi \in K$. Therefore, if g is a solution of the equation $\operatorname{Ric}(g) = R$, by (1.3) we see that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K$.

We denote by K' the kernel of the morphism of vector bundles

$$\iota : K \to S^2T^*,$$

which sends $\xi \in K$ into Q_ξ. If Q vanishes identically, that is, if $K' = K$, then by Lemma 2.1,(iii) we see that K is integrable.

LEMMA 2.2. *Suppose that the morphism ι has constant rank.*

(i) *A section ξ of K over X is a section of K' if and only if $\mathcal{L}_\xi R = 0$.*

(ii) *If ξ_1 is a section of K and ξ_2 is a section of K' over X, then $[\xi_1, \xi_2]$ is a section of K.*

(iii) *The sub-bundle K' of T is integrable.*

PROOF: Assertion (i) follows directly from Lemma 2.1,(i) or (2.6). If ξ_1 is a section of K and ξ_2 is a section of K' over X, then by (2.2) we have

$$0 = Q_{\xi_2}(\eta, \xi_1) = -\tfrac{1}{2} R(\eta, [\xi_1, \xi_2]),$$

for all $\eta \in T$, and so we see that $[\xi_1, \xi_2]$ is a section of K. If ξ_1 is also a section of K', then by (i) we have

$$\mathcal{L}_{[\xi_1,\xi_2]} R = (\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}) R = 0;$$

thus $[\xi_1, \xi_2]$ is a section of K', and so (iii) holds.

Let K'' be the sub-bundle of T, with possibly varying fiber, consisting of all elements $\xi \in T$ satisfying

$$Q(\eta, \xi, \zeta) = 0,$$

for all $\eta \in T$, $\zeta \in K$. If K is integrable, then according to Lemma 2.1,(iii) we see that $K \subset K''$.

LEMMA 2.3. *Let $x \in X$ and $\xi \in K_x$; assume that Q_ξ does not vanish. Then there exists a Riemannian metric g on X such that*

$$\operatorname{Tr}_g Q_\xi = 0$$

if and only if Q_ξ is not semi-definite.

PROOF: If g is a Riemannian metric on X satisfying the equation $\operatorname{Tr}_g Q_\xi = 0$, we may choose an orthonormal basis $\{t_1, \dots, t_n\}$ of T_x which diagonalizes Q_ξ. If $Q_\xi(t_i, t_j) = q_j \delta_{ij}$, then we have $\sum_{j=1}^n q_j = 0$. Since $Q_\xi \neq 0$, this equality implies that Q_ξ is not semi-definite. We now prove the converse and assume that Q_ξ is not semi-definite. According to (2.5), we may choose a basis $\{t'_1, \dots, t'_n\}$ of T'_x such that $t'_n = \xi(x)$ and $Q_\xi(t'_i, t'_j) = q'_j \delta_{ij}$, for $1 \leq i, j \leq n$, with $q'_n = 0$. Since Q_ξ is not semi-definite, we may suppose that $q'_1 > 0$ and $q'_2 < 0$. Choose $\lambda_2 > 0$ so that

$$\lambda_2 q'_2 + \sum_{j=3}^{n-1} q'_j < 0;$$

set

$$\lambda_1 = -\frac{1}{q'_1}\left(\lambda_2 q'_2 + \sum_{j=3}^{n-1} q'_j\right),$$

and $\lambda_j = 1$ for $3 \leq j \leq n$. Then we have $\lambda_j > 0$ for $1 \leq j \leq n$, and we consider

$$t_j = \sqrt{\lambda_j}\, t'_j,$$

for $1 \leq j \leq n$. Then $\{t_1, \dots, t_n\}$ is a basis of T_x; we see that

$$Q_\xi(t_i, t_j) = q_j \delta_{ij},$$

for $1 \leq i, j \leq n$, where $q_j = \lambda_j q'_j$ for $1 \leq j \leq n-1$, and $q_n = 0$. By the definition of the λ_j, we have $\sum_{j=1}^{n} q_j = 0$. A Riemannian metric g on X whose value at x is determined by

$$g(t_i, t_j) = \delta_{ij}$$

satisfies the required condition $\mathrm{Tr}_g\, Q_\xi = 0$.

THEOREM 2.1. *Let $x \in X$ and $\xi \in K_x$. If Q_ξ does not vanish and is semi-definite, then there does not exist a Riemannian metric g on any neighborhood of x such that*

$$\mathrm{Ric}(g) = R. \tag{2.8}$$

PROOF: A Riemannian metric g on X solution of the equation (2.8) on a neighborhood of x satisfies the equation $\mathrm{Tr}_g\, Q_\xi = 0$. Since $Q_\xi \neq 0$, by Lemma 2.3 this equality implies that Q_ξ is not semi-definite.

We choose a fixed complement T' of K in T. If $\rho : T^* \to T'^*$ is the natural restriction mapping, we denote by $R^\sharp$ the inverse of the isomorphism

$$\rho R^\flat : T' \to T'^*.$$

If R is non-degenerate, then $T' = T$ and $R^\sharp$ is indeed the inverse of $R^\flat$. If $i : T' \to T$ is the inclusion mapping, we set

$$S = i R^\sharp \rho : T^* \to T.$$

Then the endomorphism $SR^\flat$ of T is equal to the projection π' of T onto T' corresponding to the decomposition $T = T' \oplus K$. Therefore if $\eta \in T$ and $\alpha \in T^*$ are related by

$$\alpha = R^\flat(\eta),$$

then we have

$$S\alpha = \pi'\eta.$$

According to (1.3) and (2.7), a Riemannian metric g on X which is a solution of the equation $\mathrm{Ric}(g) = R$ also satisfies the equations

$$\mathrm{Tr}_g\, Q_\xi = 0, \qquad g^\flat S B_g R = 0, \tag{2.9}$$

for all sections ξ of K. It is easily seen that these two relations are in fact equivalent to the Bianchi identity (1.3).

Let g be a Riemannian metric on X and h be a section of S^2T^* over X. We consider the section

$$L^g(h) = (\mathrm{id} \otimes g^\sharp)\lambda\nabla^g h$$

of $S^2T^* \otimes T$. From the definitions of B_g and $L^g(h)$, it follows directly that

$$B_g h = g^\flat(\mathrm{Tr}_g \otimes \mathrm{id})L^g(h). \tag{2.10}$$

Let $L^g(h)R$ be the section of $T^* \otimes S^2T^*$ defined by

$$(L^g(h)R)(\xi, \xi_1, \xi_2) = -(R(L^g(h)(\xi, \xi_1), \xi_2) + R(\xi_1, L^g(h)(\xi, \xi_2)),$$

for $\xi, \xi_1, \xi_2 \in T$. It is easily verified that

$$\langle \xi, \sigma(B_g)(L^g(h)R)\rangle = -R((\mathrm{Tr}_g \otimes \mathrm{id})L^g(h), \xi),$$

for $\xi \in T$; from this equality, we infer that

$$S\sigma(B_g)(L^g(h)R) = -\pi'(\mathrm{Tr}_g \otimes \mathrm{id})L^g(h). \tag{2.11}$$

Let $x \in X$ and assume that $h(x) = 0$; then $g + h$ is a Riemannian metric on a neighborhood U of x and

$$\nabla^{g+h} - \nabla^g : \mathcal{T} \to \mathcal{T}^* \otimes \mathcal{T}$$

is a differential operator of order zero on U which arises from a section of $S^2T^* \otimes T$. In fact, we have

$$\nabla^{g+h} - \nabla^g = L^g(h)$$

at x. Thus we obtain

$$(\nabla^{g+h}R)(x) = (\nabla^g R)(x) + (L^g(h)R)(x),$$

and so

$$B_{g+h}(R) = B_g(R) + \sigma(B_g)(L^g(h)R)$$

at x. Therefore, by (2.10) and (2.11), we have

$$S(B_{g+h}(R) - B_g(R)) = -\pi'(\mathrm{Tr}_g \otimes \mathrm{id})L^g(h) = -\pi' g^\sharp B_g(h)$$

at x, and hence

$$g^\flat SB_{g+h}(R)(x) - g^\flat SB_g(R)(x) = -g^\flat \pi' g^\sharp \sigma(B_g)\varepsilon^{-1} j_1(h)(x). \tag{2.12}$$

(cf. [2, pp. 405–406]).

We consider the morphism of fibered manifolds

$$\psi : J_1(S^2_+T^*) \to T^*$$

defined by

$$\psi(j_1(g)(x)) = (g^\flat SB_g R)(x),$$

for $x \in X$. According to (2.12), ψ is quasi-linear and its symbol

$$\sigma(\psi) : T^* \otimes S^2T^* \to T^*$$

over $S^2_+T^*$ is determined by the relation

$$\sigma(\psi)_g = -g^\flat \pi' g^\sharp \sigma(B_g). \tag{2.13}$$

3. The symbols of the Ricci and Bianchi operators

We choose a fixed complement T' of K in T and consider the morphism ψ associated to T'. Let g be a Riemannian metric on X and let

$$\mu_g : (S^kT^* \otimes S^2T^*) \oplus (S^{k+1}T^* \otimes T^*) \to S^{k-1}T^* \otimes T^*$$

be the morphism of vector bundles sending $u \oplus v$, with $u \in S^kT^* \otimes S^2T^*$ and $v \in S^{k+1}T^* \otimes T^*$, into $\sigma_{k-1}(B_g)u$.

Lemma 3.1. *Let $x \in X$ and g be a Riemannian metric on X. Assume that T'_x is the orthogonal complement of K_x in T_x. Then the sequences*

$$
\begin{aligned}
(3.1)\qquad & S^{k+2}T^* \otimes S^2T^* \xrightarrow{\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g} \\
& (S^kT^* \otimes S^2T^*) \oplus (S^{k+1}T^* \otimes g^\flat(T')) \xrightarrow{\mu_g} S^{k-1}T^* \otimes T^* \to 0
\end{aligned}
$$

are exact at x for $k \geq 0$, and the kernel of $\sigma(\varphi)_g \oplus \sigma_1(\psi)_g$ is involutive at x. If $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x such that $\{t_1, \dots, t_r\}$ is a basis of T'_x and $\{t_{r+1}, \dots, t_n\}$ is a basis of K_x, then the mapping

$$
(3.2)\quad \sigma(\varphi)_g \oplus \sigma_1(\psi)_g : S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x \to S^2T^*_x \oplus (T^*_{\{t_1,\dots,t_k\}} \otimes g^\flat(T'_x))
$$

is surjective for $0 \leq k \leq n-2$, and the sequence

$$
\begin{aligned}
(3.3)\qquad & 0 \to \alpha_n^2 \otimes \alpha_n \cdot T^*_{\{t_1,\dots,t_r\}} \to S^2T^*_{\{t_1,\dots,t_{n-1}\}} \otimes S^2T^*_x \\
& \xrightarrow{\sigma(\varphi)_g \oplus \sigma_1(\psi)_g} S^2T^*_x \oplus (T^* \otimes g^\flat(T'))_x
\end{aligned}
$$

is exact, where α_n is a generator of $T^*_{\{t_1,\dots,t_{n-1}\}}$.

Proof: According to Lemma 6.8 of [3] (see also Lemma 2.11, Chapter IX of [2]), the morphism

$$
\sigma(\varphi)_g \oplus \sigma_1(B_g) : S^2T^* \otimes S^2T^* \to S^2T^* \oplus (T^* \otimes T^*)
$$

is surjective; therefore by (2.13), so is the morphism $\sigma(\varphi)_g \oplus \sigma_1(\psi)_g$. By Lemma 6.10 of [3] (see also Lemma 2.12, Chapter IX of [2]), the sequence

$$
S^3T^* \otimes S^2T^* \xrightarrow{\sigma_1(\varphi)_g \oplus \sigma_2(B_g)} (T^* \otimes S^2T^*) \oplus (S^2T^* \otimes T^*) \xrightarrow{\mu_g} T^* \to 0
$$

is exact. To prove the exactness of (3.1) when $k = 1$, let v be an element of $T^* \otimes S^2T^*$ satisfying $\sigma(B_g)v = 0$, and let w be an arbitrary element of $S^2T^* \otimes g^\flat(T')$. By the exactness of the above sequence, there is an element $u \in S^3T^* \otimes S^2T^*$ such that $\sigma_1(\varphi)_g u = v$ and $\sigma_2(B_g)u = -w$. Since $g^\flat \pi' g^\sharp \alpha = \alpha$, for $\alpha \in g^\flat(T')$, by (2.13) we see that

$$
\sigma_2(\psi)_g u = -(\mathrm{id} \otimes g^\flat \pi' g^\sharp)\sigma_2(B_g)u = (\mathrm{id} \otimes g^\flat \pi' g^\sharp)w = w,
$$

and so (3.1) is exact when $k = 0$. If

$$
\tau'_g : S^2T^* \otimes S^2T^* \to S^2T^*
$$

is the morphism $\mathrm{Tr}_g \otimes \mathrm{id}$, then it is easily verified that the kernel of

$$\tau_g' \oplus \sigma_1(B_g) : S^2T^* \otimes S^2T^* \to S^2T^* \oplus (T^* \otimes T^*)$$

is equal to the kernel of $\sigma(\varphi)_g \oplus \sigma_1(B_g)$. By Lemma 6.9,(a) of [3], the mappings

$$\tau_g' \oplus \sigma_1(B_g) : S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x \to S^2T^*_x \oplus (T^*_{\{t_1,\dots,t_k\}} \otimes T^*_x)$$

are surjective for $0 \leq k \leq n-2$. Therefore by the preceding remarks, we deduce that the mappings

$$\sigma(\varphi)_g \oplus \sigma_1(B_g) : S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x \to S^2T^*_x \oplus (T^*_{\{t_1,\dots,t_k\}} \otimes T^*_x)$$

are also surjective for $0 \leq k \leq n-2$. By (2.13), this implies the surjectivity of the mappings (3.2) for $0 \leq k \leq n-2$. Let $\{\alpha_1, \dots, \alpha_n\}$ be the basis of T^*_x dual to $\{t_1, \dots, t_n\}$ and let u be an element of $S^2T^*_x$. We have

$$\sigma(\varphi)_g(\alpha_n^2 \otimes u)(t_i, t_j) = -u(t_i, t_j), \tag{3.4}$$

for $1 \leq i, j < n$, and

$$\sigma(\varphi)_g(\alpha_n^2 \otimes \alpha_n \cdot \beta) = 0, \qquad \sigma_1(B_g)(\alpha_n^2 \otimes \alpha_n \cdot \beta) = 2\alpha_n \otimes \beta,$$

for $\beta \in T^*_x$. Thus by (2.13), we obtain

$$\sigma_1(\psi)_g(\alpha_n^2 \otimes \alpha_n \cdot \alpha_i) = -2\alpha_n \otimes \alpha_i, \qquad \sigma_1(\psi)_g(\alpha_n^2 \otimes \alpha_n \cdot \alpha_j) = 0,$$

for $1 \leq i \leq r$ and $r+1 \leq j \leq n$. We denote by g_{k+2} the kernel of $\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g$. We now suppose that $\alpha_n^2 \otimes u$ belongs to $g_{2,x}$. From (3.4), we infer that u is an element of $\alpha_n \cdot T^*_x$. The other equalities show that u belongs to $\alpha_n \cdot T^*_{\{t_1,\dots,t_r\}}$ and give us the exactness of the sequence (3.3). To prove that $g_{2,x}$ is involutive, we proceed to show that $\{t_1, \dots, t_n\}$ is a quasi-regular basis for $g_{2,x}$. For $0 \leq k \leq n$, we set

$$g_{2,\{t_1,\dots,t_k\}} = g_{2,x} \cap (S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x).$$

According to the preceding results, by dimension-counting we obtain

$$\begin{aligned}\dim g_{2,\{t_1,\dots,t_k\}} = {}& \frac{(n-k)(n-k+1)}{2} \cdot \frac{n(n+1)}{2} \\ & - \frac{n(n+1)}{2} - r(n-k)\end{aligned}$$

for $0 \leq k \leq n-2$, and

$$\dim g_{2,\{t_1,\dots,t_{n-1}\}} = n - r.$$

Thus, we have

$$\sum_{k=0}^{n-1} \dim g_{2,\{t_1,\dots,t_k\}} = \frac{n^2(n+1)(n+5)(n-2)}{12} + \frac{n(n+1)(n-r)}{2} + n.$$

By the exactness of the sequence (3.1) at x with $k = 1$, we see that the right-hand side of the above equality is precisely equal to $\dim g_{3,x}$. Therefore $\{t_1, \dots, t_n\}$ is a quasi-regular basis for $g_{2,x}$. By a standard argument (see [6, §4] or Chapter X, §1 of [2]), from the exactness of the sequences (3.1) at x for $k = 0, 1$ and the involutivity of $g_{2,x}$, we infer the exactness of the sequences (3.1) at x for all $k \geq 0$.

4. The exact symbol sequences

Throughout the remainder of this paper, we assume that ι has constant rank. We consider the vector bundle $L = K/K'$ and we denote by $\pi : K \to L$ the natural projection. Let

$$\chi : S^2_+T^* \to L^*$$

be the morphism of fibered manifolds determined by

$$\langle \pi(\xi), \chi(g) \rangle = \mathrm{Tr}_g\, Q_\xi,$$

where $\xi \in K$ and g is a Riemannian metric on X. The symbol

$$\sigma(\chi) : S^2T^* \to L^*$$

of χ is the morphism of vector bundles over $S^2_+T^*$ determined by

$$\langle \pi(\xi), \sigma(\chi)_g h \rangle = -\langle h, Q_\xi \rangle_g,$$

for $h \in S^2T^*$ and $\xi \in K$, where g is a Riemannian metric on X. If $u \in S^kT^* \otimes L^*$ and $\eta \in L$, let $\langle \eta, u \rangle$ be the element of S^kT^* defined by

$$\langle \eta, u \rangle(\xi_1, \dots, \xi_k) = u(\xi_1, \dots, \xi_k, \eta),$$

for $\xi_1, \dots, \xi_k \in T$. For $k \geq 1$, the morphism

$$p_k(\chi) : J_k(S^2_+T^*) \to J_k(L^*)$$

is quasi-linear and so, if g is a Riemannian metric on a neighborhood of $x \in X$ and h is a section of S^2T^* satisfying $j_{k-1}(h)(x) = 0$, we have

$$j_k(\langle \eta, \chi(g+h) \rangle)(x) = j_k(\langle \eta, \chi(g) \rangle)(x) + \langle \eta(x), \sigma_k(\chi)\varepsilon^{-1} j_k(h)(x) \rangle,$$

for all sections η of K.

From Lemma 2.3, we easily deduce the following result:

LEMMA 4.1. *Assume that* $\operatorname{rank} L = 1$. *Let* $x \in X$ *and* $\xi \in K_x$. *Suppose that* Q_ξ *does not vanish. Then the following conditions are equivalent:*

(i) Q_ξ *is not semi-definite;*
(ii) *there exists a Riemannian metric* g *on* X *such that* $\operatorname{Tr}_g Q_\xi = 0$;
(iii) *there exists a Riemannian metric* g *on* X *such that* $\chi(g)(x) = 0$.

Throughout the remainder of this paper, we assume that K is integrable. If g is a Riemannian metric on X and α is an element of T^*_x orthogonal to $K_x^\perp$, with $x \in X$, then by (2.4) we have

$$\sigma(\chi)_g(\alpha \cdot T^*_x) = 0, \tag{4.1}$$

and we see that the equality

$$\sigma(\chi)_g g = 0$$

at x is equivalent to the condition that $\operatorname{Tr}_g Q_\xi = 0$ for all $\xi \in K_x$.

We choose a fixed complement T' of K in T and consider the morphisms S and ψ corresponding to T'. If g is a Riemannian metric on X, let

$$\beta_g : S^2T^* \oplus \bigotimes^2 T^* \to L^*$$

be the morphism of vector bundles defined by

$$\langle \pi(\xi), \beta_g(u \oplus v)\rangle = \langle u + v, Q_\xi\rangle_g,$$

for $\xi \in K$, $u \in S^2T^*$, $v \in \bigotimes^2 T^*$. We consider the restriction

$$\beta_{g,k} : (S^kT^* \otimes S^2T^*) \oplus (S^{k+1}T^* \otimes T^*) \to S^kT^* \otimes L^*$$

of $\mathrm{id} \otimes \beta_g$, where id is the identity mapping of S^kT^*. For $k \geq 0$, we consider the morphism $\nu_{g,k}$ of vector bundles

$$\begin{aligned}(S^kT^* \otimes S^2T^*) \oplus (S^{k+1}T^* \otimes T^*) \oplus (S^{k+2}T^* \otimes L^*) \oplus (S^{k+1}T^* \otimes L^*) \\ \to (S^{k-1}T^* \otimes T^*) \oplus (S^kT^* \otimes L^*)\end{aligned}$$

defined by

$$\nu_{g,k}(u \oplus v \oplus w \oplus z) = \sigma_{k-1}(B_g)u \oplus \big(\beta_{g,k}(u \oplus v) - \tfrac{1}{2}(\operatorname{Tr}_g \otimes \mathrm{id})w\big),$$

for all elements $u \in S^kT^* \otimes S^2T^*$, $v \in S^{k+1}T^* \otimes T^*$, $w \in S^{k+2}T^* \otimes L^*$ and $z \in S^{k+1}T^* \otimes L^*$. Then $\nu_{g,k+1}$ is the k-th prolongation of $\nu_{g,1}$. We also denote by $\nu_{g,k}$ the restriction of $\nu_{g,k}$ to the sum of the first three factors. From the exactness of the sequences (3.1), it follows that $\sigma_{k-1}(B_g)$ is a surjective morphism; since $\operatorname{Tr}_g : S^{k+2}T^* \to S^kT^*$ is surjective, we see that the morphisms $\nu_{g,k}$ are also surjective.

LEMMA 4.2. *If K is integrable and g is a Riemannian metric on X, the sequences*

$$
\begin{aligned}
S^{k+2}T^* \otimes S^2T^* &\xrightarrow{\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g \oplus \sigma_{k+2}(\chi)_g} \\
(S^kT^* \otimes S^2T^*) &\oplus (S^{k+1}T^* \otimes g^\flat(T')) \oplus (S^{k+2}T^* \otimes L^*) \\
&\xrightarrow{\nu_{g,k}} (S^{k-1}T^* \otimes T^*) \oplus (S^kT^* \otimes L^*) \to 0
\end{aligned}
\tag{4.2}
$$

of vector bundles are complexes for $k \geq 0$.

PROOF: For $u \in S^2T^* \otimes S^2T^*$ and $h \in S^2T^*$, it is easily verified that

$$\langle \sigma(\varphi)_g u - \sigma_1(B_g)u + \tfrac{1}{2}(\mathrm{Tr}_g \otimes \mathrm{id})u, h\rangle_g = 0 \tag{4.3}$$

and that

$$\langle g^\flat(\eta) \otimes g^\flat(\zeta), h\rangle_g = h(\eta,\zeta), \tag{4.4}$$

where $\eta, \zeta \in T$. For $\xi \in K$, from (4.3) we infer that

$$\langle \sigma(\varphi)_g u - \sigma_1(B_g)u, Q_\xi\rangle_g - \tfrac{1}{2}\mathrm{Tr}_g\langle \pi(\xi), \sigma_2(\chi)_g u\rangle = 0. \tag{4.5}$$

Since K is integrable, by Lemma 2.1,(iii) and (4.4) we see that

$$\langle g^\flat(\eta) \otimes g^\flat(\zeta), Q_\xi\rangle_g = Q(\eta,\zeta,\xi) = 0,$$

for $\eta \in T$ and $\zeta \in K$. Therefore if π'' is the projection of T onto K corresponding to the decomposition $T = T' \oplus K$, for $v \in T^* \otimes T^*$ we have

$$\langle (\mathrm{id} \otimes g^\flat \pi'' g^\sharp)v, Q_\xi\rangle_g = 0. \tag{4.6}$$

By (2.13), we know that

$$\sigma_1(\psi)_g = -\sigma_1(B_g) + (\mathrm{id} \otimes g^\flat \pi'' g^\sharp)\sigma_1(B_g),$$

and so by (4.6) we obtain

$$\langle \sigma_1(\psi)_g u, Q_\xi\rangle_g = -\langle \sigma_1(B_g)u, Q_\xi\rangle_g.$$

From this equality and (4.5), we deduce the fact that (4.2) is a complex when $k = 0$. By Lemma 3.1, we now see that (4.2) is a complex when $k = 1$, and hence for all $k \geq 1$.

We define a morphism

$$\gamma_k : S^kT^* \otimes L^* \to S^kT^* \otimes S^2T^*$$

satisfying

$$\sigma_k(\chi)\gamma_k = -\mathrm{id} \tag{4.7}$$

as follows. Let $x \in X$ and let $\{\xi_1, \ldots, \xi_m\}$ be a set of elements of K_x for which $\{Q_{\xi_1}, \ldots, Q_{\xi_m}\}$ is an orthonormal basis of $\iota(K_x)$. If $u \in (S^kT^* \otimes L^*)_x$, we set

$$\gamma_k u = \sum_{j=1}^{m} \langle \pi(\xi_j), u \rangle \otimes Q_{\xi_j}.$$

Notice that $\gamma_k u$ does not depend on the choice of the set $\{\xi_1, \ldots, \xi_m\}$ of elements of K_x as long as it satisfies the above condition. The relation (4.7) follows directly from the definitions of the mappings $\sigma_k(\chi)$ and γ_k.

Let

$$\rho_g : T^* \otimes S^2T^* \to K^* \otimes L^*$$

be the morphism of vector bundles defined by

$$(\rho_g u)(\xi, \pi(\eta)) = \langle u_\xi, Q_\eta \rangle_g,$$

for $u \in T^* \otimes S^2T^*$, $\xi, \eta \in K$, where u_ξ is the element of $\bigotimes^2 T^*$ determined by

$$u_\xi(\zeta_1, \zeta_2) = u(\zeta_1, \zeta_2, \xi),$$

for $\zeta_1, \zeta_2 \in T$.

Lemma 4.3. *Let g be a Riemannian metric on X. Then the morphism*

$$\sigma(\psi)_g \oplus \sigma_1(\chi)_g : T^* \otimes S^2T^* \to g^\flat(T') \oplus (T^* \otimes L^*)$$

is surjective. If $x \in X$ and T'_x is the orthogonal complement of K_x in T_x, then the morphism

$$\sigma(\psi)_g \oplus \sigma_1(\chi)_g \oplus \rho_g : T^* \otimes S^2T^* \to g^\flat(T') \oplus (T^* \otimes L^*) \oplus (K^* \otimes L^*)$$

is surjective at x.

Proof: By (2.13), since $\sigma(B_g)$ is surjective, so is the morphism $\sigma(\psi)_g$. We may now assume that K is non-zero. We first show that

$$\sigma_1(\chi)_g : \mathrm{Ker}\, \sigma(\psi)_g \to T^* \otimes L^*$$

is surjective at $x \in X$. Let α be a non-zero element of T^*_x orthogonal to $K_x^\perp$ and consider the subspace $V_1 = \alpha \cdot T^*_x$ of $S^2T^*_x$. Let $v \in (T^* \otimes L^*)_x$; according to Lemma 1.1, there is an element $u' \in T^*_x \otimes V_1$ satisfying

$$\sigma(B_g)u' = \sigma(B_g)\gamma_1 v.$$

Then $u = u' - \gamma_1 v$ belongs to $\operatorname{Ker}\sigma(B_g)$ and hence to $\operatorname{Ker}\sigma(\psi)_g$. By (4.1) and (4.7), we also have $\sigma_1(\chi)_g u = v$, and so the morphism

$$\sigma(\psi)_g \oplus \sigma_1(\chi)_g : T^* \otimes S^2T^* \to g^\flat(T') \oplus (T^* \otimes L^*)$$

is surjective at x. To complete the proof of the lemma, under the assumption that T'_x is the orthogonal complement of K_x in T_x, we now verify that the morphism

$$\rho_g : \operatorname{Ker}\sigma(\psi)_g \cap \operatorname{Ker}\sigma_1(\chi)_g \to K^* \otimes L^*$$

is surjective at x. In fact, let $\{\xi_1, \dots, \xi_m\}$ be a set of elements of K_x for which $\{Q_{\xi_1}, \dots, Q_{\xi_m}\}$ is an orthonormal basis of $\iota(K_x)$, and let $\{t_1, \dots, t_n\}$ be an orthonormal basis of T_x such that $\{t_1, \dots, t_r\}$ is a basis of T'_x and $\{t_{r+1}, \dots, t_n\}$ is a basis of K_x. We denote by $\{\alpha_1, \dots, \alpha_n\}$ the basis of T^*_x dual to $\{t_1, \dots, t_n\}$. If $w \in (K^* \otimes L^*)_x$, we consider the element

$$u = \sum_{l=1}^{m} \sum_{j=r+1}^{n} \sum_{i,k=1}^{r} w(t_j, \pi(\xi_l)) Q_{\xi_l}(t_i, t_k) \alpha_i \otimes \alpha_k \cdot \alpha_j$$

of $(T^* \otimes S^2T^*)_x$. According to (4.1), we have $\sigma_1(\chi)_g u = 0$. Moreover for $r+1 \le j \le n$, by (2.4) we obtain the equality

$$u_{t_j} = \sum_{l=1}^{m} w(t_j, \pi(\xi_l)) Q_{\xi_l}$$

and therefore

$$(\rho_g u)(t_j, \pi(\eta)) = w(t_i, \pi(\eta)),$$

for $\eta \in K_x$, or $\rho_g u = w$. We easily see that $\sigma(B_g)u$ is a linear combination of $\alpha_{r+1}, \dots, \alpha_n$, and so by (2.13) we have $\sigma(\psi)_g u = 0$.

Proposition 4.1. *Let $x \in X$ and g_0 be a Riemannian metric on X. Suppose that T'_x is the orthogonal complement of K_x in T_x and that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. Let ∇ be a connection in T and v be an element of $(K^* \otimes L^*)_x$. Then there exists a Riemannian metric g on X such that $g(x) = g_0(x)$ and*

$$\psi(j_1(g)(x)) = 0, \qquad p_1(\chi)(j_1(g)(x)) = 0, \qquad (\rho_g \nabla g)(x) = v. \tag{4.8}$$

Proof: According to Lemma 4.3, we can find an element u of $(T^* \otimes S^2T^*)_x$ which is a solution of the system

$$\sigma(\psi)_{g_0} u = -\psi(j_1(g_0)(x)), \qquad \sigma_1(\chi)_{g_0} u = -\varepsilon^{-1} p_1(\chi)(j_1(g_0)(x)),$$
$$\rho_{g_0} u = v - (\rho_{g_0} \nabla g_0)(x).$$

Since ψ and $p_1(\chi)$ are quasi-linear morphisms, if g is a Riemannian metric on X satisfying $g(x) = g_0(x)$ and

$$\varepsilon^{-1} j_1(g - g_0)(x) = u,$$

we easily see that g satisfies (4.8).

Lemma 4.4. *Let $x \in X$ and let g be a Riemannian metric on X. Assume that K is integrable, that T'_x is the orthogonal complement of K_x in T_x and that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. Then the sequences (4.2) are exact at x for all $k \geq 0$, and the kernel of $\sigma(\varphi)_g \oplus \sigma_1(\psi)_g \oplus \sigma_2(\chi)_g$ is involutive at x. Moreover, if $\{t_1, \ldots, t_n\}$ is an orthonormal basis of T_x such that $\{t_1, \ldots, t_r\}$ is a basis of T'_x and $\{t_{r+1}, \ldots, t_n\}$ is a basis of K_x, then the sequences*

$$\begin{aligned} &S^2T^*_{\{t_1,\ldots,t_k\}} \otimes S^2T^*_x \xrightarrow{\sigma(\varphi)_g \oplus \sigma_1(\psi)_g \oplus \sigma_2(\chi)_g} \\ &\qquad S^2T^*_x \oplus (T^*_{\{t_1,\ldots,t_k\}} \otimes g^\flat(T'_x)) \oplus (S^2T^*_{\{t_1,\ldots,t_k\}} \otimes L^*_x) \xrightarrow{\nu_{g,0}} L^*_x \to 0 \end{aligned} \tag{4.9}$$

are exact for $0 \leq k \leq n-2$, and the sequence

$$\begin{aligned} 0 \to \alpha_n^2 \otimes \alpha_n \cdot T^*_{\{t_1,\ldots,t_r\}} &\to S^2T^*_{\{t_1,\ldots,t_{n-1}\}} \otimes S^2T^*_x \\ &\xrightarrow{\sigma(\varphi)_g \oplus \sigma_1(\psi)_g \oplus \sigma_2(\chi)_g} S^2T^*_x \oplus (T^* \otimes g^\flat(T'))_x \oplus (S^2T^*_{\{t_1,\ldots,t_{n-1}\}} \otimes L^*_x) \end{aligned}$$

is exact, where α_n is a generator of $T^*_{\{t_1,\dots,t_{n-1}\}}$.

Proof: Let l be an integer equal to 2 or to 3 and W be a subspace of $S^lT^*_x$; consider the mapping

$$\begin{aligned}\sigma_l(\chi)_g : (S^lT^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x) \cap \operatorname{Ker}\sigma_{l-2}(\varphi)_g \cap \operatorname{Ker}\sigma_{l-1}(\psi)_g \\ \to S^l_0T^*_{\{t_1,\dots,t_k\}} \otimes L^*_x.\end{aligned} \tag{4.10}$$

We suppose that either $l = 3$ and $W = S^3T^*_x$, or that $l = 2$ and $W = S^2T^*_{\{t_1,\dots,t_k\}}$, with $0 \le k \le n-2$. By Lemma 3.1, we see that, in the first case, the surjectivity of (4.10) implies that (4.2) is exact when $k = 1$; in the second case, the surjectivity of (4.10) implies that the sequences (4.9) are exact. Now assume that L is non-zero and consider the subspace V of $S^2T^*_x$ generated by $\alpha_n \cdot T^*_x$ and g. Let v be an element of $(S^l_0T^*_x \cap W) \otimes L^*_x$. Then $\gamma_l v$ belongs to $(S^l_0T^*_x \cap W) \otimes S^2T^*_x$. According to Lemma 1.1 or Lemma 1.2, there exists $u' \in (S^l_0T^*_x \cap W) \otimes V$ such that

$$\sigma_{l-1}(B_g)u' = \sigma_{l-1}(B_g)\gamma_l v.$$

We set $u = u' - \gamma_l v$. By (4.1), (4.7) and our hypothesis on g, we have $\sigma_l(\chi)_g u = v$. Since $\sigma_{l-1}(B_g)u = 0$ and $(\operatorname{Tr}_g \otimes \operatorname{id})u = 0$, we know that $\sigma(\varphi)_g u = 0$, and hence the mapping (4.10) is surjective. The exactness of the last sequence of the lemma is a direct consequence of the exactness of (3.3) and (4.1). We denote by g'_{k+2} the kernel of $\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g \oplus \sigma_{k+2}(\chi)_g$. To prove that $g'_{2,x}$ is involutive, we proceed to show that $\{t_1,\dots,t_n\}$ is a quasi-regular basis for $g'_{2,x}$. We consider the kernel g_{k+2} of $\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g$. For $0 \le k \le n$, we set

$$\begin{aligned}g_{2,\{t_1,\dots,t_k\}} &= g_{2,x} \cap (S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x),\\ g'_{2,\{t_1,\dots,t_k\}} &= g'_{2,x} \cap (S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x).\end{aligned}$$

According to the preceding results and Lemma 3.1, by dimension-counting we obtain

$$\dim g'_{2,\{t_1,\dots,t_k\}} = \dim g_{2,\{t_1,\dots,t_k\}} - (\dim S^2T^*_{\{t_1,\dots,t_k\}} - 1)\dim L_x$$

for $0 \le k \le n-2$, and

$$\begin{aligned}\dim g'_{2,\{t_1,\dots,t_{n-1}\}} &= \dim g_{2,\{t_1,\dots,t_{n-1}\}},\\ \dim g'_{3,x} &= \dim g_{3,x} - \dim S^3_0T^*_x \cdot \dim L_x.\end{aligned}$$

It is easily verified that

$$\dim S_0^3 T_x^* = \sum_{k=0}^{n-2} (\dim S^2 T^*_{\{t_1,\ldots,t_k\}} - 1) = \frac{n(n-1)(n+4)}{6}.$$

By Lemma 3.1, $\{t_1,\ldots,t_n\}$ is a quasi-regular basis for $g_{2,x}$; from the above equalities, it follows that it is also quasi-regular for $g'_{2,x}$. By a standard argument (see [6, §4] or Chapter X, §1 of [2]), from the exactness of the sequences (4.2) at x for $k = 0, 1$ and the involutivity of $g'_{2,x}$, we infer the exactness of the sequences (4.2) at x for all $k \geq 0$.

5. Algebraic results on quadratic forms

If E is a real vector space, we denote by $\bigwedge^k E$ and $S^l E$ the k-th exterior product and the l-th symmetric product of E. Let V and W be real vector spaces of dimension p and m, respectively. Assume that $m \geq 2$ and that V is endowed with a positive-definite inner product $\langle\ ,\ \rangle$.

We consider the trace mappings

$$\mathrm{Tr} : S^2V^* \to \mathbb{R}, \qquad \mathrm{Tr}_l : \bigwedge^l V^* \otimes S^2V^* \to \bigwedge^{l-1} V^* \otimes V^*$$

defined by

$$\mathrm{Tr}\, h = \sum_{i=1}^p h(v_i, v_i), \quad (\mathrm{Tr}_l\, u)(\eta_1, \ldots, \eta_{l-1}, \zeta) = \sum_{i=1}^p u(\eta_1, \ldots, \eta_{l-1}, v_i, v_i, \zeta),$$

for $h \in S^2V^*$, $u \in \bigwedge^l V^* \otimes S^2V^*$, $\eta_1, \ldots, \eta_{l-1}, \zeta \in V$, where $\{v_1, \ldots, v_p\}$ is an orthonormal basis of V.

Let A be a section of $S^2V^* \otimes W$; we consider the element

$$\mathrm{Tr}\, A = (\mathrm{Tr} \otimes \mathrm{id})A$$

of W. If η is a vector of V, we set

$$V_\eta = \{\, \zeta \in V \mid \langle \zeta, \eta \rangle = 0,\ \ A(\zeta, \eta) = 0 \,\};$$

clearly the dimension of V_η is $\geq p - m - 1$. We denote by V'_η the subspace of V generated by V_η and η.

We consider the two following properties that A can possess:

(i) the equality

$$\{\, \zeta \in V \mid A(\zeta, \eta) = 0, \text{ for all } \eta \in V \,\} = 0$$

holds;

(ii) under the natural identification of $S^2V^* \otimes W$ with $\mathrm{Hom}\,(S^2V, W)$, the element A corresponds to a surjective mapping from S^2V to W.

Let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V. For $1 \leq i, j \leq p$, we consider the element $A_{ij} = A(v_i, v_j)$ of W. If $\mathrm{Tr}\, A = 0$, then we have

$$\sum_{j=1}^{p} A_{jj} = 0.$$

Conditions (ii) can be restated as follows: the vectors $\{A_{ij}\}$ span all of W. On the other hand, condition (i) implies that, for all $1 \leq i \leq p$, there exists an integer $1 \leq j \leq p$ such that $A_{ij} \neq 0$. If $\{v_2, \ldots, v_q\}$ is a basis for V_η, where $\eta = v_1$, clearly we have $A_{1j} = 0$ for $2 \leq j \leq q$, and we see that

$$A_{1q+1} \wedge \ldots \wedge A_{1p} \neq 0.$$

Indeed, if the vectors $A_{1q+1}, \ldots, A_{1p}$ of W were linearly independent, we could find a non-trivial linear combination of the vectors $v_{q+1}, \ldots, v_p$ belonging to V_η.

Lemma 5.1. *Suppose that $m \geq 2$ and that the element A of $S^2V^* \otimes W$ satisfies condition* (ii). *Let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V. Let $1 \leq q \leq p$ and $Y_1, \ldots, Y_q$ be elements of W. If the equalities*

$$Y_1 \wedge \ldots \wedge Y_q \wedge A_{j_{q+1}k_{q+1}} \wedge \ldots \wedge A_{j_m k_m} = 0$$

hold for all integers $1 \leq j_{q+1}, k_{q+1}, \ldots, j_m, k_m \leq p$, then we have

$$Y_1 \wedge \ldots \wedge Y_q = 0.$$

Proof: If $Y_1 \wedge \ldots \wedge Y_q$ does not vanish, according to condition (ii), there exist integers $1 \leq j_{q+1}, k_{q+1}, \ldots, j_m, k_m \leq p$ such that

$$\{Y_1, \ldots, Y_q, A_{j_{q+1}k_{q+1}}, \ldots, A_{j_m k_m}\}$$

is a basis for W. This clearly contradicts our hypothesis.

Let η be a unit vector of V and let $-1 \leq d \leq m-1$ be a given integer. We say that η possesses property $\mathcal{P}_d$ if

$$A(\eta, \zeta_1) \wedge A(\eta_2, \zeta_2) \wedge \ldots \wedge A(\eta_{m-d}, \zeta_{m-d}) = 0,$$

for all $\eta_2, \ldots, \eta_{m-d} \in V'_\eta$ and $\zeta_1, \ldots, \zeta_{m-d} \in V$. Let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V, with $v_1 = \eta$. Set $q = \dim V_\eta + 1$; if $q \geq 2$, suppose that $\{v_2, \ldots, v_q\}$ is a basis of V_η. Then η possesses property $\mathcal{P}_d$ if and only if the relations

$$A_{1k_1} \wedge A_{i_2k_2} \wedge \ldots \wedge A_{i_{m-d}k_{m-d}} = 0$$

hold for all $1 \leq i_2, \ldots, i_{m-d} \leq q$ and $1 \leq k_1, \ldots, k_{m-d} \leq p$. Clearly, property $\mathcal{P}_d$ implies that $q \geq p - m + d + 1$. We remark that property $\mathcal{P}_{-1}$ always holds, since W is m-dimensional.

LEMMA 5.2. *Suppose that $m \geq 2$ and $p \geq 2$ and that the element A of $S^2V^* \otimes W$ satisfies condition* (ii). *Let η be a unit vector of V which possesses property $\mathcal{P}_0$. If either $V'_\eta = V$, or if $\dim V'_\eta = p-1$ and $\operatorname{Tr} A = 0$, then we have*

$$A(\eta, \zeta) = 0,$$

for all $\zeta \in V$.

PROOF: Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V, with $v_1 = \eta$; we set $A_{jr} = A(v_j, v_r)$. We shall show that our hypotheses imply that

$$A_{1k} \wedge A_{j_2 l_2} \wedge \dots \wedge A_{j_m l_m} = 0, \tag{5.1}$$

for all $1 \leq k, j_2, l_2, \dots, j_m, l_m \leq p$. By Lemma 5.1, from these relations we infer that $A_{1k} = 0$, for $1 \leq k \leq p$, which is the desired conclusion. First, if $V_\eta = V$, then property $\mathcal{P}_0$ for η tells us that (5.1) holds. On the other hand, if $\dim V'_\eta = p-1$ and $\{v_1, \dots, v_{p-1}\}$ is a basis of V'_η, then property $\mathcal{P}_0$ implies that (5.1) holds for all $1 \leq k, l_2, \dots, l_m \leq p$ and $1 \leq j_2, \dots, j_m \leq p-1$. If moreover $\operatorname{Tr} A = 0$, then we have $A_{pp} = -\sum_{j=1}^{p-1} A_{jj}$, and so in this case (5.1) is also true.

If $\alpha \in V^*$ and $\beta \in \bigwedge^2 V^*$, we define elements $E(\alpha)$ of $V^* \otimes W$ and $F(\beta)$ of $V^* \otimes V^* \otimes W$ by

$$E(\alpha) = (\operatorname{Tr}_1 \otimes \mathrm{id})(\alpha \otimes A), \qquad F(\beta)(\eta_1, \eta_2) = (\operatorname{Tr}_2 \otimes \mathrm{id})(\beta \otimes A)(\eta_2, \eta_1),$$

for $\eta_1, \eta_2 \in V$. Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V. For $1 \leq i, j \leq p$, we write $\alpha_i = \alpha(v_i)$ and $\beta_{ij} = \beta(v_i, v_j)$; then the elements $E(\alpha)_i = E(\alpha)(v_i)$ and $F(\beta)_{ij} = F(\beta)(v_i, v_j)$ of W are given by

$$E(\alpha)_i = \sum_{k=1}^{p} \alpha_k A_{ik}, \qquad F(\beta)_{ij} = \sum_{k=1}^{p} \beta_{jk} A_{ik}.$$

Let a_l, b_l be given real numbers, with $l = 1, 2, 3$; for $\beta \in \bigwedge^2 V^*$, we define elements $F_l(\beta)$ of $V^* \otimes V^* \otimes W$ by

$$F_l(\beta)(\xi, \eta) = a_l F(\beta)(\xi, \eta) + b_l F(\beta)(\eta, \xi)$$

for $\xi, \eta \in V$, and, for $1 \leq i, j \leq p$ and $l = 1, 2, 3$, we set

$$F_l(\beta)_{ij} = F_l(\beta)(v_i, v_j).$$

For $\beta_1,\dots,\beta_r \in \bigwedge^2 V^*$ and $1 \le i_0, i_1, \dots, i_r \le p$, we consider the elements

$$\Phi'(\beta_1,\dots,\beta_r; v_{i_0}, v_{i_1},\dots,v_{i_r}) = F(\beta_1)_{i_0 i_1} \wedge F_1(\beta_2)_{i_0 i_2} \wedge \dots \wedge F_1(\beta_r)_{i_0 i_r},$$

$$\Phi''(\beta_1,\dots,\beta_r; v_{i_0}, v_{i_1},\dots,v_{i_r}) = F_3(\beta_1)_{i_0 i_1} \wedge F_1(\beta_2)_{i_0 i_2} \wedge \dots \wedge F_1(\beta_r)_{i_0 i_r},$$

$$\Phi(\beta_2,\dots,\beta_r; v_{i_0}, v_{i_2},\dots,v_{i_r})$$
$$= F(\beta_2)_{i_0 i_2} \wedge F(\beta_2)_{i_2 i_0} \wedge F_1(\beta_3)_{i_0 i_3} \wedge \dots \wedge F_1(\beta_r)_{i_0 i_r}$$

of $\bigwedge^r W$; if $i_1 = i_2$ and $\beta_1 = \beta_2$, we see that

$$\Phi'(\beta_2,\beta_2,\dots,\beta_r; v_{i_0}, v_{i_2}, v_{i_2},\dots,v_{i_r}) = b_1 \Phi(\beta_2,\dots,\beta_r; v_{i_0}, v_{i_2},\dots,v_{i_r}),$$

$$\Phi''(\beta_2,\beta_2,\dots,\beta_r; v_{i_0}, v_{i_2}, v_{i_2},\dots,v_{i_r})$$
$$= (a_3 b_1 - a_1 b_3)\, \Phi(\beta_2,\dots,\beta_r; v_{i_0}, v_{i_2},\dots,v_{i_r}).$$

LEMMA 5.3. *Suppose that $m \ge 2$ and $p \ge 2$. Let a_1, b_1 be given non-zero real numbers. Let η be a unit vector of V for which V_η is non-zero. Let $\{v_1,\dots,v_p\}$ be an orthonormal basis of V, with $v_1 = \eta$; suppose that $\{v_2,\dots,v_q\}$ is a basis for V_η, where $2 \le q \le p$. Then the following conditions are equivalent:*

(a) *For all $\beta_2,\dots,\beta_m \in \bigwedge^2 V^*$ and all integers $2 \le i_2,\dots,i_m \le q$, the element*

$$\Phi(\beta_2,\dots,\beta_m; v_1, v_{i_2},\dots,v_{i_m}) \tag{5.2}$$

of $\bigwedge^m W$ vanishes.

(b) *The equalities*

$$A_{1k_1} \wedge A_{ik_2} \wedge A_{j_3k_3} \wedge \dots \wedge A_{j_mk_m} = 0$$

hold for all $2 \le i \le q$, $1 \le k_1,\dots,k_m \le p$ and $1 \le j_3,\dots,j_m \le q$.

PROOF: For $2 \le i_2,\dots,i_m \le q$, we view the expression (5.2), where $\beta_2,\dots,\beta_m$ are elements of $\bigwedge^2 V^*$, as a polynomial in the variables $\beta_{l,ij} = \beta_l(v_i,v_j)$, where $i \ne j$, with coefficients in $\bigwedge^m W$. The vanishing of this polynomial is equivalent to the vanishing of all its coefficients and hence, since $A_{1i_2} = 0$ and $a_1, b_1 \ne 0$, to the relations

$$A_{1k} \wedge A_{i_2 l} \wedge Y_3 \wedge \dots \wedge Y_m = 0,$$

for all $1 \le k, l \le p$, where Y_j is equal either to A_{1l_j}, $A_{i_jl_j}$ or $b_1 A_{i_ji_j} - a_1 A_{11}$, with $1 \le l_j \le p$ and $l_j \ne 1, i_j$, for $3 \le j \le m$. Since the integers $2 \le i_3,\dots,i_m \le q$ are arbitrary, the equivalence of conditions (a) and (b) is a consequence of the preceding observation.

Lemma 5.4. *Suppose that $m \geq 2$ and $p \geq 2$. Let a_1, b_1, a_3, b_3 be given real numbers satisfying $a_1, a_3 \neq 0$. Let η be a unit vector of V for which V_η is non-zero. Let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V, with $v_1 = \eta$; suppose that $\{v_2, \ldots, v_q\}$ is a basis for V_η, where $2 \leq q \leq p$. Assume that condition* (b) *of Lemma 5.3 holds. Let $2 \leq i_1, \ldots, i_m \leq q$ be given integers. Suppose that, for all vectors $\beta_1, \ldots, \beta_m$ of $\bigwedge^2 V^*$, one of the two elements*

$$(5.3)\quad \Phi'(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}), \qquad \Phi''(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m})$$

of $\bigwedge^m W$ vanishes. Then η possesses property $\mathcal{P}_0$.

Proof: If $\beta_2, \ldots, \beta_m$ are elements of $\bigwedge^2 V^*$, we write

$$P(\beta_2, \ldots, \beta_m) = F_1(\beta_2)_{1i_2} \wedge \ldots \wedge F_1(\beta_m)_{1i_m}.$$

If $\beta_1 \in \bigwedge^2 V^*$, then we have

$$\begin{aligned}
\Phi'(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}) &= \sum_{k=1}^{p} \beta_{i_1 k} A_{1k} \wedge P(\beta_2, \ldots, \beta_m), \\
\Phi''(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}) &= a_3 \Phi'(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}) \\
&\qquad + b_3 \sum_{k=1}^{p} \beta_{1k} A_{i_1 k} \wedge P(\beta_2, \ldots, \beta_m),
\end{aligned}$$

where $\beta_{ij} = \beta_1(v_i, v_j)$. For fixed $\beta_2, \ldots, \beta_m \in \bigwedge^2 V^*$, we view the expression

$$\Phi'(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}) \otimes \Phi''(\beta_1, \ldots, \beta_m; v_1, v_{i_1}, \ldots, v_{i_m}),$$

where β_1 is an element of $\bigwedge^2 V^*$, as a polynomial in the variables $\beta_{ij} = \beta_1(v_i, v_j)$, with $i \neq j$, with coefficients in $\bigwedge^m W \otimes \bigwedge^m W$. Our hypotheses imply that this polynomial vanishes identically; in particular, this means that the coefficient of $\beta_{i_1 k}^2$ in its expansion vanishes, where $k \neq 1, i_1$. It is easily seen that this coefficient is equal to

$$a_3 (A_{1k} \wedge P(\beta_2, \ldots, \beta_m)) \otimes (A_{1k} \wedge P(\beta_2, \ldots, \beta_m)).$$

Since $a_3 \neq 0$, we therefore obtain the equality

$$(5.4)\qquad A_{1k} \wedge P(\beta_2, \ldots, \beta_m) = 0,$$

for all $q+1 \leq k \leq p$ and $\beta_2, \dots, \beta_m \in \bigwedge^2 V^*$. We now view the left-hand side of (5.4), where $\beta_2, \dots, \beta_m \in \bigwedge^2 V^*$, as a polynomial in the variables $\beta_{l,ij} = \beta_l(v_i, v_j)$, with $2 \leq l \leq m$ and $i \neq j$, with coefficients in $\bigwedge^m W$. Since $a_1 \neq 0$, from the vanishing of the coefficients of this polynomial we deduce the relations

$$A_{1k} \wedge Y_2 \wedge \ldots \wedge Y_m = 0,$$

for all $q+1 \leq k \leq p$, where the vector Y_j of W is equal either to A_{1l_j} or $b_1 A_{i_j i_j} - a_1 A_{11}$, with $1 \leq l_j \leq p$ and $l_j \neq 1, i_j$, for $2 \leq j \leq m$. By condition (b) of Lemma 5.3, we see that these equalities imply that

$$A_{1k_1} \wedge \ldots \wedge A_{1k_m} = 0,$$

for all integers $1 \leq k_1, \dots, k_m \leq p$. From condition (b) of Lemma 5.3, we deduce that η possesses property $\mathcal{P}_0$.

We consider the following property which a unit vector η of V might possess:

(A) The space V_η is non-zero, and, if $\{v_1, \dots, v_p\}$ is an orthonormal basis of V for which $v_1 = \eta$ and $\{v_2, \dots, v_q\}$ is a basis of V_η, where $2 \leq q \leq p$, there exist integers $2 \leq i_1, \dots, i_m \leq q$ and elements $\beta_1, \dots, \beta_m$ of $\bigwedge^2 V^*$ such that the two elements (5.3) of $\bigwedge^m W$ do not vanish.

Lemma 5.5. *Suppose that $m \geq 2$ and $p \geq 3$. Let a_1, b_1, a_3, b_3 be given real numbers satisfying $a_1, b_1, a_3 \neq 0$ and $a_3 b_1 - a_1 b_3 \neq 0$. Assume that the element A of $S^2 V^* \otimes W$ satisfies conditions* (i) *and* (ii) *and* $\operatorname{Tr} A = 0$. *Let η be a unit vector of V for which* $\dim V_\eta \geq p-2$. *Then η possesses property* (A).

Proof: Assume that the assertion of the lemma is not true. According to Lemma 5.3, we easily see that the hypotheses of Lemma 5.4 hold. Thus η possesses property $\mathcal{P}_0$; then by Lemma 5.2, we deduce that $A(\eta, \zeta) = 0$, for all $\zeta \in V$, which contradicts condition (i).

Lemma 5.6. *Let $d, r \geq 0$ be given integers and Y be an element of $\bigwedge^r W$. Assume that the element A of $S^2 V^* \otimes W$ satisfies* $\operatorname{Tr} A = 0$. *Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V. Suppose that, for all integers $2 \leq i_1, \dots, i_d \leq p$ and all vectors $\alpha_1, \dots, \alpha_d$ of V^*, the element*

$$Y \wedge E(\alpha_1)_{i_1} \wedge \ldots \wedge E(\alpha_d)_{i_d} \tag{5.5}$$

of $\bigwedge^{r+d} W$ vanishes. Then we have

$$Y \wedge A_{j_1 k_1} \wedge \ldots \wedge A_{j_d k_d} = 0, \tag{5.6}$$

for all integers $1 \leq j_1, k_1 \dots, j_d, k_d \leq p$. *If moreover* A *satisfies condition* (ii) *and* Y *is a decomposable element of* $\bigwedge^r W$, *with* $r \geq 1$, *then* Y *vanishes.*

Proof: We view the expression (5.5), where $\alpha_1, \dots, \alpha_d$ are elements of V^*, as a polynomial in the variables $\alpha_{l,j} = \alpha_l(v_j)$, with coefficients in $\bigwedge^{r+d} W$. For $1 \leq k_1 \dots, k_d \leq p$, the coefficient of $\alpha_{1,k_1} \dots \alpha_{d,k_d}$ in this polynomial is equal to

$$Y \wedge A_{i_1 k_1} \wedge \dots \wedge A_{i_d k_d}.$$

Thus the vanishing of these coefficients implies that (5.6) holds for all integers $2 \leq i_1, \dots, i_d \leq p$ and $1 \leq k_1 \dots, k_d \leq p$. Since $A_{11} = -\sum_{j=1}^{p} A_{jj}$, the equality (5.6) is always true. If A satisfies condition (ii) and Y is a decomposable element of $\bigwedge^r W$, where $r \geq 1$, the vanishing of Y is a consequence of Lemma 5.1.

Lemma 5.7. *Suppose that* $m \geq 2$ *and* $p \geq 2$. *Assume that the element* A *of* $S^2 V^* \otimes W$ *satisfies condition* (ii) *and* $\operatorname{Tr} A = 0$. *Let* $\{v_1, \dots, v_p\}$ *be an orthonormal basis of* V. *Then there exist integers* $2 \leq i_1, \dots, i_m \leq p$ *and vectors* $\alpha_1, \dots, \alpha_m$ *of* V^* *such that the element*

$$E(\alpha_1)_{i_1} \wedge \dots \wedge E(\alpha_m)_{i_m} \tag{5.7}$$

of $\bigwedge^m W$ *does not vanish.*

Proof: We suppose that the elements (5.7) of $\bigwedge^m W$ vanish for all integers $2 \leq i_1, \dots, i_m \leq p$ and all vectors $\alpha_1, \dots, \alpha_m$ of V^*. According to Lemma 5.6, with $Y = 1$ and $d = m$, we see that

$$A_{j_1 k_1} \wedge \dots \wedge A_{j_m k_m} = 0,$$

for all integers $1 \leq j_1, k_1 \dots, j_m, k_m \leq p$. This clearly contradicts condition (ii).

Lemma 5.8. *Let* $r \geq 0$ *be a given integer and* Y *be an element of* $\bigwedge^r W$. *Let* $\{v_1, \dots, v_p\}$ *be an orthonormal basis of* V.

(i) *Let* $1 \leq q \leq p$ *be a given integer. Suppose that, for all integers* $q \leq i, j \leq p$ *and all vectors* α *of* V^*, *the element*

$$Y \wedge E(\alpha)_i \wedge E(\alpha)_j \tag{5.8}$$

of $\bigwedge^{r+2} W$ *vanishes. Then we have*

$$Y \wedge A_{ij} \wedge A_{kl} = 0,$$

for all $q \leq i, j, k, l \leq p$.

(ii) *Let $d \geq 1$ be a given integer. Suppose that, for all integers $2 \leq i_1, \dots, i_d \leq p$ and all vectors α of V^*, the element*

$$Y \wedge E(\alpha)_{i_1} \wedge \dots \wedge E(\alpha)_{i_d} \tag{5.9}$$

of $\bigwedge^{r+d} W$ vanishes. Then we have

$$Y \wedge A_{1i_1} \wedge \dots \wedge A_{1i_d} = 0,$$

for all integers $2 \leq i_1, \dots, i_d \leq p$.

Proof: (i) We suppose that the elements (5.8) of $\bigwedge^{r+2} W$ vanish for all vectors α of V^* and all integers $q \leq i, j \leq p$. For $1 \leq i, j \leq p$ and $\alpha \in V^*$, we have

$$E(\alpha)_i \wedge E(\alpha)_j = \tfrac{1}{2} \sum_{k,l=1}^{p} (A_{ik} \wedge A_{jl} + A_{il} \wedge A_{jk}) \alpha_k \alpha_l.$$

Thus if we view the expression (5.8), where $\alpha \in V^*$, as a polynomial in the variables $\alpha_k = \alpha(v_k)$ with coefficients in $\bigwedge^{r+2} W$, we easily see that our assumption is equivalent to the relations

$$Y \wedge (A_{ik} \wedge A_{jl} + A_{il} \wedge A_{jk}) = 0,$$

for all $1 \leq k, l \leq p$ and $q \leq i, j \leq p$. Therefore, for $q \leq i, j, k, l \leq p$, we have

$$Y \wedge A_{ij} \wedge A_{kl} = -Y \wedge A_{il} \wedge A_{jk} = Y \wedge A_{ik} \wedge A_{jl} = -Y \wedge A_{ij} \wedge A_{kl}.$$

From these equalities, we deduce the desired conclusion.

(ii) We suppose that the elements (5.9) of $\bigwedge^2 W$ vanish for all integers $2 \leq i_1, \dots, i_d \leq p$ and all vectors α of V^*. If we view the expression (5.9), where $\alpha \in V^*$, as a polynomial in the variables $\alpha_k = \alpha(v_k)$ with coefficients in $\bigwedge^{r+d} W$, the coefficient coefficient of α_1^d is equal to

$$Y \wedge A_{1i_1} \wedge \dots \wedge A_{1i_d}.$$

Since this coefficient vanishes, we obtain assertion (ii).

Lemma 5.9. *Suppose that $m = 2$ and $p \geq 2$. Assume that the element A of $S^2V^* \otimes W$ satisfies condition* (ii). *Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V. Then there exist distinct integers $1 \leq i, j \leq p$ and a vector α of V^* and such that the element*

$$E(\alpha)_i \wedge E(\alpha)_j \tag{5.10}$$

of $\bigwedge^2 W$ does not vanish.

Proof: We suppose that the elements (5.10) of $\bigwedge^2 W$ vanish for all vectors α of V^* and all integers $1 \leq i, j \leq p$. According to Lemma 5.8,(i) with $q = 1$, we know that the equality

$$A_{ij} \wedge A_{kl} = 0 \tag{5.11}$$

holds for all integers $1 \leq i, j, k, l \leq p$. Since $m = 2$, this clearly contradicts condition (ii).

Lemma 5.10. *Suppose that $m = 3$. Assume that the element A of $S^2V^* \otimes W$ satisfies condition* (ii) *and* $\operatorname{Tr} A = 0$. *Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V. Then there exist integers $2 \leq i_1, i_2, i_3 \leq p$ and vectors α_1, α_2 of V^* such that the element*

$$E(\alpha_1)_{i_1} \wedge E(\alpha_1)_{i_2} \wedge E(\alpha_2)_{i_3} \tag{5.12}$$

of $\bigwedge^3 W$ does not vanish.

Proof: We suppose that the elements (5.12) of $\bigwedge^3 W$ vanish for all integers $2 \leq i_1, i_2, i_3 \leq p$ and all vectors α_1, α_2 of V^*. By Lemma 5.6 and condition (ii), we obtain the equality

$$E(\alpha)_{i_1} \wedge E(\alpha)_{i_2} = 0,$$

for all integers $2 \leq i_1, i_2 \leq p$ and all vectors α of V^*. According to Lemma 5.8, we deduce that

$$A_{1i_1} \wedge A_{1i_2} = A_{ij} \wedge A_{kl} = 0,$$

for all integers $2 \leq i_1, i_2, i, j, k, l \leq p$. Since $A_{11} = -\sum_{j=2}^{p} A_{jj}$, these last relations imply that the subspace of W generated by the vectors A_{ij}, with $1 \leq i, j \leq p$, is of dimension ≤ 2; this contradicts condition (ii).

LEMMA 5.11. *Suppose that $m \geq 4$ and $p \geq 3$. Assume that the element A of $S^2V^* \otimes W$ satisfies condition* (ii) *and* $\mathrm{Tr}\, A = 0$. *Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V. If either $m = 4$ and $\dim V_{v_1} < p - 2$ or if $m \geq 5$, then there exist integers $2 \leq i_1, \dots, i_m \leq p$ and a system $\{\alpha_1, \dots, \alpha_m\}$ of vectors of V^*, where at most $m - 2$ of these vectors are distinct, such that the element* (5.7) *of $\bigwedge^m W$ does not vanish.*

PROOF: We suppose that the elements

$$E(\alpha_1)_{i_1} \wedge E(\alpha_1)_{i_2} \wedge E(\alpha_1)_{i_3} \wedge E(\alpha_4)_{i_4} \wedge \ldots \wedge E(\alpha_m)_{i_m},$$

$$E(\alpha_1)_{i_1} \wedge E(\alpha_1)_{i_2} \wedge E(\alpha_2)_{i_3} \wedge E(\alpha_2)_{i_4} \wedge E(\alpha_5)_{i_5} \wedge \ldots \wedge E(\alpha_m)_{i_m}$$

of $\bigwedge^m W$ vanish for all vectors $\alpha_1, \alpha_2, \alpha_4, \dots, \alpha_m$ of V^* and all integers $2 \leq i_1, \dots, i_m \leq p$. We denote by W' the subspace of W generated by the vectors A_{ij}, with $1 \leq i, j \leq p$. By Lemmas 5.6 and 5.8,(ii) and condition (ii), we obtain the equalities

$$(5.13)\quad A_{1i_1} \wedge A_{1i_2} \wedge A_{1i_3} = 0, \quad E(\alpha_1)_{i_1} \wedge E(\alpha_1)_{i_2} \wedge E(\alpha_2)_{i_3} \wedge E(\alpha_2)_{i_4} = 0,$$

for all integers $2 \leq i_1, i_2, i_3, i_4 \leq p$ and all vectors α_1, α_2 of V^*. According to Lemma 5.8, from the second equalities of (5.13) we infer that

$$(5.14)\qquad A_{1i_1} \wedge A_{1i_2} \wedge A_{i_3j_3} \wedge A_{i_4j_4} = A_{i_1j_1} \wedge A_{i_2j_2} \wedge A_{i_3j_3} \wedge A_{i_4j_4} = 0,$$

for all integers $2 \leq i_1, j_1, \dots, i_4, j_4 \leq p$. First, suppose that there exist integers $2 \leq i_1, i_2 \leq p$ such that $A_{1i_1} \wedge A_{1i_2} \neq 0$; this assumption is equivalent to the fact that the dimension of V_{v_1} is $< p - 2$. Then if there are integers $2 \leq i, j \leq p$ such that $A_{1i_1} \wedge A_{1i_2} \wedge A_{ij} \neq 0$, since $A_{11} = -\sum_{k=2}^{p} A_{kk}$, according to (5.13) and (5.14) the vectors $A_{1i_1}, A_{1i_2}, A_{ij}$ generate the subspace W'. Since $m \geq 4$, this clearly contradicts condition (ii). On the other hand, if $A_{1i_1} \wedge A_{1i_2} \wedge A_{ij} = 0$, for all $2 \leq i, j \leq p$, then by (5.13) the subspace W' is of dimension ≤ 2, which again contradicts condition (ii). By (5.14), we know that the subspace of W generated by the vectors A_{ij}, with $2 \leq i, j \leq p$, is of dimension ≤ 3. If the relation $A_{1i_1} \wedge A_{1i_2} = 0$ holds for all integers $2 \leq i_1, i_2 \leq p$, it follows that the subspace W' is of dimension ≤ 4. Thus when $m \geq 5$, we obtain a contradiction.

If ω is an element of W^*, we consider the element A_ω of S^2V^* determined by

$$A_\omega(\eta_1, \eta_2) = \langle A(\eta_1, \eta_2), \omega \rangle,$$

for $\eta_1, \eta_2 \in V$. If ω is non-zero and $\{v_1, \dots, v_p\}$ is an orthonormal basis of V which diagonalizes the quadratic form A_ω, then we remark that the dimension of V_{v_i} is $\geq p - m$, for all $1 \leq i \leq p$.

Lemma 5.12. *Suppose that $m = 3$ and $p \geq 5$. Let a_2, b_2 be given real numbers satisfying $a_2 > 0$, $b_2 \geq 0$. Assume that the element A of $S^2V^* \otimes W$ satisfies condition* (ii) *and* $\operatorname{Tr} A = 0$. *Let ω be a non-zero element of W^* and let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V which diagonalizes A_ω; for $1 \leq j \leq p$, we set $V_j = V_{v_j}$. Assume that for all integers $2 \leq i_1, i_2, j \leq p$, for all vectors ζ of V_j orthogonal to v_1 and for all vectors β of $\bigwedge^2 V^*$ and α of V^*, the elements*

$$E(\alpha)_{i_1} \wedge E(\alpha)_{i_2} \wedge F_2(\beta)(v_j, \zeta) \tag{5.15}$$

of $\bigwedge^3 W$ vanish. If the vector $\eta = v_1$ possesses property $\mathcal{P}_0$, then the dimension of V_1 is $\geq p - 2$.

Proof: Assume that $\eta = v_1$ possesses property $\mathcal{P}_0$ and that the dimension of V_1 is $< p - 2$. Since the basis $\{v_1, \dots, v_p\}$ diagonalizes the quadratic form A_ω, we know that $\dim V_i \geq p - 3 \geq 2$, for $2 \leq i \leq p$. Thus the dimension of V_1 is equal to $p - 3$ and there exist integers $2 \leq i_1, i_2 \leq p$ such that $Y = A_{1i_1} \wedge A_{1i_2} \neq 0$. Let $2 \leq j \leq p$ be a given integer; then there exist a unit vector ζ of V_j orthogonal to v_1. According to Lemma 5.8,(ii), our hypotheses imply that

$$Y \wedge F_2(\beta)(v_j, \zeta) = 0, \tag{5.16}$$

for all $\beta \in \bigwedge^2 V^*$. We first choose an orthonormal basis $\{v'_1, \dots, v'_p\}$ of V, with $v'_1 = v_j$ and $v'_2 = \zeta$ and view the left-hand side of (5.16) as a polynomial in the variables $\beta'_{kl} = \beta(v'_k, v'_l)$, where $k \neq l$, with coefficients in $\bigwedge^m W$; the coefficient of β'_{2l}, with $3 \leq l \leq p$, in its expansion is equal to

$$a_2 Y \wedge A(v_j, v'_l).$$

Since a_2 is non-zero and $A(v_j, v'_2) = 0$, the vanishing of this coefficient implies that

$$Y \wedge A_{jl} = 0, \tag{5.17}$$

for all integers $1 \leq l \leq p$ different from j. Since the integer $2 \leq j \leq p$ was arbitrary, we see that (5.17) holds for all integers $2 \leq j \leq p$ and $1 \leq l \leq p$, with $j \neq l$. Set $d = p - 4 \geq 1$ and let $2 \leq j_1, \dots, j_d \leq p$ be distinct integers. There exists a unit vector $\xi = \sum_{l=2}^{p} c_l v_l$ of V_1 orthogonal to $v_{j_1}, \dots, v_{j_d}$. Since η possesses property $\mathcal{P}_0$, we know that

$$Y \wedge A(\xi, v_l) = 0,$$

for all $1 \leq l \leq p$. By (5.17), the preceding equalities imply that

$$c_l Y \wedge A_{ll} = 0,$$

for all $2 \leq l \leq p$. Hence, since ξ is a unit vector, these relations give us an integer $2 \leq k \leq p$ different from $j_1, \ldots, j_d$ such that

$$Y \wedge A_{kk} = 0.$$

Since the integers $2 \leq j_1, \ldots, j_d \leq p$ can be arbitrary, from this last fact we deduce the existence of $d+1$ distinct integers $2 \leq l_1, \ldots, l_{d+1} \leq p$ such that

$$Y \wedge A_{l_i l_i} = 0,$$

for $1 \leq i \leq d+1$. Without loss of generality, we may suppose that $l_i = i+1$, for $1 \leq i \leq p-3$. As η possesses property $\mathcal{P}_0$, we have $Y \wedge A_{11} = 0$. Therefore, the equality

$$Y \wedge A_{ll} = 0 \tag{5.18}$$

holds for $1 \leq l \leq p-2$. We may write $\zeta = \sum_{l=2}^{p} c'_l v_l$, where $c'_j = 0$. We now view the left-hand side of (5.16) as a polynomial in the variables $\beta_{kl} = \beta(v_k, v_l)$, where $k \neq l$, with coefficients in $\bigwedge^m W$; by (5.17), the coefficient of β_{jl}, with $2 \leq l \leq p$ and $l \neq j$, is equal to

$$c'_l Y \wedge (b_2 A_{ll} - a_2 A_{jj}).$$

Since ξ is a unit vector, the vanishing of these coefficients gives us an integer $2 \leq k \leq p$ different from j such that

$$Y \wedge (a_2 A_{jj} - b_2 A_{kk}) = 0.$$

Since $a_2 \neq 0$, the previous observation, with $j = p-1$, and the relations (5.18) tell us that either $Y \wedge A_{p-1,p-1} = 0$ or that

$$Y \wedge (a_2 A_{p-1,p-1} - b_2 A_{pp}) = 0.$$

Since $\operatorname{Tr} A = 0$, by (5.18) we see that

$$Y \wedge (A_{p-1,p-1} + A_{pp}) = 0.$$

Since $a_2 > 0$ and $b_2 \geq 0$, from these relations we deduce that (5.18) holds for all $1 \leq l \leq p$ and hence that (5.17) is true for all $1 \leq j, l \leq p$. By condition (ii) and Lemma 5.1, it follows that $Y = 0$, which is a contradiction.

LEMMA 5.13. *Suppose that $m = 3$ and $p \geq 5$. Let a_l, b_l be given real numbers, with $l = 1, 2, 3$, satisfying $a_1, b_1, a_3 \neq 0$, $a_2 > 0$, $b_2 \geq 0$ and $a_3 b_1 - a_1 b_3 \neq 0$. Assume that the element A of $S^2V^* \otimes W$ satisfies conditions* (i) *and* (ii) *and* $\operatorname{Tr} A = 0$. *Let ω be a non-zero element of W^* and let $\{v_1, \dots, v_p\}$ be an orthonormal basis of V which diagonalizes A_ω; for $1 \leq j \leq p$, we set $V_j = V_{v_j}$. Then either the vector v_1 possesses property* (A) *or the following assertion holds: there exist integers $2 \leq i_1, i_2, j \leq p$, a vector ζ of V_j orthogonal to v_1 and vectors β of $\bigwedge^2 V^*$ and α of V^* such that the element* (5.15) *of $\bigwedge^3 W$ does not vanish.*

PROOF: Assume that the assertion of the lemma is not true. According to Lemma 5.5, we must have $\dim V_1 < p-2$. Also according to Lemma 5.3, the hypotheses of Lemma 5.4 hold. Thus η has property $\mathcal{P}_0$. Lemma 5.12 now implies that $\dim V_1 \geq p-2$; we thus obtain a contradiction.

We identify $S^l V^*$ with a subspace of the l-th tensor product $\bigotimes^l V^*$ of V^* and we consider the inner product $\langle\ ,\ \rangle$ on $\bigotimes^2 V^*$ induced by the inner-product $\langle\ ,\ \rangle$ of V. If $\beta \in \bigwedge^2 V^*$ and $h \in S^2V^*$, we consider the element $\langle F(\beta), h\rangle$ of W obtained from $F(\beta)$ and h via the inner-product on $\bigotimes^2 V^*$.

If η is a vector of V, we consider the subspace

$$H_\eta = \left\{ h \in S^2V^* \;\middle|\; \begin{array}{c} \operatorname{Tr} h = h(\eta, \zeta) = 0, \text{ for all } \zeta \in V, \\ \langle h, A_\omega \rangle = 0, \text{ for all } \omega \in W^* \end{array} \right\}$$

of S^2V^*.

LEMMA 5.14. *Suppose that $m = 3$ and $p = 4$. Let a_1, b_1 be given real numbers, with $a_1 \neq 0$. Assume that the element A of $S^2V^* \otimes W$ satisfies condition* (ii) *and* $\operatorname{Tr} A = 0$. *Let η be a unit vector of V for which V_η is non-zero. Assume that, for all vectors ζ of V_η, for all elements h of H_η and for all vectors $\beta_1, \beta_2, \beta_3$ of $\bigwedge^2 V^*$, the element*

$$F_1(\beta_1)(\eta, \zeta) \wedge F_1(\beta_2)(\eta, \zeta) \wedge \langle F(\beta_3), h\rangle \tag{5.19}$$

of $\bigwedge^3 W$ vanishes. If the vector η possesses property $\mathcal{P}_0$, then the dimension of V_η is ≥ 2.

PROOF: Assume that η possesses property $\mathcal{P}_0$ and that V_η is one-dimensional. Let $\{v_1, \dots, v_4\}$ be an orthonormal basis of V, for which $v_1 = \eta$ and v_2 is a generator of V_η. Then we know that $Y = A_{13} \wedge A_{14} \neq 0$. From property $\mathcal{P}_0$ of η, we obtain the relations

$$A_{1k_1} \wedge A_{i_2k_2} \wedge A_{i_3k_3} = 0, \tag{5.20}$$

for all integers $1 \leq i_1, i_2 \leq 2$ and $1 \leq k_1, k_2, k_3 \leq 4$. From the condition $\operatorname{Tr} A = 0$, it therefore follows that

$$Y \wedge (A_{33} + A_{44}) = 0. \tag{5.21}$$

We now consider a relation of the form

$$c_1 A_{33} + c_2 A_{34} + \sum_{i=2}^{4} d_i A_{2i} = 0, \tag{5.22}$$

with $c_j, d_i \in \mathbb{R}$ and $d_2 = -c_1$. The element h of S^2V^* determined by $h(v_j, v_l) = h_{jl}$, where $h_{33} = c_1$, $h_{34} = c_2$, $h_{2i} = d_i$ for $2 \leq i \leq 4$ and $h_{jl} = 0$ when $1 \leq j \leq l \leq 4$ and $(jl) \neq (33)$, (34) or $(2i)$, with $2 \leq i \leq 4$, belongs to H_η. We view the expression (5.19), where $\beta_1, \beta_2, \beta_3 \in \bigwedge^2 V^*$, as a polynomial in the variables $\beta_{l,ij} = \beta_l(v_i, v_j)$, with $i \neq j$, with coefficients in $\bigwedge^3 W$. Our hypotheses imply that the coefficients of $\beta_{1,23}\beta_{2,24}\beta_{3,34}$ and $\beta_{1,23}\beta_{2,24}\beta_{3,2j}$, with $j = 3, 4$, in its expansion vanish; thus by (5.20), since $a_1 \neq 0$, we obtain the equalities

$$Y \wedge (c_2(A_{33} - A_{44}) - c_1 A_{34}) = 0$$

and

$$Y \wedge \left(\sum_{i=2}^{4} d_i A_{3i} \right) = Y \wedge \left(\sum_{i=2}^{4} d_i A_{4i} \right) = 0. \tag{5.23}$$

From the relations (5.22) and (5.20), we infer that

$$Y \wedge (c_1 A_{33} + c_2 A_{34}) = 0.$$

The determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ c_1 & 0 & c_2 \\ c_2 & -c_2 & -c_1 \end{pmatrix}$$

is equal to $c_1^2 + 2c_2^2$. Therefore whenever $c_1^2 + c_2^2 \neq 0$, according to the preceding relations and (5.21), we obtain the equalities

$$Y \wedge A_{33} = Y \wedge A_{34} = Y \wedge A_{44} = 0. \tag{5.24}$$

Thus if $c_1^2 + c_2^2 \neq 0$, by (5.20) we have

$$Y \wedge A_{jl} = 0, \tag{5.25}$$

for all $1 \leq j, l \leq 4$, and so, by condition (ii) and Lemma 5.1, we see that $Y = 0$; this leads to a contradiction. Now suppose that the space V_{v_2} is one-dimensional. Then we have $A_{23} \wedge A_{24} \neq 0$; by (5.20), we see that $A_{23} \wedge A_{24}$ is a multiple of Y and hence we may write

$$A_{22} = aA_{23} + bA_{24}, \tag{5.26}$$

where $a^2 + b^2 = 1$. If $A_{34} \wedge A_{23} \wedge A_{24} = 0$, then we have a relation of the form

$$A_{34} = c_1' A_{23} + c_2' A_{24};$$

from the preceding discussion, we obtain a contradiction. On the other hand, if $A_{34} \wedge A_{23} \wedge A_{24} \neq 0$, then we can write

$$A_{33} = c_1'' A_{34} + c_2'' A_{23} + c_3'' A_{24},$$

and by (5.26) we obtain the relation

$$A_{33} = c_1'' A_{34} + A_{22} + (c_2'' - a)A_{23} + (c_3'' - b)A_{24};$$

once again the preceding discussion leads us to a contradiction. Thus we know that the dimension of the space V_{v_2} is ≥ 2 and we may assume without loss of generality that the orthonormal basis $\{v_1, \dots, v_4\}$ of V is chosen in such a way that $A_{23} = 0$. Then the relation (5.22) holds with $c_1 = c_2 = d_1 = d_2 = 0$ and $d_3 = 1$; by (5.23) and (5.21), we see that (5.24) holds. From (5.20), we obtain the equalities (5.25), which by condition (ii) and Lemma 5.1 lead us to a contradiction.

Lemma 5.15. *Suppose that $m = 3$ and $p = 4$. Let a_1, b_1, a_3, b_3 be given real numbers satisfying $a_1, b_1, a_3 \neq 0$ and $a_3 b_1 - a_1 b_3 \neq 0$. Assume that the element A of $S^2V^* \otimes W$ satisfies conditions* (i) *and* (ii) *and* $\operatorname{Tr} A = 0$. *Let η be a unit vector of V for which V_η is non-zero. Then either the vector v_1 possesses property* (A) *or the following assertion holds: there exist a vector ζ of V_η, an element h of H_η and vectors $\beta_1, \beta_2, \beta_3$ of $\bigwedge^2 V^*$ such that the element* (5.19) *of $\bigwedge^3 W$ does not vanish.*

Proof: Assume that the assertion of the lemma is not true. According to Lemma 5.5, we must have $\dim V_1 < p - 2$. Also according to Lemma 5.3, the hypotheses of Lemma 5.4 hold. Thus η has property $\mathcal{P}_0$. Lemma 5.14 now implies that $\dim V_1 \geq p-2$; we thus obtain a contradiction.

We now return to the situation considered in §4. Let x be a point of X. We consider the subspace

$$K''_x = \{\, \xi \in T_x \mid Q(\eta, \xi, \zeta) = 0, \text{ for all } \eta \in T_x,\, \zeta \in K_x \,\}$$

of T_x and denote by H_x the annihilator of K''_x. From the definition of K''_x, it follows that Q_ξ is an element of $S^2 H_x$, for all $\xi \in K_x$.

Let g be a Riemannian metric on X and let T''_x be the orthogonal complement of K''_x in T_x. We consider the trace mapping

$$\mathrm{Tr}^2_g : S^2T^* \otimes \textstyle\bigwedge^2 T^* \to T^* \otimes T^*$$

defined by

$$(\mathrm{Tr}^2_g\, u)(\eta_1, \eta_2) = \sum_{j=1}^{n} u(\eta_1, t_j, \eta_2, t_j),$$

for $u \in (S^2T^* \otimes \bigwedge^2 T^*)_x$, where $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. If η is a vector of T''_x, we set

$$W_\eta = \{\, \xi \in T''_x \mid g(\xi, \eta) = Q_\zeta(\xi, \eta) = 0, \text{ for all } \zeta \in K_x \,\}.$$

If μ is an element of H_x and λ is an element of $\bigwedge^2 H_x$, we define elements $E(\mu)$ of $(T^* \otimes L^*)_x$ and $F(\lambda)$ of $(T^* \otimes T^* \otimes L^*)_x$ by

$$\langle \pi(\xi), E(\mu) \rangle = \mathrm{Tr}^1_g(\mu \otimes Q_\xi), \qquad \langle \pi(\xi), F(\lambda) \rangle = \mathrm{Tr}^2_g(Q_\xi \otimes \lambda),$$

for $\xi \in K_x$. Since Q_ξ belongs to $S^2 H_x$, for $\xi \in K_x$, we easily verify that $E(\mu)$ belongs to $H_x \otimes L^*_x$ and that $F(\lambda)$ belongs to $H_x \otimes H_x \otimes L^*_x$.

Clearly, the element A of $S^2 T''^*_x \otimes L^*_x$, which is well-defined by

$$A(\eta_1, \eta_2, \pi(\xi)) = Q(\eta_1, \eta_2, \xi),$$

for $\eta_1, \eta_2 \in T''_x$, $\xi \in K_x$, satisfies conditions (i) and (ii). If $\mathrm{Tr}_g\, Q_\xi = 0$, for all $\xi \in K_x$, then we see that $\mathrm{Tr}\, A = 0$, with respect to the inner product on T''_x induced by g. For $\eta \in T''_x$, the subspace W_η of T''_x is equal to the subspace V_η corresponding to A. If $\rho : H_x \to T''^*_x$ is the isomorphism sending an element of H_x into its restriction to T''_x, then we see that

$$\rho E(\mu) = E(\rho\mu), \qquad \rho F(\lambda) = F(\rho\lambda). \tag{5.27}$$

6. Analyzing the operator P

We assume throughout the remainder of this paper that K is integrable and that ι has constant rank. We set $m = \operatorname{rank} L$ and $r = n - \operatorname{rank} K$.

For $k \geq 1$, let N_k be the subset of $J_k(S^2_+T^*)$ consisting of those elements $j_k(g)(x)$ of $J_k(S^2_+T^*)$, with $x \in X$ and where g is a Riemannian metric on X, satisfying

$$j_{k-2}(\operatorname{Ric}(g) - R)(x) = 0, \qquad j_{k-1}(B_g R)(x) = 0, \qquad j_k(\chi(g))(x) = 0.$$

If $k = 1$, the first relation does not impose any condition on $j_1(g)(x)$. We also set

$$N_0 = \{\, q \in S^2_+T^* \mid \chi(q) = 0 \,\}.$$

Let T' be a fixed orthogonal complement of K in T and consider the morphisms S and ψ corresponding to T'. According to (2.7), if g is a Riemannian metric on X and $x \in X$, the equality

$$j_{k-1}(B_g R)(x) = 0 \tag{6.1}$$

is equivalent to the relations

$$j_{k-1}(g^\flat S B_g R)(x) = 0, \qquad j_{k-1}(\chi(g))(x) = 0. \tag{6.2}$$

Thus N_k is equal to the subset of $J_k(S^2_+T^*)$ consisting of those k-jets q of Riemannian metrics at $x \in X$ satisfying

$$p_{k-2}(\varphi)(q) = j_{k-2}(R)(x), \qquad p_{k-1}(\psi)(q) = 0, \qquad p_k(\chi)(q) = 0;$$

if $k = 1$, the first relation does not impose any condition on q.

Proposition 6.1. *The mapping $\pi_0 : N_1 \to N_0$ is surjective.*

Proof: Since ψ and $p_1(\chi)$ are quasi-linear morphisms, the result follows from the preceding description of N_1, the surjectivity of the morphism $\sigma(\psi)_g \oplus \sigma_1(\chi)_g$ for any Riemannian metric g on X, given by Lemma 4.3, and the proof of Proposition 4.1.

Proposition 6.2. *Let $x \in X$ and let g be a Riemannian metric on X. Suppose that T'_x is the orthogonal complement of K_x in T_x and that $j_{k+1}(g)(x)$ is an element of N_{k+1}, with $k \geq 0$. Then $j_{k+1}(g)(x)$ belongs to $\pi_{k+1}N_{k+2}$ if and only if*

$$\nu_{g,k}(\varepsilon^{-1} j_k(\operatorname{Ric}(g) - R)(x) \oplus \varepsilon^{-1} j_{k+1}(g^\flat S B_g R)(x) \oplus \varepsilon^{-1} j_{k+2}(\chi(g))(x)) = 0.$$

PROOF: Since $p_k(\varphi)$, $p_{k+1}(\psi)$ and $p_{k+2}(\chi)$ are quasi-linear morphisms, by Lemma 4.2 the left-hand side of the equality occurring in the proposition depends only on $j_{k+1}(g)(x)$. We deduce the desired result from the exactness of the sequence (4.2) with $k = 0$, given by Lemma 4.4.

Since we are interested in the local solvability of the equation $\mathrm{Ric}(g) = R$, by Lemmas 2.1,(iii) and 2.2, without loss of generality we may suppose that $X = \mathbb{R}^n$ with a system of linear coordinates $(x^1, \dots, x^n)$, and that

$$\{\partial/\partial x^{r+1}, \dots, \partial/\partial x^n\}$$

is a frame for K over X and

$$\{\eta_1, \dots, \eta_{n-m-r}\}$$

is a frame for K' over X, where each η_j is a linear combination (with real coefficients) of $\partial/\partial x^{r+1}, \dots, \partial/\partial x^n$. In fact, if we wish, we could also suppose that $\eta_j = \partial/\partial x^{r+j}$.

We write $\partial_j = \partial/\partial x^j$. We now choose T' to be the integrable complement of K in T generated by the vector fields $\{\partial_1, \dots, \partial_r\}$. We use the standard tensor notation in terms of this system of coordinates $(x^1, \dots, x^n)$. In expressions written in terms of these coordinates, we shall use the summation convention, unless it is explicitly stated otherwise. The summation should be performed over all roman indices from 1 to n and over the index α from $r+1$ to n, which are repeated at least once, only after all the terms have been multiplied out. Even if an index is repeated more than once, the sum over that index should be taken only once.

Given a Riemannian metric g on X and $x \in X$, if we make an appropriate linear change of coordinates, we may suppose that

$$g_{ij}(x) = \delta_{ij} \tag{6.3}$$

in terms of the coordinates $(x^1, \dots, x^n)$; if T' is the complement of K defined in terms of this coordinate system, we have $g(T'_x, K_x) = 0$.

Let ∇ be the flat connection on T satisfying

$$\nabla \partial_j = 0,$$

for $1 \leq j \leq n$; notice that ∇ preserves the sub-bundles K and K' of T and so it induces a connection on L^*. If ξ is a section of K over X and g is a Riemannian metric on X, we set

$$P_\xi(g) = \langle \mathrm{Ric}(g) - R + \nabla g^\flat S B_g R, Q_\xi \rangle_g + \tfrac{1}{2}\Delta_g(\mathrm{Tr}_g\, Q_\xi),$$

where Δ_g is the Laplacian corresponding to the metric g. We define a differential operator P sending a Riemannian metric g on X into the section

$$P(g) = \nu_{g,0}\big((\mathrm{Ric}(g) - R) \oplus (\nabla g^{\flat} SB_g R) \oplus \nabla^2 \chi(g)\big)$$

of L^*. We remark that the operator P depends only on R, T' and the connection ∇. By (2.10), a Riemannian metric g on X solution of the equation $\mathrm{Ric}(g) = R$ satisfies

$$P(g) = 0.$$

If ξ is a section of K satisfying $\nabla\xi = 0$ (in particular if $\xi = \partial_j$, with $r+1 \leq j \leq n$) and g is a Riemannian metric on X, we have the equality

$$\langle \pi(\xi), P(g) \rangle = P_\xi(g).$$

Thus if $\{\xi_1, \ldots, \xi_m\}$ are linearly independent sections of K over X, where each ξ_j is a linear combination (with real coefficients) of $\partial_{r+1}, \ldots, \partial_n$, such that $\{\pi(\xi_1), \ldots, \pi(\xi_m)\}$ is a frame for L over X, then a Riemannian metric g on X satisfies $P(g) = 0$ if and only if $P_{\xi_j}(g) = 0$ for $1 \leq j \leq m$.

Let $(y^1, \ldots, y^n)$ be a coordinate system on X, where y^i is a linear combination (with real coefficients) of $(x^1, \ldots, x^r)$, for $1 \leq i \leq r$, and y^j is a linear combination (with real coefficients) of $(x^{r+1}, \ldots, x^n)$, for $r+1 \leq j \leq n$. If we replace the coordinate system $(x^1, \ldots, x^n)$ by $(y^1, \ldots, y^n)$, then the bundle T' and the connection ∇ do not change, nor does the operator P.

From the fact that the morphism φ is quasi-linear, we easily deduce that the morphism of fibered manifolds

$$\widetilde{\Phi} : J_2(S^2_+ T^*) \to L^*,$$

sending $j_2(g)(x)$ into $P(g)(x)$, where g is a Riemannian metric on X, is also quasi-linear and that its symbol

$$\sigma(\widetilde{\Phi}) : S^2 T^* \otimes S^2 T^* \to L^*$$

is the morphism over $J_1(S^2_+ T^*)$ determined by

$$\sigma(\widetilde{\Phi})_g = \nu_{g,0} \circ (\sigma(\varphi)_g \oplus \sigma_1(\psi)_g \oplus \sigma_2(\chi)_g)\,.$$

Thus by Lemma 4.2, $\sigma(\widetilde{\Phi})$ vanishes identically and we infer that $P(g)(x)$ depends only on the 1-jet $j_1(g)(x)$. Hence there is a morphism of fibered manifolds

$$\Phi : J_1(S^2_+T^*) \to L^*$$

such that $\widetilde{\Phi} = \Phi \circ \pi_1$.

Let g be a Riemannian metric on X. We consider the morphisms

$$\sigma_k(\Phi)_g : S^{k+1}T^* \otimes S^2T^* \to S^kT^* \otimes L^*.$$

For $k \geq 0$, the morphism

$$\lambda_{k,g} = \sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g \oplus \sigma_{k+2}(\chi)_g \oplus \sigma_{k+1}(\Phi)_g$$

at $x \in X$ depends only on $j_1(g)(x)$. We consider the sequence
(6.4)

$$\begin{aligned} &S^{k+2}T^* \otimes S^2T^* \xrightarrow{\lambda_{k,g}} \\ &\quad (S^kT^* \otimes S^2T^*) \oplus (S^{k+1}T^* \otimes g^\flat(T')) \oplus (S^{k+2}T^* \otimes L^*) \oplus (S^{k+1}T^* \otimes L^*) \\ &\qquad\qquad \xrightarrow{\nu_{g,k}} (S^{k-1}T^* \otimes T^*) \oplus (S^kT^* \otimes L^*) \to 0. \end{aligned}$$

If $x \in X$ and $p = j_1(g)(x)$, the kernel $g''_{k+2,p}$ of $\lambda_{k,g}$ at $x \in X$ is equal to the k-th prolongation of $g''_{2,p}$. For $k \geq 0$, we consider the kernel g'_{k+2} of $\sigma_k(\varphi)_g \oplus \sigma_{k+1}(\psi)_g \oplus \sigma_{k+2}(\chi)_g$ and the restriction

$$\sigma_{k+1}(\Phi)_g : g'_{k+2} \to S^{k+1}T^* \otimes L^*.$$

For $x \in X$, the subspace $g'_{k+2,x}$ of $(S^{k+2}T^* \otimes S^2T^*)_x$ depends only on $g(x)$; we set

$$V_x = \sigma_2(\Phi)_g(g'_{3,x}).$$

Proposition 6.3. *Let $x \in X$ and let g be a Riemannian metric on X. Assume that T'_x is the orthogonal complement of K_x in T_x and that $\mathrm{Tr}_g\, Q_\xi = 0$, for all $\xi \in K_x$. Let $\{t_1, \dots, t_n\}$ be an orthonormal basis of T_x such that $\{t_1, \dots, t_r\}$ is a basis of T'_x and $\{t_{r+1}, \dots, t_n\}$ is a basis of K_x. Suppose that the following conditions hold:*

(i) *the mapping*

$$\sigma_1(\Phi)_g : g'_{2,x} \cap (S^2T^*_{\{t_1,\dots,t_k\}} \otimes S^2T^*_x) \to T^*_{\{t_1,\dots,t_k\}} \otimes L^*_x \tag{6.5}$$

is surjective for $0 \leq k \leq n-1$;

(ii) $V_x = (S^2T^* \otimes L^*)_x$.

Then, if $p = j_1(g)(x)$, the subspace $g''_{2,p}$ of $(S^2T^* \otimes S^2T^*)_x$ is involutive and $\{t_1, \ldots, t_n\}$ is a quasi-regular basis for $g''_{2,p}$; moreover the sequences (6.4) *are exact at x for all $k \geq 0$.*

PROOF: To prove that $g''_{2,p}$ is involutive, we proceed to show that $\{t_1, \ldots, t_n\}$ is a quasi-regular basis for $g''_{2,p}$. For $0 \leq k \leq n$, we set

$$\begin{aligned} g''_{2,p,\{t_1,\ldots,t_k\}} &= g''_{2,p} \cap (S^2T^*_{\{t_1,\ldots,t_k\}} \otimes S^2T^*_x), \\ g'_{2,\{t_1,\ldots,t_k\}} &= g'_{2,x} \cap (S^2T^*_{\{t_1,\ldots,t_k\}} \otimes S^2T^*_x). \end{aligned}$$

According to condition (i), by dimension-counting we obtain

$$\dim g''_{2,p,\{t_1,\ldots,t_k\}} = \dim g'_{2,\{t_1,\ldots,t_k\}} - (n-k) \dim L_x,$$

for $0 \leq k \leq n-1$; similarly, from condition (ii) we deduce that

$$\dim g''_{3,p} = \dim g'_{3,x} - \dim S^2T^*_x \cdot \dim L_x.$$

By Lemma 4.4, $\{t_1, \ldots, t_n\}$ is a quasi-regular basis for $g'_{2,x}$; from the above equalities, it follows that $\{t_1, \ldots, t_n\}$ is also quasi-regular for $g''_{2,p}$. By condition (i) with $k = 0$ and condition (ii), from Lemma 4.4 we obtain the exactness of the sequences (6.4) at x for $k = 0, 1$. By a standard argument (see [4, §4] or Chapter X, §1 of [1]), from the exactness of the sequences (6.4) at x for $k = 0, 1$ and the involutivity of $g''_{2,p}$, we infer the exactness of the sequences (6.4) at x for all $k \geq 0$.

For $k \geq 1$, let N'_k be the subset of N_k consisting of those elements q of N_k satisfying $p_{k-1}(\Phi)(q) = 0$. Clearly, if

$$\Psi = \varphi \oplus p_1(\psi) \oplus p_2(\chi) \oplus p_1(\Phi)$$

and s' is the section $R \oplus 0 \oplus 0 \oplus 0$ of the vector bundle

$$S^2T^* \oplus J_1(T^*) \oplus J_2(L^*) \oplus J_1(L^*),$$

then we have

$$N'_{k+2} = \{q \in J_{k+2}(S^2_+T^*)_x \mid p_k(\Psi)q = j_k(s')(x),\, x \in X\}$$

for $k \geq 0$. If Q vanishes identically, then $N_k = N'_k$ for $k \geq 1$.

PROPOSITION 6.4. *Let $x \in X$ and let g be a Riemannian metric on X. Suppose that T'_x is the orthogonal complement of K_x in T_x and that $j_1(g)(x)$ is an element of N_1. Then $j_1(g)(x)$ belongs to $\pi_1 N_2$ if and only if one of the following three equivalent conditions holds:*

(i) $\nu_{g,0}((\mathrm{Ric}(g) - R)(x) \oplus \varepsilon^{-1} j_1(g^\flat S B_g R)(x) \oplus \varepsilon^{-1} j_2(\chi(g))(x)) = 0$;

(ii) $P(g)(x) = 0$;

(iii) $j_1(g)(x) \in N'_1$.

PROOF: The equivalence of conditions (i)–(iii) is an immediate consequence of the definitions of $\nu_{g,0}$, P and N'_1. The desired result follows from Proposition 6.2.

PROPOSITION 6.5. *Let $x \in X$. Let $k \geq 0$ and g be a Riemannian metric on X satisfying $j_{k+1}(g)(x) \in N'_{k+1}$. If the sequence (6.4) is exact at x, there exists $q' \in N'_{k+2,x}$ such that $\pi_{k+1} q' = j_{k+1}(g)(x)$.*

PROOF: We write $q = j_{k+1}(g)(x)$. Since $p_k(\varphi)$, $p_{k+1}(\psi)$, $p_{k+2}(\chi)$ and $p_{k+1}(\Phi)$ are quasi-linear morphisms, by the exactness of the sequence (6.4) there exists an element q' of $N'_{k+2,x}$ satisfying $\pi_{k+1} q' = q$ if and only if

$$\begin{aligned} \nu_{g,k}(\varepsilon^{-1} j_k(\mathrm{Ric}(g) - R)(x) \oplus \varepsilon^{-1} p_{k+1}(\psi) q'' \\ \oplus \varepsilon^{-1} p_{k+2}(\chi) q'' \oplus \varepsilon^{-1} p_{k+1}(\Phi) q'') = 0, \end{aligned}$$

where $q'' = j_{k+2}(g)(x)$. By the definitions of $\nu_{g,k}$ and P, the above relation is equivalent to the pair of equations

$$\sigma_{k-1}(B_g) \varepsilon^{-1} j_k(\mathrm{Ric}(g) - R)(x) = 0, \qquad \varepsilon^{-1} j_k(P(g))(x) = 0.$$

The second one holds automatically, since q belongs to N'_{k+1}. If $k = 0$, the first equality is always satisfied. If $k \geq 1$, by the Bianchi identity (1.3), we have

$$\begin{aligned} \sigma_{k-1}(B_g) \varepsilon^{-1} j_k(\mathrm{Ric}(g) - R)(x) &= j_{k-1}(B_g(\mathrm{Ric}(g) - R))(x) \\ &= -j_{k-1}(B_g R)(x), \end{aligned}$$

and so the first equality is equivalent to (6.1) and hence also to (6.2). Since q belongs to N'_{k+1}, the equalities (6.2) are satisfied.

PROPOSITION 6.6. *Suppose that Q vanishes identically. Let $x \in X$ and let g be a Riemannian metric on X. Assume that T'_x is the orthogonal* *complement of K_x in T_x. Let $k \geq 1$; if $j_k(g)(x) \in N_k$, then there exists $q' \in N_{k+1,x}$ such that $\pi_k q' = j_k(g)(x)$.*

PROOF: Since T'_x is the orthogonal complement of K_x in T_x, by Lemma 4.4 the sequences (4.2) or (6.4) are exact for $k \geq 0$, and the result follows from Propositions 6.4 and 6.5.

From Propositions 6.1 and 6.6 and Malgrange's strong prolongability theorem (see [6]) applied to the mapping $\varphi \oplus p_1(\psi)$ and the section $R \oplus 0$ of $S^2T^* \oplus J_1(T^*)$, we now obtain the following:

THEOREM 6.1. *Assume that X is a real-analytic manifold, that R is a real-analytic symmetric 2-form on X and that $R^\flat$ has constant rank. If the tensor Q associated to R vanishes identically, then for $x \in X$ there exists a real-analytic Riemannian metric g on a neighborhood of x such that*

$$\mathrm{Ric}(g) = R.$$

The above theorem includes the case of non-degenerate sections R of S^2T^* treated in [2] (see also [1, Chapter IX]).

Let $\varpi : T^* \to K^*$ be the natural restriction mapping. If g is a Riemannian metric on X, we shall consider the section

$$\omega(g) = \rho_g \nabla g - \tfrac{1}{4}(\varpi \otimes \mathrm{id})\sigma_1(\chi)_g(\nabla g)$$

of $K^* \otimes L^*$ over X.

We now proceed to give an expression for the operator P in terms of the coordinates $(x^1, \ldots, x^n)$. If $S(dx^i) = S^{ij}\partial_j$ for $1 \leq i \leq n$, we have $S^{ij} = 0$ if $r+1 \leq i \leq n$ or if $r+1 \leq j \leq n$, and

$$R_{ij}S^{jk} = \delta_i^k - \delta_i^\alpha \delta_\alpha^k \tag{6.6}$$

for $1 \leq i, k \leq n$. Let C be the section $\lambda \nabla R$ of $S^2T^* \otimes T^*$ and B be the section of $S^2T^* \otimes T$ equal to the composition $S \circ \lambda \nabla R$; we have

$$B(\partial_i, \partial_j) = B^l_{ij}\partial_l = C_{ijk}S^{kl}\partial_l.$$

Let g be a Riemannian metric on X. We set

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

and consider the Christoffel symbols $\Gamma^k_{ij} = g^{kl}\Gamma_{ijl}$ of g. We have

$$(B_gR)_j = g^{ik}(C_{ikj} - \Gamma^l_{ik}R_{lj});$$

therefore by (6.6), we obtain

$$(g^\flat SB_gR)_i = g_{ij}g^{kl}B^j_{kl} - g^{jk}\Gamma_{jki} + g_{i\alpha}g^{jk}\Gamma^\alpha_{jk}.$$

Let ξ be a section of K over X. We set

$$Q^\xi_{ij} = Q_\xi(\partial_i, \partial_j), \qquad Q^{pq}_\xi = g^{pi}g^{qj}Q^\xi_{ij}.$$

Let

$$\theta : T^* \otimes S^2T^* \to T^* \otimes S^2T^* \otimes L^*$$

be the morphism of vector bundles depending only on Q and g determined by

$$\begin{aligned}\langle \pi(\xi), \theta(u)\rangle_{ijl} = \tfrac{1}{2}g^{kp}(Q^\xi_{ik}(u_{lpj} + u_{jpl} - 3u_{pjl}) + Q^\xi_{jk}(u_{pil} - u_{lpi} - u_{ipl})\\ + Q^\xi_{lk}(u_{pij} - u_{jpi} - u_{ipj})),\end{aligned}$$

for $u \in T^* \otimes S^2T^*$. If we compute Q in terms of the connection ∇, we easily see that

$$Q^{\partial_\alpha}_{ij} = -\frac{1}{2}\frac{\partial R_{ij}}{\partial x^\alpha},$$

for $1 \le i, j \le n$, $r+1 \le \alpha \le n$. By (2.4), we have

$$Q^\xi_{i\alpha} = 0, \tag{6.7}$$

for $1 \le i \le n$, $r+1 \le \alpha \le n$, and so

$$g_{i\alpha}Q^{ij}_\xi = 0,$$

for $1 \le j \le n$, $r+1 \le \alpha \le n$. According to this relation, we see that

$$\begin{aligned}&\langle \nabla g^\flat SB_gR, Q_\xi\rangle_g\\ &\qquad = Q^{ik}_\xi \partial_k(g^\flat SB_gR)_i\\ &\qquad = Q^{ik}_\xi\left\{\frac{1}{2}g^{jl}\left(\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ij}}{\partial x^l\partial x^k} - \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}\right) - \frac{\partial g^{jl}}{\partial x^k}\Gamma_{jli}\right.\\ &\qquad\qquad \left. + \frac{\partial g_{i\alpha}}{\partial x^k}g^{jl}\Gamma^\alpha_{jl} + \left(\frac{\partial g_{ij}}{\partial x^k}g^{pq} + g_{ij}\frac{\partial g^{pq}}{\partial x^k}\right)B^j_{pq}\right\} + O(g);\end{aligned}$$

here and in subsequent formulas $O(g)$ stands for a term which does not depend on the derivatives of g. On the other hand, we have

$$\begin{aligned}\langle \mathrm{Ric}(g), Q_\xi\rangle_g &= Q_\xi^{ik}\mathrm{Ric}(g)_{ik} \\ &= \frac{1}{2}Q_\xi^{ik}g^{jl}\left(\frac{\partial^2 g_{ji}}{\partial x^l\partial x^k} + \frac{\partial^2 g_{li}}{\partial x^j\partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}\right) \\ &\quad + Q_\xi^{ik}(\Gamma^j_{ik}\Gamma^l_{jl} - \Gamma^j_{il}\Gamma^l_{jk}),\end{aligned}$$

and

$$\Delta_g(\mathrm{Tr}_g\, Q_\xi) = -g^{ij}\left(Q^\xi_{kl}\frac{\partial^2 g^{kl}}{\partial x^i\partial x^j} + 2\frac{\partial Q^\xi_{kl}}{\partial x^j}\frac{\partial g^{kl}}{\partial x^i}\right) + O(g).$$

Since

$$\frac{\partial^2 g^{kl}}{\partial x^i\partial x^j} = -\frac{\partial g^{kr}}{\partial x^i}\frac{\partial g_{rs}}{\partial x^j}g^{sl} - g^{kr}\frac{\partial g_{rs}}{\partial x^j}\frac{\partial g^{sl}}{\partial x^i} - g^{kr}\frac{\partial^2 g_{rs}}{\partial x^i\partial x^j}g^{sl},$$

we see that

$$\Delta_g(\mathrm{Tr}_g\, Q_\xi) = g^{ij}\left(Q_\xi^{kl}\frac{\partial^2 g_{kl}}{\partial x^i\partial x^j} + 2Q^\xi_{kl}g^{ls}\frac{\partial g^{kr}}{\partial x^i}\frac{\partial g_{rs}}{\partial x^j} - 2\frac{\partial Q^\xi_{kl}}{\partial x^j}\frac{\partial g^{kl}}{\partial x^i}\right) + O(g).$$

Hence, we obtain the following equality:

$$\begin{aligned}P_\xi(g) &= \langle \mathrm{Ric}(g) + \nabla g^\flat SB_gR, Q_\xi\rangle_g + \tfrac{1}{2}\Delta_g(\mathrm{Tr}_g\, Q_\xi) + O(g) \\ &= Q_\xi^{ik}\Bigg(\Gamma^j_{ik}\Gamma^l_{jl} - \Gamma^j_{il}\Gamma^l_{jk} + \left(\frac{\partial g_{ij}}{\partial x^k}g^{pq} + g_{ij}\frac{\partial g^{pq}}{\partial x^k}\right)B^j_{pq} \\ &\qquad\qquad - \frac{\partial g^{jl}}{\partial x^k}\Gamma_{jli} + \frac{\partial g_{i\alpha}}{\partial x^k}g^{jl}\Gamma^\alpha_{jl}\Bigg) \\ &\quad + g^{ij}\left(Q^\xi_{kl}g^{ls}\frac{\partial g^{kr}}{\partial x^i}\frac{\partial g_{rs}}{\partial x^j} - \frac{\partial Q^\xi_{kl}}{\partial x^j}\frac{\partial g^{kl}}{\partial x^i}\right) + O(g).\end{aligned} \tag{6.8}$$

We now fix a point x of X. We write $t_i = \partial_i(x)$ and denote by α_i the value of the section dx^i of T^* at x, and by $\tilde{\alpha}_i$ the restriction of α_i to K_x. Let $\{\xi_1, \dots, \xi_m\}$ be a system of linearly independent sections of K over X, where each ξ_j is a linear combination (with real coefficients) of $\partial_{r+1}, \dots, \partial_n$, such that $\{\pi(\xi_1), \dots, \pi(\xi_m)\}$ is a frame for L over X. Let

$\{\beta_1, \dots, \beta_m\}$ be the basis of L_x^* dual to the basis $\{\pi(\xi_1)(x), \dots, \pi(\xi_m)(x)\}$ of L_x. For $1 \leq s \leq m$, we write

$$Q_{ij}^s = Q_{ij}^{\xi_s}.$$

If $w \in (T^* \otimes S^2T^* \otimes L^*)_x$, we set

$$w_{ijk}^s = \langle \pi(\xi_s), w \rangle_{ijk},$$

where $1 \leq i, j, k \leq n$.

We now assume that $g_{ij}(x) = \delta_{ij}$; then we have

$$\frac{\partial g^{ij}}{\partial x^k}(x) = -\frac{\partial g_{ij}}{\partial x^k}(x).$$

Let $1 \leq s \leq m$; for $u \in (T^* \otimes S^2T^*)_x$, we obtain

$$\begin{aligned} \theta(u)_{ijk}^s = \tfrac{1}{2}(&Q_{il}^s(u_{kjl} + u_{jkl} - 3u_{ljk}) + Q_{jl}^s(u_{lik} - u_{kil} - u_{ikl}) \\ &+ Q_{kl}^s(u_{lij} - u_{jil} - u_{ijl})). \end{aligned} \tag{6.9}$$

From (6.8), we obtain

$$\begin{aligned} P_\xi(g) = Q_{ik}^\xi \bigg\{ &\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\frac{\partial g_{ik}}{\partial x^j}\right)\frac{\partial g_{ll}}{\partial x^j} + \frac{\partial g_{i\alpha}}{\partial x^k}\left(\frac{\partial g_{j\alpha}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{jj}}{\partial x^\alpha}\right) \\ &- \frac{1}{4}\left(\frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right)\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \\ &+ \frac{1}{2}\frac{\partial g_{jl}}{\partial x^k}\left(\frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^i}\right) - \frac{\partial g_{il}}{\partial x^j}\frac{\partial g_{kl}}{\partial x^j} \\ &+ B_{ll}^j \frac{\partial g_{ij}}{\partial x^k} - B_{jl}^i \frac{\partial g_{jl}}{\partial x^k}\bigg\} + \frac{\partial Q_{jk}^\xi}{\partial x^i}\frac{\partial g_{jk}}{\partial x^i} + O(g) \end{aligned} \tag{6.10}$$

at x. Therefore, using (6.7), we see that

$$P_\xi(g) = Q_{ik}^\xi \left(\frac{\partial g_{i\alpha}}{\partial x^k} - \frac{1}{4}\frac{\partial g_{ik}}{\partial x^\alpha}\right)\frac{\partial g_{\alpha\alpha}}{\partial x^\alpha} + \phi_\xi(g) \tag{6.11}$$

at x, where $\phi_\xi(g)$ is an expression which does not involve the derivatives $\partial g_{\alpha\alpha}/\partial x^\alpha$, with $r + 1 \leq \alpha \leq n$.

We set $c = (\nabla g)(x)$ and thus we have

$$c_{ijk} = \frac{\partial g_{jk}}{\partial x^i}(x).$$

If

$$\omega(g)(x) = \sum_{\substack{r+1\leq j\leq n\\ 1\leq s\leq m}} \omega_j^s \tilde{\alpha}_j \otimes \beta_s,$$

we have

$$\omega_j^s = \sum_{i,k=1}^{r} Q_{ik}^s(x)\big(c_{ikj} - \tfrac{1}{4}c_{jik}\big), \tag{6.12}$$

for $r+1 \leq j \leq n$, $1 \leq s \leq m$. Thus by (6.11), we obtain

$$P(g)(x) = \sum_{\substack{r+1\leq j\leq n\\ 1\leq s\leq m}} \omega_j^s \frac{\partial g_{jj}}{\partial x^j}(x)\beta_s + \phi(g), \tag{6.13}$$

where $\phi(g)$ is an element of L_x^* which does not involve the derivatives

$$\frac{\partial g_{jj}}{\partial x^j}(x),$$

with $r+1 \leq j \leq n$.

If $p_1(\chi)(j_1(g)(x)) = 0$, by (6.3) we see that

$$\frac{\partial Q_{ll}^\xi}{\partial x^j}(x) = \left(Q_{ik}^\xi \frac{\partial g_{ik}}{\partial x^j}\right)(x), \tag{6.14}$$

for all sections ξ of K over X and $1 \leq j \leq n$.

Proposition 6.7. *Let g_0 be a Riemannian metric on X satisfying $g_{0,ij}(x) = \delta_{ij}$. Suppose that $\mathrm{Tr}_{g_0} Q_\xi = 0$, for all $\xi \in K_x$. Let z be an element of $(K^* \otimes L^*)_x$.*

(i) *There exists a Riemannian metric g on X such that $g(x) = g_0(x)$ and*

$$\psi(j_1(g)(x)) = 0, \qquad p_1(\chi)(j_1(g)(x)) = 0, \qquad \omega(g)(x) = z. \tag{6.15}$$

(ii) *Let g be a Riemannian metric on X satisfying the conditions of* (i) *and let u be an element of $(T^* \otimes S^2T^*)_x$ satisfying*

$$\sigma(\psi)_g u = 0, \qquad \sigma_1(\chi)_g u = 0, \qquad \rho_g u = 0. \tag{6.16}$$

Then a Riemannian metric $\tilde{g}$ on X for which $j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon u$ also satisfies the conditions of (i).

(iii) *Assume that z is of maximal rank. Let g be a Riemannian metric on X satisfying the conditions of* (i). *Then there exists an element*

$$u = \sum_{j=r+1}^{n} a_j \alpha_j \otimes \alpha_j^2 \tag{6.17}$$

of $(T^* \otimes S^2T^*)_x$, with $a_{r+1}, \ldots, a_n \in \mathbb{R}$, satisfying (6.16) *and*

$$j_1(g)(x) + \varepsilon u \in N_1'.$$

Proof: (i) Consider the element

$$v = z + \frac{1}{4} \sum_{\substack{r+1 \le j \le n \\ 1 \le s \le m}} \sum_{l=1}^{n} \frac{\partial Q_{ll}^s}{\partial x^j}(x) \tilde{\alpha}_j \otimes \beta_s$$

of $(K^* \otimes L^*)_x$. By Proposition 4.1, there exists a Riemannian metric g on X such that $g(x) = g_0(x)$ and (4.8) holds. By (6.14), we see that

$$(\rho_g \nabla g)(x) = z + \tfrac{1}{4}(\varpi \otimes \mathrm{id})\sigma_1(\chi)_g(\nabla g)(x),$$

and so $\omega(g)(x) = z$.

(ii) Since ψ and $p_1(\chi)$ are quasi-linear morphisms, we have

$$\psi(j_1(\tilde{g})(x)) = \psi(j_1(g)(x)) + \sigma(\psi)_g u = 0,$$
$$p_1(\chi)(j_1(\tilde{g})(x)) = p_1(\chi)(j_1(g)(x)) + \sigma_1(\chi)_g u = 0.$$

On the other hand, we see that

$$\omega(\tilde{g})(x) = \omega(g)(x) + \rho_g u - \tfrac{1}{4}(\varpi \otimes \mathrm{id})\sigma_1(\chi)_g u = \omega(g)(x).$$

(iii) According to (2.14), (4.1) and (6.7), the element u of $(T^*\otimes S^2T^*)_x$ given by (6.17) satisfies the relations (6.16). From (6.13), we obtain

$$\Phi(j_1(g)(x)+\varepsilon u) = \Phi(j_1(g)(x)) + 2\sum_{\substack{r+1\leq j\leq n\\ 1\leq s\leq m}} \omega^s_j a_j \beta_s,$$

where ω^s_j are the coefficients of $\omega(g)(x)$. Since $\omega(g)(x)$ has maximal rank, we may choose $a_{r+1},\dots,a_n\in\mathbb{R}$ in such a way that

$$\sum_{\substack{r+1\leq j\leq n\\ 1\leq s\leq m}} \omega^s_j a_j \beta_s = -\tfrac{1}{2}\,\Phi(j_1(g)(x)).$$

Therefore by (ii), we see that $j_1(g)(x)+\varepsilon u$ belongs to N_1'.

For $k\geq 0$ and $u\in(S^{k+1}T^*\otimes S^2T^*)_x$, we write

$$\sigma_k(\Phi)_g u = \sum_{s=1}^m (\sigma_k(\Phi)_g u)_s \otimes \beta_s,$$

where $(\sigma_k(\Phi)_g u)_s$ is an element of $S^kT^*_x$; for $u\in(T^*\otimes S^2T^*)_x$, we set $(\sigma(\Phi)_g u)_s = (\sigma_0(\Phi)_g u)_s$. By (6.10), for $u\in(T^*\otimes S^2T^*)_x$ we see that

$$\begin{aligned}(\sigma(\Phi)_g u)_s = Q^s_{ik}(x)\big\{&\tfrac{1}{2}c_{jll}\big(u_{ikj}-\tfrac{1}{2}u_{jik}\big)+\tfrac{1}{2}\big(c_{ikj}-\tfrac{1}{2}c_{jik}\big)u_{jll}\\ &+\big(c_{jj\alpha}-\tfrac{1}{2}c_{\alpha jj}\big)u_{ik\alpha}+c_{ik\alpha}\big(u_{jj\alpha}-\tfrac{1}{2}u_{\alpha jj}\big)\\ &-\tfrac{1}{4}(c_{kjl}+c_{jkl}-c_{ljk})(u_{lij}+u_{ijl}-u_{jil})\\ &-\tfrac{1}{4}(c_{lij}+c_{ijl}-c_{jil})(u_{kjl}+u_{jkl}-u_{ljk})\\ &+\tfrac{1}{2}(c_{lij}+c_{jil}-c_{ijl})u_{kjl}+\tfrac{1}{2}c_{kjl}(u_{lij}+u_{jil}-u_{ijl})\\ &-2c_{jil}u_{jkl}+B^j_{ll}(x)u_{ikj}-B^i_{jl}(x)u_{kjl}\big\}+\frac{\partial Q^s_{jk}}{\partial x^i}(x)u_{ijk}.\end{aligned}$$

For $1\leq i,j,k\leq n$ and $1\leq s\leq m$, we set

$$b^s_{ijk} = (\sigma(\Phi)_g(\alpha_i\otimes\alpha_j\cdot\alpha_k))_s$$

and

$$a^s_{ijk} = \left(\langle\pi(\xi_s),\theta(\nabla g)\rangle_{ijk} - Q^s_{il}B^l_{jk} + \frac{\partial Q^s_{jk}}{\partial x^i}\right)(x). \tag{6.18}$$

For $u \in (T^* \otimes S^2T^*)_x$, we obtain

$$
\begin{aligned}
(\sigma(\Phi)_g u)_s = Q^s_{ik}(x)\bigl\{ & \tfrac{1}{2}c_{jll}\bigl(u_{ikj} - \tfrac{1}{2}u_{jik}\bigr) + \tfrac{1}{2}\bigl(c_{ikj} - \tfrac{1}{2}c_{jik}\bigr)u_{jll} \\
& + \bigl(c_{jj\alpha} - \tfrac{1}{2}c_{\alpha jj}\bigr)u_{ik\alpha} + c_{ik\alpha}\bigl(u_{jj\alpha} - \tfrac{1}{2}u_{\alpha jj}\bigr) \\
& + B^j_{ll}(x)u_{ikj}\bigr\} + a^s_{ijk}u_{ijk}.
\end{aligned} \tag{6.19}
$$

In particular, if an element $u \in (T^* \otimes S^2T^*)_x$ satisfies $u_{ijj} = u_{jji} = 0$ for all $1 \leq i \leq n$, we have

$$
\begin{aligned}
(\sigma(\Phi)_g u)_s = Q^s_{ik}(x)\bigl\{ & \tfrac{1}{2}c_{jll}\bigl(u_{ikj} - \tfrac{1}{2}u_{jik}\bigr) + \bigl(c_{jj\alpha} - \tfrac{1}{2}c_{\alpha jj}\bigr)u_{ik\alpha} \\
& + B^j_{ll}(x)u_{ikj}\bigr\} + a^s_{ijk}u_{ijk}.
\end{aligned} \tag{6.20}
$$

Thus if $1 \leq i, j, k \leq n$ are distinct integers, we have

$$
\begin{aligned}
b^s_{ijk} = {} & Q^s_{ij}(x)\bigl\{\tfrac{1}{2}c_{kll} + \bigl(c_{ll\alpha} - \tfrac{1}{2}c_{\alpha ll}\bigr)\delta^k_\alpha + B^k_{ll}(x)\bigr\} \\
& + Q^s_{ik}(x)\bigl\{\tfrac{1}{2}c_{jll} + \bigl(c_{ll\alpha} - \tfrac{1}{2}c_{\alpha ll}\bigr)\delta^j_\alpha + B^j_{ll}(x)\bigr\} + 2a^s_{ijk}.
\end{aligned} \tag{6.21}
$$

If $1 \leq i, j, k \leq n$ are distinct integers, with $r+1 \leq k \leq n$, by (6.21), (6.18) and (6.9) we obtain

$$
\begin{aligned}
b^s_{ijk} = {} & Q^s_{ij}(x)\bigl\{\tfrac{1}{2}c_{kll} + \bigl(c_{ll\alpha} - \tfrac{1}{2}c_{\alpha ll}\bigr)\delta^k_\alpha + B^k_{ll}(x)\bigr\} \\
& + Q^s_{il}(x)(c_{jkl} + c_{kjl} - 3c_{ljk} - 2B^l_{jk}(x)) \\
& + Q^s_{jl}(x)(c_{lik} - c_{ikl} - c_{kil}).
\end{aligned} \tag{6.22}
$$

We consider the subspace

$$
K''_x = \{\, \xi \in T_x \mid Q(\eta, \xi, \zeta) = 0, \text{ for all } \eta \in T_x,\, \zeta \in K_x \,\}
$$

of T_x. We know that K_x is a subspace of K''_x, and we set $p = n - \dim K''_x$; then we have $\dim(T'_x \cap K''_x) = r - p$. We denote by H_x the annihilator of K''_x. We continue to assume that (6.3) holds and further suppose that $\{t_{p+1}, \ldots, t_r\}$ is a basis for $T'_x \cap K''_x$. Then $\{\alpha_1, \ldots, \alpha_p\}$ is a basis of H_x. Moreover we have

$$
Q^\xi_{ij}(x) = 0,
$$

for $1 \leq i \leq n$, $p+1 \leq j \leq n$ and all $\xi \in K_x$, and

$$
\sigma(\chi)_g(\alpha_j \cdot T^*_x) = 0, \tag{6.23}
$$

for $p+1 \leq j \leq n$. From (6.22), if $p+1 \leq i,j,k \leq n$ are distinct integers, with $r+1 \leq k \leq n$, we deduce that

$$b^s_{ijk} = 0, \tag{6.24}$$

for all $1 \leq s \leq m$.

If u is an element of $T^*_{\{t_1,\dots,t_p\}} \otimes S^2T^*_{\{t_1,\dots,t_p\}}$, by (6.19) and (6.18) we see that

$$\begin{aligned}(\sigma(\Phi)_g u)_s = Q^s_{ik}(x)\big\{\tfrac{1}{2}\big(c_{ikj} - \tfrac{1}{2}c_{jik}\big)u_{jll} + c_{ik\alpha}\big(u_{jj\alpha} - \tfrac{1}{2}u_{\alpha jj}\big)\big\} \\ + \frac{\partial Q^s_{jk}}{\partial x^i}(x)u_{ijk}.\end{aligned} \tag{6.25}$$

Thus the restriction of $\sigma(\Phi)_g$ to $T^*_{\{t_1,\dots,t_p\}} \otimes S^2T^*_{\{t_1,\dots,t_p\}}$ depends only on $g(x)$, Q and the expressions

$$\sum_{i,k=1}^p Q^s_{ik}(x)c_{ikj}, \qquad \sum_{i,k=1}^p Q^s_{ik}(x)c_{jik},$$

with $p+1 \leq j \leq n$ and $1 \leq s \leq m$. If $T'^{\perp}$ denotes the annihilator of T' and if u belongs to $(T'^{\perp} \otimes S^2T'^{\perp})_x$, from (6.25) we obtain

$$(\sigma(\Phi)_g u)_s = \sum_{j,l=r+1}^n \sum_{i,k=1}^p Q^s_{ik}(x)\big(c_{ikj}u_{llj} - \tfrac{1}{4}c_{jik}u_{jll}\big). \tag{6.26}$$

By (6.26) and (6.12), we have

$$\sigma(\Phi)_g(\alpha_j \otimes \alpha_j^2) = 2\sum_{s=1}^m \omega^s_j \beta_s, \tag{6.27}$$

for $r+1 \leq j \leq n$.

If w is an element of $T^*_x \otimes S^2H_x$ satisfying

$$(\mathrm{id} \otimes \mathrm{Tr}_g)w = 0, \qquad \sigma_1(\chi)_g w = 0, \tag{6.28}$$

by (6.19) we have

$$(\sigma(\Phi)_g w)_s = Q^s_{ik}(x)\big(\tfrac{1}{2}c_{jll} + B^j_{ll}(x)\big)w_{ikj} + a^s_{ijk}w_{ijk}. \tag{6.29}$$

Throughout the remainder of this paper, we no longer use the summation convention which we had established at the beginning of this section.

Lemma 6.1. *Assume that* (6.3) *holds. If* $\omega(g)(x)$ *is an element of maximal rank of* $(K^* \otimes L^*)_x$ *and if the mapping* (6.5) *is surjective when* $k = r$, *then the mappings* (6.5) *are surjective for all* $0 \leq k \leq r$.

Proof: If $1 \leq i, j \leq n$, the elements $\alpha_i \cdot \alpha_j \otimes \alpha_j^2$ and $\alpha_j^2 \otimes \alpha_i \cdot \alpha_j$ of $(S^2T^* \otimes S^2T^*)_x$ belong to $\operatorname{Ker} \sigma(\varphi)_g$. By (2.13) and (4.1), we see that

$$v_{ij} = \alpha_i \cdot \alpha_j \otimes \alpha_j^2 + \frac{1}{4}\alpha_j^2 \otimes \alpha_i \cdot \alpha_j$$

belongs to $g'_{2,x}$ whenever $r + 1 \leq j \leq n$. By (6.27), we obtain the desired result from the formula

$$\sigma_1(\Phi)_g v_{ij} = \alpha_i \otimes \sigma(\Phi)_g(\alpha_j \otimes \alpha_j^2) + \alpha_j \otimes \sigma(\Phi)_g\big(\alpha_i \otimes \alpha_j^2 + \tfrac{1}{2}\alpha_j \otimes \alpha_i \cdot \alpha_j\big).$$

For $1 \leq i, j \leq n$, we write

$$A_{ij}^s = b_{iij}^s.$$

If $1 \leq j \leq r$ and $1 \leq i \leq n$ are distinct integers, by (6.19) and (6.18) we see that

$$A_{ij}^s = \sum_{k=1}^{n} \big\{ Q_{ii}^s(x)\big(\tfrac{1}{2}c_{jkk} + B_{kk}^j(x)\big) + Q_{ij}^s B_{kk}^i(x) \big\} + 2a_{iij}^s \tag{6.30}$$

when $1 \leq i \leq r$, and that

$$A_{ij}^s = 2\theta(c)_{iij}^s \tag{6.31}$$

when $r + 1 \leq i \leq n$.

Let u be an element of $(T^* \otimes S^2T^*)_x$ and let $\tilde{g}$ be a Riemannian metric on X such that

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon u; \tag{6.32}$$

we set

$$\tilde{b}_{ijk}^s = (\sigma(\Phi)_{\tilde{g}}(\alpha_i \otimes \alpha_j \cdot \alpha_k))_s, \qquad \tilde{A}_{ij}^s = \tilde{b}_{iij}^s,$$

for $1 \leq i, j, k \leq n$.

If u can be written in the form

$$u = \sum_{k=r+1}^{n} a_k \alpha_k \otimes \alpha_k^2, \tag{6.33}$$

with $a_{r+1}, \dots, a_n \in \mathbb{R}$, then by (6.9) we have $\theta(u) = 0$. Hence by (6.21), if $1 \le i, j, k \le n$ are distinct integers, the equality

$$\tilde{b}^s_{ijk} = b^s_{ijk} \tag{6.34}$$

holds whenever $p + 1 \le i \le n$, or whenever $p + 1 \le j, k \le n$, or whenever $1 \le j, k \le p$; moreover, if $1 \le i, j \le p$ and $p + 1 \le k \le n$ are distinct integers and if $\sigma(\chi)_g(\alpha_i \cdot \alpha_j) = 0$, then we see that (6.34) holds in this case. If w is an element of $T^* \otimes S^2 H_x$ satisfying (6.28), by (6.18) and (6.29) we obtain the relation

$$\sigma(\Phi)_{\tilde{g}} w = \sigma(\Phi)_g w. \tag{6.35}$$

Also according to (6.30) and (6.31), we have

$$\tilde{A}^s_{ij} = A^s_{ij}, \tag{6.36}$$

whenever $1 \le i, j \le n$ are distinct integers, with $1 \le j \le r$. By (6.25), we know that the restriction of $\sigma(\Phi)_{\tilde{g}}$ to $T^*_{\{t_1,\dots,t_p\}} \otimes S^2 T^*_{\{t_1,\dots,t_p\}}$ is equal to the restriction of $\sigma(\Phi)_g$. By (2.13) and (4.1), we see that u satisfies (6.16). Hence by Proposition 6.7,(ii), we have $\omega(\tilde{g})(x) = \omega(g)(x)$.

On the other hand, if u satisfies $u_{ijj} = u_{jji} = 0$ for all $1 \le i, j \le n$, then, if $1 \le i, j, k \le n$ are distinct integers, by (6.21) we see that

$$\tilde{b}^s_{ijk} = b^s_{ijk} + 2\theta(u)^s_{ijk}. \tag{6.37}$$

If w is an element of $T^* \otimes S^2 H_x$ satisfying (6.28), then by (6.18) and (6.29) we have

$$(\sigma(\Phi)_{\tilde{g}} w)_s = (\sigma(\Phi)_g w)_s + \sum_{i,j,k=1}^{n} \theta(u)^s_{ijk} w_{ijk}. \tag{6.38}$$

Moreover by (6.30) and (6.31), we have

$$\tilde{A}^s_{ij} = A^s_{ij} + 2\theta(u)^s_{iij}, \tag{6.39}$$

whenever $1 \le i, j \le n$ are distinct integers, with $1 \le j \le r$; thus if $p + 1 \le i \le n$ and $1 \le j \le r$, with $i \ne j$, by (6.9) we see that (6.36) holds. If u also satisfies the relations $u_{i_1 i_2 i_3} = 0$ for all $1 \le i_1, i_2, i_3 \le p$, then by (6.9) we see that the equality (6.36) holds whenever $1 \le i, j \le n$ are distinct integers, with $1 \le j \le p$.

We have a mapping

$$\gamma : \bigwedge^2 H_x \otimes T^*_x \to (T^* \otimes S^2T^*)_x,$$

sending $v_1 \wedge v_2 \otimes \alpha$, with $v_1, v_2 \in H_x$, $\alpha \in T^*_x$, into $v_1 \otimes v_2 \cdot \alpha - v_2 \otimes v_1 \cdot \alpha$. We now suppose that the element u of $(T^* \otimes S^2T^*)_x$ can be written in the form

$$u = \sum_{k=p+1}^{n} \gamma(u_k \otimes \alpha_k) + \sum_{p+1 \leq j < k \leq n} u_{jk} \otimes \alpha_j \cdot \alpha_k, \tag{6.40}$$

where $u_k \in \bigwedge^2 H_x$ and $u_{jk} \in H_x$. Clearly we have

$$\sum_{i,j=1}^{n} Q^s_{ij}(x) u_{ijk} = 0,$$

for all $1 \leq k \leq n$ and $1 \leq s \leq m$; thus by (2.13) and (4.1), we see that u satisfies (6.16). Hence by Proposition 6.7,(ii), we have $\omega(\tilde{g})(x) = \omega(g)(x)$. By (6.25), we know that the restriction of $\sigma(\Phi)_{\tilde{g}}$ to $T^*_{\{t_1,\dots,t_p\}} \otimes S^2T^*_{\{t_1,\dots,t_p\}}$ is equal to the restriction of $\sigma(\Phi)_g$. Such an element u satisfies the conditions $u_{ijj} = u_{jji} = 0$ for all $1 \leq i, j \leq n$, and $u_{i_1i_2i_3} = 0$ for all $1 \leq i_1, i_2, i_3 \leq p$. Hence (6.36) holds whenever $1 \leq i, j \leq n$ are distinct integers, with $1 \leq j \leq p$.

From the above discussion, Proposition 6.7,(ii) and Lemma 6.1, we obtain the following result:

Lemma 6.2. *Assume that* (6.3) *holds, that* $\{t_{p+1}, \dots, t_r\}$ *is a basis for* $T'_x \cap K''_x$. *Suppose that* g *satisfies the following two conditions:*

(i) *we have*

$$\psi(j_1(g)(x)) = 0, \qquad p_1(\chi)(j_1(g)(x)) = 0,$$

and $\omega(g)(x)$ *is an element of maximal rank of* $(K^* \otimes L^*)_x$*;*

(ii) *the mapping*

$$\sigma_2(\Phi)_g : g'_{3,x} \cap (S^3T^*_{\{t_1,\dots,t_p\}} \otimes S^2T^*_{\{t_1,\dots,t_p\}}) \to S^2T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x \tag{6.41}$$

and the mappings (6.5) *are surjective for* $0 \leq k \leq n-1$.
Let u *be an element of* $(T^* \otimes S^2T^*)_x$ *and* $\tilde{g}$ *be a Riemannian metric on* X *satisfying* (6.32). *If* u *is either of the form* (6.33) *or of the form* (6.40), *then the conditions* (i) *and* (ii) *hold for* $\tilde{g}$*; in particular, we have* $\omega(\tilde{g})(x) = \omega(g)(x)$. *Moreover, the relation* (6.36) *holds whenever* $1 \leq i, j \leq n$ *are distinct integers, with* $1 \leq j \leq p$.

In the following lemma, λ_j is a non-negative real number for $r+1 \leq j \leq n$.

Lemma 6.3. *Assume that* (6.3) *holds, that* $\{t_{p+1}, \ldots, t_r\}$ *is a basis for* $T'_x \cap K''_x$ *and that*

$$\begin{aligned} &\sum_{i,k=1}^{n} Q^s_{ik}(x) c_{ikj} = (1+\lambda_j)\delta_j^{n-m+s}, \\ &\frac{1}{4}\sum_{i,k=1}^{n} Q^s_{ik}(x) c_{jik} = \lambda_j \delta_j^{n-m+s}, \end{aligned} \tag{6.42}$$

for $r+1 \leq j \leq n$, $1 \leq s \leq m$. *Then the mappings* (6.41) *and* (6.5) *are surjective for* $0 \leq k \leq n-1$.

Proof: For $n-m+1 \leq j \leq n$, by (6.27) and our hypothesis, we see that

$$\sigma(\Phi)_g(\alpha_j \otimes \alpha_j^2) = 2\beta_{j-n+m}$$

and by (6.25) we also have

$$\sigma(\Phi)_g(\alpha_i \otimes \alpha_i \cdot \alpha_j) = (1+\lambda_j)\beta_{j-n+m},$$

if $p+1 \leq i \leq n$, with $i \neq j$. Moreover from this last relation and (6.24), we infer that

$$\sigma(\Phi)_g((\alpha_i + \alpha_k) \otimes (\alpha_i + \alpha_k)\cdot \alpha_j) = 2(1+\lambda_j)\beta_{j-n+m},$$

if $p+1 \leq i < k \leq n$, with $i, k \neq j$. Hence we have

$$\sigma_1(\Phi)_g(\alpha_j^2 \otimes \alpha_j^2) = 4\alpha_j \otimes \beta_{j-n+m}, \quad \sigma_2(\Phi)_g(\alpha_j^3 \otimes \alpha_j^2) = 6\alpha_j^2 \otimes \beta_{j-n+m}$$

for $n-m+1 \leq j \leq n$, and

$$\begin{aligned} \sigma_1(\Phi)_g(\alpha_i^2 \otimes \alpha_i \cdot \alpha_j) &= 2(1+\lambda_j)\alpha_i \otimes \beta_{j-n+m}, \\ \sigma_2(\Phi)_g(\alpha_i^3 \otimes \alpha_i \cdot \alpha_j) &= 3(1+\lambda_j)\alpha_i^2 \otimes \beta_{j-n+m}, \end{aligned}$$

for $p+1 \leq i \leq n$ and $n-m+1 \leq j \leq n$, with $i \neq j$. If i, j, k are distinct integers, with $n-m+1 \leq j \leq n$ and $p+1 \leq i, k \leq n$, then we see that

$$\sigma_2(\Phi)_g((\alpha_i + \alpha_k)^3 \otimes (\alpha_i + \alpha_k)\cdot \alpha_j) = 6(1+\lambda_j)(\alpha_i + \alpha_k)^2 \otimes \beta_{j-n+m}.$$

On the other hand, for $1 \leq i \leq n$ and $n-m+1 \leq j \leq n$ we obtain

$$\begin{aligned} &\sigma_2(\Phi)_g\big(\alpha_i \cdot \alpha_j^2 \otimes \alpha_j^2 + \tfrac{1}{6}\alpha_j^3 \otimes \alpha_i \cdot \alpha_j\big) \\ &\qquad = 4\alpha_i \cdot \alpha_j \otimes \beta_{j-n+m} + \alpha_j^2 \otimes \sigma(\Phi)_g\big(\alpha_i \otimes \alpha_j^2 + \tfrac{1}{2}\alpha_i \otimes \alpha_i \cdot \alpha_j\big). \end{aligned}$$

If $\alpha, \beta \in T^*_x$, the element $\alpha^2 \otimes \alpha \cdot \beta$ of $(S^2T^* \otimes S^2T^*)_x$ belongs to $\operatorname{Ker} \sigma(\varphi)_g$; therefore the elements $\alpha \cdot \beta^2 \otimes \beta^2$ and $\alpha^3 \otimes \alpha \cdot \beta$ of $(S^3T^* \otimes S^2T^*)_x$ are contained in $\operatorname{Ker} \sigma_1(\varphi)_g$. By (2.13) and (4.1), if $r+1 \leq j \leq n$ and if α is orthogonal to α_j we see that $\alpha^2 \otimes \alpha \cdot \alpha_j$ and $\alpha_j^2 \otimes \alpha_j^2$ are elements of $g'_{2,x}$, while

$$\alpha^3 \otimes \alpha \cdot \alpha_j, \qquad \alpha_j^3 \otimes \alpha_j^2, \qquad \alpha \cdot \alpha_j^2 \otimes \alpha_j^2 + \frac{1}{6}\alpha_j^3 \otimes \alpha \cdot \alpha_j$$

belong to $g'_{3,x}$. From the above equalities, we infer that the mappings (6.5) are surjective for $p \leq k \leq n-1$ and that (6.41) is surjective. By (6.42), we see that $\omega_j^s = \delta_j^{n-m+s}$, for $r+1 \leq j \leq n$, $1 \leq s \leq m$; thus $\omega(g)(x)$ is an element of maximal rank of $(K^* \otimes L^*)_x$. Therefore by Lemma 6.1, we obtain the surjectivity of all the mappings (6.5).

In the following lemma, we do not make any assumptions on the coordinate system $(x^1, \dots, x^n)$ relative to the metric g or to K''_x.

LEMMA 6.4. *Let $x \in X$ and let g_0 be a Riemannian metric on X satisfying $g_{0,ij}(x) = \delta_{ij}$. Suppose that $\operatorname{Tr}_{g_0} Q_\xi = 0$, for all $\xi \in K_x$, and that $\{t_{p+1}, \dots, t_r\}$ is a basis of $T'_x \cap K''_x$. Then there exist a system of linearly independent coordinate functions $(y^{r+1}, \dots, y^n)$ on X, which are linear combinations* (*with real coefficients*) *of the coordinate functions $(x^{r+1}, \dots, x^n)$, and a Riemannian metric g on X such that, with respect to the coordinate system $(y^1, \dots, y^n)$ on X, where $y^i = x^i$ for $1 \leq i \leq r$, conditions* (i) *and* (ii) *of Lemma 6.2 hold and we have $g_{ij}(x) = \delta_{ij}$, and such that*

$$\operatorname{Tr}_g Q_\xi = 0, \qquad g_{|T'_x} = g_{0|T'_x}, \qquad g(T'_x, K_x) = 0,$$

for all $\xi \in K_x$.

PROOF: We consider the $(n-r) \times m$ matrix (M_s^j) defined by

$$M_s^j = \sum_{l=1}^{r} \frac{\partial Q_{ll}^{\xi_s}}{\partial x^j}(x),$$

for $r+1 \leq j \leq n$, $1 \leq s \leq m$. There exist a system of linearly independent coordinate functions $(y^{r+1}, \dots, y^n)$ on X which are linear combinations (with real coefficients) of the coordinate functions $(x^{r+1}, \dots, x^n)$ and a system $\{\zeta_1, \dots, \zeta_m\}$ of linearly independent sections of K over X, where each ζ_j is a linear combination (with real coefficients) of $\{\xi_1, \dots, \xi_m\}$, such that

$$\frac{1}{4}\sum_{l=1}^{r} \frac{\partial Q_{ll}^{\zeta_s}}{\partial y^j}(x) = \lambda_j \delta_j^{n-m+s}$$

holds for $r+1 \leq j \leq n$, $1 \leq s \leq m$, where $\lambda_j = 1$ or 0. We set $y^i = x^i$ for $1 \leq i \leq r$. We then replace $(x^1, \dots, x^n)$ by the coordinate system $(y^1, \dots, y^n)$; the bundle T' and the connection ∇ remain the same. Let g' be a Riemannian metric on X such that $g'_{ij}(x) = \delta_{ij}$; then by Lemma 2.1,(iii), we see that

$$\operatorname{Tr}_{g'} Q_\xi = \operatorname{Tr}_{g_0} Q_\xi = 0,$$

for all $\xi \in K_x$. Let $\{\beta'_1, \dots, \beta'_m\}$ be the basis of L^*_x dual to the basis $\{\pi(\zeta_1)(x), \dots, \pi(\zeta_m)(x)\}$ of L_x and consider the element

$$z = \sum_{\substack{r+1 \leq j \leq n \\ 1 \leq s \leq m}} \delta_j^{n-m+s} \tilde{\alpha}_j \otimes \beta'_s$$

of maximal rank of $(K^* \otimes L^*)_x$. According to Proposition 6.7,(i), there exists a Riemannian metric g on X satisfying $g(x) = g'(x)$ and (6.15). By (6.14), we easily see that the equalities (6.42) hold, with $Q^s_{ik} = Q^{\zeta_s}_{ik}$. Lemma 6.3 now tells us that g satisfies condition (ii) of Lemma 6.2, with respect to the coordinate system $(y^1, \dots, y^n)$.

7. Determinants and variations of 1-jets

We continue to assume that X is equal to $\mathbb{R}^n$ with a system of linear coordinates $(x^1, \dots, x^n)$, which satisfy the conditions of §6 relative to K and K'. We consider the objects associated in §6 to R and to the point x of X; these include the complement T' of K and the connection ∇ determined by the coordinate system $(x^1, \dots, x^n)$. We continue to use the notations introduced there. Throughout this section, the morphisms ψ and Φ are the ones corresponding to this sub-bundle T' and this connection ∇, and we suppose that the rank m of L is ≥ 1. Let $\{\xi_1, \dots, \xi_m\}$ be a system of linearly independent sections of K over X, where each ξ_j is a linear combination (with real coefficients) of $\partial_{r+1}, \dots, \partial_n$, such that $\{\pi(\xi_1), \dots, \pi(\xi_m)\}$ is a frame for L over X. Let $\{\beta_1, \dots, \beta_m\}$ be the basis of L^*_x dual to the basis $\{\pi(\xi_1)(x), \dots, \pi(\xi_m)(x)\}$ of L_x.

We further assume that $\{t_{p+1}, \dots, t_r\}$ is a basis for $T'_x \cap K''_x$. The annihilator H_x of K''_x is generated by $\{\alpha_1, \dots, \alpha_p\}$. Let g be a Riemannian metric on X satisfying (6.3). The orthogonal complement T''_x of K''_x in T_x is the subspace of T'_x generated by $\{t_1, \dots, t_p\}$. We consider the objects associated to g in §6, and, in particular, the coefficients b^s_{ijk}.

Let ϕ be an orthogonal transformation of T''_x, which we extend to T_x by requiring that the restriction of ϕ to K''_x be the identity mapping. We set $t'_i = \phi(t_i)$, for $1 \leq i \leq n$. Let $\{\alpha'_1, \dots, \alpha'_n\}$ be the basis of T^*_x dual to the orthonormal basis $\{t'_1, \dots, t'_n\}$ of T_x; we write $\alpha_i^\phi = \alpha'_i$, for $1 \leq i \leq n$. Then we have $\phi^*\alpha'_i = \alpha_i$, $t'_j = t_j$, $\alpha'_j = \alpha_j$, for $1 \leq i \leq p$ and $p+1 \leq j \leq n$; moreover $\{\alpha'_1, \dots, \alpha'_p\}$ is a basis of H_x. Let u be an element of $(T^* \otimes S^2T^*)_x$ and $\tilde{g}$ be a Riemannian metric on X satisfying (6.32). We set

$$\begin{aligned} b(\phi)^s_{ijk} &= (\sigma(\Phi)_g(\alpha'_i \otimes \alpha'_j \cdot \alpha'_k))_s, \qquad & A(\phi)^s_{ij} &= b(\phi)^s_{iij}, \\ \tilde{b}(\phi)^s_{ijk} &= (\sigma(\Phi)_{\tilde{g}}(\alpha'_i \otimes \alpha'_j \cdot \alpha'_k))_s, \qquad & \tilde{A}(\phi)^s_{ij} &= \tilde{b}(\phi)^s_{iij}, \end{aligned}$$

for $1 \leq i, j, k \leq n$ and $1 \leq s \leq m$. We consider the coordinate system $(y^1, \dots, y^n)$ on X, where $y^j = x^j$ for $p+1 \leq j \leq n$ and y^i is the unique linear combination of the coordinates $(x^1, \dots, x^p)$ determined by the equality $(dy^i)(x) = \alpha'_i$, for $1 \leq i \leq p$. Since (6.3) holds in this coordinate system, the analogues of the relations (6.34), (6.36) and (6.37) are also valid in this system. Namely, if u is of the form (6.33), the equality

$$\tilde{b}(\phi)^s_{ijk} = b(\phi)^s_{ijk} \tag{7.1}$$

holds for all distinct integers $1 \leq i, j, k \leq n$, whenever $p+1 \leq i \leq n$, or whenever $p+1 \leq j, k \leq n$, or whenever $1 \leq j, k \leq p$. Moreover if $1 \leq i, j \leq p$ and $p+1 \leq k \leq n$ are distinct integers and if $\sigma(\chi)_g(\alpha'_i \cdot \alpha'_j) = 0$, then the relation (7.1) also holds in this case. Furthermore, if $1 \leq i, j \leq n$ are distinct integers, with $1 \leq j \leq r$, we have

$$\tilde{A}(\phi)^s_{ij} = A(\phi)^s_{ij}. \tag{7.2}$$

On the other hand, if u is of the form (6.40), we see that

$$\tilde{b}(\phi)^s_{ijk} = b(\phi)^s_{ijk} + 2\theta(u)(t'_i, t'_j, t'_k, \pi(\xi_s)) \tag{7.3}$$

if $1 \leq i, j, k \leq n$ are distinct integers and $1 \leq s \leq m$, and that (7.2) holds whenever $1 \leq i, j \leq n$ are distinct integers, with $1 \leq j \leq p$.

If $\{\alpha, \beta\}$ is an orthonormal system of elements of T^*_x, we consider the element

$$\theta(\alpha, \beta) = \frac{1}{2}\,\alpha^2 \cdot \beta - \frac{1}{6}\,\beta^3$$

of $S^3_0 T^*_x$. If $1 \leq i, j \leq n$ are distinct integers, we set $\theta_{ij} = \theta(\alpha_i, \alpha_j)$ and $\theta'_{ij} = \theta(\phi)_{ij} = \theta(\alpha'_i, \alpha'_j)$. If h is an element of $S^2T^*_x$, we have

$$\sigma_2(\Phi)_g(\theta(\alpha, \beta) \otimes h) = \tfrac{1}{2}(\alpha^2 - \beta^2) \otimes \sigma(\Phi)_g(\beta \otimes h) + \alpha \cdot \beta \otimes \sigma(\Phi)_g(\alpha \otimes h). \tag{7.4}$$

If $h(g^\sharp(\alpha),\zeta) = h(g^\sharp(\beta),\zeta) = 0$, for all vectors ζ of T_x, and if $\mathrm{Tr}_g\, h = 0$, then the element $\theta(\alpha,\beta)\otimes h$ of $(S^3T^*\otimes S^2T^*)_x$ belongs to the kernels of $\sigma_1(\varphi)_g$ and $\sigma_2(B_g)$; if moreover $\sigma(\chi)_g h = 0$, it follows that $\theta(\alpha,\beta)\otimes h$ is an element of $g'_{3,x}$. In particular, if ϑ_1,ϑ_2 are elements of T^*_x such that $\{\alpha,\beta,\vartheta_1,\vartheta_2\}$ is an orthonormal system of elements of T^*_x and $\sigma(\chi)_g(\vartheta_1\cdot\vartheta_2) = 0$, then $\theta(\alpha,\beta)\otimes\vartheta_1\cdot\vartheta_2$ is an element of $g'_{3,x}$.

If $1 \le i,j,k \le n$ are distinct integers, the elements

$$\begin{aligned} v'_{ijk} &= v(\phi)_{ijk} = \theta'_{ij}\otimes\alpha'_i\cdot\alpha'_k - \theta'_{ji}\otimes\alpha'_j\cdot\alpha'_k, \\ w'_{ijk} &= w(\phi)_{ijk} = \theta'_{ij}\otimes\alpha'_j\cdot\alpha'_k - \theta'_{ji}\otimes\alpha'_i\cdot\alpha'_k \end{aligned}$$

of $(S^3T^*\otimes S^2T^*)_x$ belong to the kernels of $\sigma_1(\varphi)_g$ and $\sigma_2(B_g)$; if

$$\sigma(\chi)_g(\alpha'_i\cdot\alpha'_k) = \sigma(\chi)_g(\alpha'_j\cdot\alpha'_k) = 0,$$

then v'_{ijk} and w'_{ijk} belong to $g'_{3,x}$. If ϕ is the identity mapping of T''_x, we write $w_{ijk} = w'_{ijk}$. We have the relations

$$\begin{aligned} (\sigma_2(\Phi)_g v'_{ijk})_s &= \tfrac{1}{2}(b(\phi)^s_{ijk} + b(\phi)^s_{jik})(\alpha'^2_i - \alpha'^2_j) \\ &\qquad + (A(\phi)^s_{ik} - A(\phi)^s_{jk})\alpha'_i\cdot\alpha'_j, \\ (\sigma_2(\Phi)_g w'_{ijk})_s &= \tfrac{1}{2}(A(\phi)^s_{ik} + A(\phi)^s_{jk})(\alpha'^2_i - \alpha'^2_j) \\ &\qquad + (b(\phi)^s_{ijk} - b(\phi)^s_{jik})\alpha'_i\cdot\alpha'_j. \end{aligned} \tag{7.5}$$

If $1 \le i,j,k,l,q \le n$ are distinct integers, the elements

$$\alpha'_i\cdot\alpha'_j\cdot\alpha'_k\otimes\alpha'_l\cdot\alpha'_q, \qquad \alpha'_i\cdot(\alpha'^2_j - \alpha'^2_k)\otimes\alpha'_l\cdot\alpha'_q$$

of $(S^3T^*\otimes S^2T^*)_x$ belong to the kernels of $\sigma_1(\varphi)_g$ and $\sigma_2(B_g)$; if

$$\sigma(\chi)_g(\alpha'_l\cdot\alpha'_q) = 0,$$

then they are elements of $g'_{3,x}$. We have

$$\begin{aligned} &(\sigma_2(\Phi)_g\alpha'_i\cdot\alpha'_j\cdot\alpha'_k\otimes\alpha'_l\cdot\alpha'_q)_s \\ &\qquad = b(\phi)^s_{klq}\alpha'_i\cdot\alpha'_j + b(\phi)^s_{jlq}\alpha'_i\cdot\alpha'_k + b(\phi)^s_{ilq}\alpha'_j\cdot\alpha'_k, \\ &(\sigma_2(\Phi)_g\alpha'_i\cdot(\alpha'^2_j - \alpha'^2_k)\otimes\alpha'_l\cdot\alpha'_q)_s \\ &\qquad = 2b(\phi)^s_{jlq}\alpha'_i\cdot\alpha'_j - 2b(\phi)^s_{klq}\alpha'_i\cdot\alpha'_k + b(\phi)^s_{klq}(\alpha'^2_j - \alpha'^2_k) \end{aligned} \tag{7.6}$$

Let $1 \leq i, k, j, l \leq n$ be given integers and let h be an element of $S^2T^*_x$. We consider the $2m \times 2$ matrices

$$M(g, \phi; i, k, j) = (M^a_b), \qquad N(g, \phi; i, k, h) = (N^a_b),$$

defined by

$$\begin{aligned} M^s_1 &= b(\phi)^s_{ikj} + b(\phi)^s_{kij}, & M^{s+m}_1 &= A(\phi)^s_{ij} - A(\phi)^s_{kj}, \\ M^s_2 &= A(\phi)^s_{ij} + A(\phi)^s_{kj}, & M^{s+m}_2 &= b(\phi)^s_{ikj} - b(\phi)^s_{kij} \end{aligned}$$

and by

$$N^s_1 = (\sigma_g(\Phi)(\alpha'_k \otimes h))_s = N^{s+m}_2, \qquad N^{s+m}_1 = (\sigma_g(\Phi)(\alpha'_i \otimes h))_s = -N^s_2,$$

for $1 \leq s \leq m$. We write

$$N(g, \phi; i, k, j, l) = N(g, \phi; i, k, \alpha'_j \cdot \alpha'_l)$$

and we consider $2m \times 2$ matrices

$$\begin{aligned} M(g, u, \phi; i, k, j) &= M(\tilde{g}, \phi; i, k, j) - M(g, \phi; i, k, j), \\ N(g, u, \phi; i, k, h) &= N(\tilde{g}, \phi; i, k, h) - N(g, \phi; i, k, h), \\ N(g, u, \phi; i, k, j, l) &= N(\tilde{g}, \phi; i, k, j, l) - N(g, \phi; i, k, j, l). \end{aligned}$$

If ϕ is the identity mapping of T''_x, we write $M(g; i, k, j) = M(g, \phi; i, k, j)$ and $M(g, u; i, k, j) = M(g, u, \phi; i, k, j)$.

If u is of the form (6.33) and if $1 \leq i, j, k \leq p$ are distinct integers, with $1 \leq j \leq p$ and $p+1 \leq k \leq n$, by (7.1) and (7.2) we see that

$$M(g, u, \phi; i, k, j) = 0, \tag{7.7}$$

whenever $\sigma(\chi)_g(\alpha'_i \cdot \alpha'_j) = 0$. Moreover, if u is of the form (6.33) and if $1 \leq i, j, l \leq n$ and $p+1 \leq k \leq n$ are distinct integers, by (7.1) we also have

$$N(g, u, \phi; i, k, j, l) = 0, \tag{7.8}$$

whenever $p+1 \leq i \leq n$, or $1 \leq j, l \leq p$, or $p+1 \leq j, l \leq n$, or whenever $1 \leq i, j \leq p$, $p+1 \leq l \leq n$ and $\sigma(\chi)_g(\alpha'_i \cdot \alpha'_j) = 0$. Furthermore, if h belongs to $S^2 H_x$ and satisfies $\mathrm{Tr}_g\, h = 0$ and $\sigma(\chi)_g h = 0$, by (6.35) we obtain the relation

$$N(g, u, \phi; i, k, h) = 0, \tag{7.9}$$

for all integers $1 \leq i, k \leq n$.

If $A_1, \ldots, A_m$ are $2m \times 2$ matrices, we denote by $A_1 * \cdots * A_m$ the $2m \times 2m$ matrix whose $(2j-1)$-th column is equal to the first column of A_j and whose $2j$-th column is equal to the second column of A_j, for $1 \leq j \leq m$.

Let $0 \leq m_1 \leq m$ and $1 \leq i, k, j_1, \ldots, j_{m_1} \leq n$ be given integers; let $h_{m_1+1}, \ldots, h_m$ be elements of $S^2 T_x^*$. We write

$$\tilde{\jmath} = (j_1, \ldots, j_{m_1}), \qquad \tilde{h} = (h_{m_1+1}, \ldots, h_m),$$

and we denote by $\mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \tilde{h})$ the $2m \times 2m$ matrix

$$M(g, \phi; i, k, j_1) * \cdots * M(g, \phi; i, k, j_{m_1}) * N(g, \phi; i, k, h_{m_1+1}) \\ * \cdots * N(g, \phi; i, k, h_m).$$

We denote by $\{w_1, \ldots, w_q\}$ the subspace of $S^k T_x^*$ spanned by the elements $w_1, \ldots, w_q$ of $S^k T_x^*$.

Lemma 7.1. *Let α be a unit vector of H_x and let $p + 1 \leq k \leq n$ be a given integer. Let $1 \leq i \leq p$ be an integer and ϕ be an orthogonal transformation of T_x'' satisfying $\alpha_i^\phi = \alpha$. Let m_1 be an integer satisfying $0 \leq m_1 \leq m$. Let $1 \leq j_1, \ldots, j_{m_1} \leq p$ be integers different from i. Suppose that the relations*

$$\sigma(\chi)_g(\alpha \cdot \alpha_{j_l}^\phi) = 0 \tag{7.10}$$

hold for all $1 \leq l \leq m_1$. Let $h_{m_1+1}, \ldots, h_m$ be elements of $S^2 T_x^*$ satisfying

$$h_q(g^\sharp(\alpha), \zeta) = h_q(t_k, \zeta) = \mathrm{Tr}_g\, h_q = 0, \qquad \sigma(\chi)_g h_q = 0,$$

for all vectors ζ of T_x and for all $m_1 + 1 \leq q \leq m$. If the matrix $\mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \tilde{h})$ is invertible, where

$$\tilde{\jmath} = (j_1, \ldots, j_{m_1}), \qquad \tilde{h} = (h_{m_1+1}, \ldots, h_m),$$

then we have the inclusion

$$\{\alpha \cdot \alpha_k, \alpha^2 - \alpha_k^2\} \otimes L_x^* \subset V_x.$$

Proof: By our hypotheses, the elements

$$v'_{ikj_l}, \qquad w'_{ikj_l}, \qquad \theta'_{ik} \otimes h_q, \qquad \theta'_{ki} \otimes h_q$$

of $(S^3T^* \otimes S^2T^*)_x$, with $1 \leq l \leq m_1$ and $m_1 + 1 \leq q \leq m$, belong to $g'_{3,x}$. If we consider their images under $\sigma_2(\Phi)_g$, we see that our assertion is a consequence of the formulas (7.4) and (7.5).

Let $1 \leq i, k \leq n$ be distinct integers, with $p + 1 \leq k \leq n$. Let $1 \leq j_1, \dots, j_m \leq p$ and $p + 1 \leq k_1, \dots, k_m \leq n$ be integers different from i and k; we write

$$\begin{aligned} \tilde{\jmath} &= (j_1, \dots, j_m), & \bar{\jmath} &= (j_2, \dots, j_m), \\ \tilde{k} &= (k_1, \dots, k_m), & \bar{k} &= (k_2, \dots, k_m). \end{aligned} \tag{7.11}$$

Let h be an element of S^2H_x satisfying $\mathrm{Tr}_g\, h = 0$ and $\sigma(\chi)_g h = 0$. We denote by $\mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \bar{k})$, $\mathcal{N}(g, \phi; i, k, \tilde{\jmath}, \tilde{k})$ and $\mathcal{N}(g, \phi; i, k, \bar{\jmath}, \bar{k}, h)$ the $2m \times 2m$ matrices

$$M(g, \phi; i, k, j_1) * N(g, \phi; i, k, j_2, k_2) * \cdots * N(g, \phi; i, k, j_m, k_m),$$
$$N(g, \phi; i, k, j_1, k_1) * N(g, \phi; i, k, j_2, k_2) * \cdots * N(g, \phi; i, k, j_m, k_m),$$
$$N(g, \phi; i, k, h) * N(g, \phi; i, k, j_2, k_2) * \cdots * N(g, \phi; i, k, j_m, k_m)$$

respectively, and we set

$$\mathcal{M}(g, u, \phi; i, k, \tilde{\jmath}, \bar{k}) = \mathcal{M}(\tilde{g}, \phi; i, k, \tilde{\jmath}, \bar{k}) - \mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \bar{k}),$$
$$\mathcal{N}(g, u, \phi; i, k, \tilde{\jmath}, \tilde{k}) = \mathcal{N}(\tilde{g}, \phi; i, k, \tilde{\jmath}, \tilde{k}) - \mathcal{N}(g, \phi; i, k, \tilde{\jmath}, \tilde{k}),$$
$$\mathcal{N}(g, u, \phi; i, k, \bar{\jmath}, \bar{k}, h) = \mathcal{N}(\tilde{g}, \phi; i, k, \bar{\jmath}, \bar{k}, h) - \mathcal{N}(g, \phi; i, k, \bar{\jmath}, \bar{k}, h).$$

According to (7.7), (7.8) and (7.9), if u is of the form (6.33) and if the equalities

$$\sigma(\chi)_g(\alpha'_i \cdot \alpha'_{j_l}) = 0$$

hold for all $1 \leq l \leq m$, we see that

$$\begin{gathered} \mathcal{M}(g, u, \phi; i, k, \tilde{\jmath}, \bar{k}) = 0, \\ \mathcal{N}(g, u, \phi; i, k, \tilde{\jmath}, \tilde{k}) = 0, \qquad \mathcal{N}(g, u, \phi; i, k, \bar{\jmath}, \bar{k}, h) = 0. \end{gathered} \tag{7.12}$$

By (6.23), for $1 \le l \le m$, the element $h_l = \alpha'_{j_l} \cdot \alpha'_{k_l}$ of $S^2T^*_x$ satisfies

$$h_l(t'_i, \zeta) = h_l(t_k, \zeta) = \mathrm{Tr}_g\, h_l = 0, \qquad \sigma(\chi)_g h_l = 0.$$

The matrices

$$\mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \bar{k}), \qquad \mathcal{N}(g, \phi; i, k, \tilde{\jmath}, \tilde{k}), \qquad \mathcal{N}(g, \phi; i, k, \bar{\jmath}, \bar{k}, h)$$

can be written in the form $\mathcal{M}(g, \phi; i, k, \tilde{\imath}, \tilde{h})$, with $m_1 = 1$, $\tilde{\imath} = (j_1)$, and $\tilde{h} = (h_2, \dots, h_m)$, or with $m_1 = 0$, $\tilde{\imath} = (\emptyset)$ and $\tilde{h} = (h_1, \dots, h_m)$ or $\tilde{h} = (h, h_2, \dots, h_m)$, respectively. Thus from Lemma 7.1, we deduce the following:

LEMMA 7.2. *Let α be a unit vector of H_x and let $p + 1 \le k \le n$ be a given integer. Let $1 \le i \le p$ be an integer and ϕ be an orthogonal transformation of T''_x satisfying $\alpha_i^\phi = \alpha$. Let $1 \le j_1, \dots, j_m \le p$ and $p + 1 \le k_2, \dots, k_m \le n$ be integers different from i and k. Set*

$$\tilde{\jmath} = (j_1, \dots, j_m), \qquad \bar{\jmath} = (j_2, \dots, j_m), \qquad \bar{k} = (k_2, \dots, k_m).$$

Let h be an element of S^2H_x satisfying

$$h(g^\sharp(\alpha), \zeta) = \mathrm{Tr}_g\, h = 0, \qquad \sigma(\chi)_g h = 0,$$

for all vectors ζ of T_x. If either the matrix $\mathcal{N}(g, \phi; i, k, \bar{\jmath}, \bar{k}, h)$ is invertible, or if $\sigma(\chi)_g(\alpha \cdot \alpha_{j_1}^\phi) = 0$ and the matrix $\mathcal{M}(g, \phi; i, k, \tilde{\jmath}, \bar{k})$ is invertible, then we have the inclusion

$$\{\alpha \cdot \alpha_k, \alpha^2 - \alpha_k^2\} \otimes L^*_x \subset V_x.$$

LEMMA 7.3. *Let $1 \le i \le p$ and $p+1 \le k_1, k_2 \le n$ be distinct integers. Let ϕ be an orthogonal transformation of T''_x. Let $1 \le j_1, \dots, j_m \le p$ and $p + 1 \le l_1, \dots, l_m \le n$ be integers different from i, k_1 and k_2. Set*

$$\tilde{\jmath} = (j_1, \dots, j_m), \qquad \tilde{l} = (l_1, \dots, l_m).$$

If the matrix $\mathcal{N}(g, \phi; k_1, k_2, \tilde{\jmath}, \tilde{l})$ is invertible, then we have the inclusion

$$\{\alpha'_i \cdot \alpha_{k_1}, \alpha'_i \cdot \alpha_{k_2}\} \otimes L^*_x \subset V_x + \alpha_{k_1} \cdot \alpha_{k_2} \otimes L^*_x.$$

PROOF: By (6.23), the elements

$$\alpha'_i \cdot \alpha'_{k_1} \cdot \alpha'_{k_2} \otimes \alpha'_{j_q} \cdot \alpha'_{l_q}, \qquad \alpha'_i \cdot (\alpha'^2_{k_1} - \alpha'^2_{k_2}) \otimes \alpha'_{j_q} \cdot \alpha'_{l_q}$$

of $(S^3T^* \otimes S^2T^*)_x$, with $1 \leq q \leq m$, belong to $g'_{3,x}$. If we consider their images under $\sigma_2(\Phi)_g$, we see that our assertion is a consequence of the formulas (7.6).

We now suppose that $m \geq 2$. If μ is an element of H_x and λ is an element of $\bigwedge^2 H_x$, we consider the elements $E(\mu)$ of $(T^* \otimes L^*)_x$ and $F(\lambda)$ of $(T^* \otimes T^* \otimes L^*)_x$ associated to μ and λ in §5, and the elements $F_l(\lambda)$ of $(T^* \otimes T^* \otimes L^*)_x$, with $l = 1, 2, 3$, defined by

$$\begin{aligned} F_1(\lambda)(\xi, \eta) &= 2F(\lambda)(\xi, \eta) - F(\lambda)(\eta, \xi), \\ F_2(\lambda)(\xi, \eta) &= F(\lambda)(\xi, \eta) + F(\lambda)(\eta, \xi), \\ F_3(\lambda)(\xi, \eta) &= F(\lambda)(\xi, \eta) - 2F(\lambda)(\eta, \xi), \end{aligned}$$

for $\xi, \eta \in T_x$. If h is an element of $S^2T^*_x$, we consider the element $\langle F(\lambda), h\rangle_g$ of L^*_x. If ξ, η are vectors of T_x, we easily verify that

$$\langle F(\lambda), g^\flat(\xi) \cdot g^\flat(\eta)\rangle_g = F_2(\lambda)(\xi, \eta). \tag{7.13}$$

We set $E(\mu, \phi) = \phi^* E(\mu)$ and

$$F(\lambda, \phi) = F(\lambda)^\phi = \phi^* F(\lambda), \qquad F_l(\lambda, \phi) = F_l(\lambda)^\phi = \phi^* F_l(\lambda),$$

for $l = 1, 2, 3$. If v is an element of $(\bigotimes^q T^* \otimes L^*)_x$, for $1 \leq j_1, \ldots, j_q \leq n$, we consider the element

$$v_{j_1 \ldots j_q} = \sum_{s=1}^{m} v^s_{j_1 \ldots j_q} \beta_s = v(t_{j_1}, \ldots, t_{j_q})$$

of L^*_x. In particular, we have

$$E(\mu, \phi)_i = E(\mu)(t'_i), \qquad F(\lambda, \phi)_{ij} = F(\lambda)(t'_i, t'_j),$$

for $1 \leq i, j \leq p$.

We recall that the element $\widetilde{Q}_{g,x}$ of $S^2T''^*_x \otimes L^*_x$, defined by

$$\widetilde{Q}_{g,x}(\eta_1, \eta_2, \pi(\xi)) = Q(\eta_1, \eta_2, \xi),$$

for $\eta_1, \eta_2 \in T''_x$, $\xi \in K_x$, satisfies conditions (i) and (ii) of §5. If $\mathrm{Tr}_g\, Q_\xi = 0$, for all $\xi \in K_x$, then the element $\widetilde{Q}_{g,x}$ also satisfies the condition $\mathrm{Tr}\, \widetilde{Q}_{g,x} = 0$, with respect to the inner product on T''_x induced by g. For $\eta \in T''_x$, the subspace

$$W_\eta = \{\, \xi \in T''_x \mid g(\xi,\eta) = Q_\zeta(\xi,\eta) = 0, \text{ for all } \zeta \in K_x \,\}$$

of T''_x is equal to the subspace V_η corresponding to $\widetilde{Q}_{g,x}$, as defined in §5. For $\alpha \in T''^*_x$ and $\beta \in \bigwedge^2 T''^*_x$, we also consider the elements $E(\alpha)$ of $T''^*_x \otimes L^*_x$ and $F(\beta)$ of $T''^*_x \otimes T''^*_x \otimes L^*_x$ associated to $\widetilde{Q}_{g,x}$ in §5. Since Q_ξ belongs to $S^2 H_x$, for $\xi \in K_x$, we know that $E(\mu)$ belongs to $H_x \otimes L^*_x$ and that $F(\lambda)$ belongs to $H_x \otimes H_x \otimes L^*_x$, for $\mu \in H_x$ and $\lambda \in \bigwedge^2 H_x$. If $\rho : H_x \to T''^*_x$ is the isomorphism sending an element of H_x into its restriction to T''_x, then by (5.27) we have

$$\rho E(\mu,\phi) = \phi^* E(\rho\mu), \qquad \rho F(\lambda,\phi) = \phi^* F(\rho\lambda). \tag{7.14}$$

If η is a vector of T''_x, we consider the subspace

$$\mathcal{H}_\eta = \{\, h \in S^2 H_x \mid \sigma(\chi)_g h = 0,\ \operatorname{Tr}_g h = h(\eta,\zeta) = 0, \text{ for all } \zeta \in T_x \,\}$$

of $S^2 H_x$; we see that $\rho\mathcal{H}_\eta$ is equal to the subspace H_η of $S^2 T''^*_x$ associated to $\widetilde{Q}_{g,x}$ in §5. Thus we are able to apply the results of §5 to $\widetilde{Q}_{g,x}$ and to replace the elements $E(\alpha)$ and $F(\beta)$, with $\alpha \in T''^*_x$ and $\beta \in \bigwedge^2 T''^*_x$, by $E(\mu)$ and $F(\lambda)$ respectively, where $\mu \in H_x$ and $\lambda \in \bigwedge^2 H_x$ are the unique elements satisfying $\rho(\mu) = \alpha$ and $\rho(\lambda) = \beta$.

Let S_1 denote the product of $\binom{n-p}{2}$ copies of H_x and let S_2 denote the product of $n-p$ copies of $\bigwedge^2 H_x$. If $\mu = (\mu_{jk})_{p+1 \le j < k \le n}$ is an element of S_1 and $\lambda = (\lambda_{p+1}, \dots, \lambda_n)$ is an element of S_2, we set $\mu_{jk} = \mu_{kj}$ for $p+1 \le j < k \le n$, and consider the elements

$$u_\mu = \sum_{p+1 \le j < k \le n} \mu_{jk} \otimes \alpha_j \cdot \alpha_k, \qquad u_\lambda = \sum_{k=p+1}^{n} \gamma(\lambda_k \otimes \alpha_k)$$

of $(T^* \otimes S^2 T^*)_x$; we write $u_{\mu,\lambda} = u_\mu + u_\lambda$. If $1 \le i \le p$ and $p+1 \le j,k \le n$ are distinct integers and if $u = u_{\mu,\lambda}$, then by (7.3) we have

$$\begin{aligned} \tilde{b}(\phi)^s_{ijk} - b(\phi)^s_{ijk} &= -3E(\mu_{jk},\phi)^s_i, \\ \tilde{b}(\phi)^s_{jik} - b(\phi)^s_{jik} &= E(\mu_{jk},\phi)^s_i, \end{aligned} \tag{7.15}$$

for $1 \leq s \leq m$. Let $1 \leq i, j \leq p$ and $p + 1 \leq k \leq n$ be distinct integers. By (7.2) and (7.3), we easily verify that the $2m \times 2$ matrix

$$M(g, u_{\mu,\lambda}, \phi; i, k, j) = (P^a_b)$$

is given by

$$P^s_1 = 2F_3(\lambda_k, \phi)^s_{ij}, \qquad P^{s+m}_2 = 6F(\lambda_k, \phi)^s_{ij}, \qquad P^s_2 = P^{s+m}_1 = 0, \tag{7.16}$$

for $1 \leq s \leq m$, and so we obtain the relation

$$M(g, tu_{\mu,\lambda}, \phi; i, k, j) = tM(g, u_{\mu,\lambda}, \phi; i, k, j), \tag{7.17}$$

for $t \in \mathbb{R}$. Moreover, if $1 \leq l \leq n$ is an integer distinct from i, j and k, by (7.3) we see that the $2m \times 2$ matrix

$$N(g, u_{\mu,\lambda}, \phi; i, k, j, l) = (Q^a_b)$$

is given by

$$Q^s_1 = Q^{s+m}_2 = -2F_2(\lambda_k, \phi)^s_{jl}, \qquad Q^s_2 = Q^{s+m}_1 = 0 \tag{7.18}$$

whenever $1 \leq l \leq p$, and by

$$Q^s_1 = Q^{s+m}_2 = E(\mu_{kl}, \phi)^s_j, \qquad Q^{s+m}_1 = -Q^s_2 = 2F_1(\lambda_l, \phi)^s_{ij} \tag{7.19}$$

whenever $p + 1 \leq l \leq n$, where $1 \leq s \leq m$, and thus we have

$$N(g, tu_{\mu,\lambda}, \phi; i, k, j, l) = tN(g, u_{\mu,\lambda}, \phi; i, k, j, l), \tag{7.20}$$

for $t \in \mathbb{R}$. On the other hand, if $p + 1 \leq k_1, k_2 \leq n$ are distinct integers different from k, the $2m \times 2$ matrix

$$N(g, u_{\mu,\lambda}, \phi; k_1, k_2, j, k) = (R^a_b)$$

is given by

$$R^s_1 = R^{s+m}_2 = E(\mu_{k_2k}, \phi)^s_j, \qquad R^{s+m}_1 = -R^s_2 = E(\mu_{k_1k}, \phi)^s_j, \tag{7.21}$$

for $1 \leq s \leq m$, and we infer that

$$N(g, tu_{\mu,\lambda}, \phi; k_1, k_2, j, k) = tN(g, u_{\mu,\lambda}, \phi; k_1, k_2, j, k), \tag{7.22}$$

for $t \in \mathbb{R}$. If h is an element of $S^2 H_x$ satisfying $\mathrm{Tr}_g\, h = 0$ and $\sigma(\chi)_g h = 0$, then by (6.38) the $2m \times 2$ matrix

$$N(g, u_{\mu,\lambda}, \phi; i, k, h) = (U^a_b)$$

is given by

$$U^s_1 = U^{s+m}_2 = -2\langle F(\lambda_k), h\rangle^s_g, \qquad U^s_2 = U^{s+m}_1 = 0, \tag{7.23}$$

for $1 \leq s \leq m$, and so we obtain the relation

$$N(g, tu_{\mu,\lambda}, \phi; i, k, h) = tN(g, u_{\mu,\lambda}, \phi; i, k, h), \tag{7.24}$$

for $t \in \mathbb{R}$.

Let $1 \leq i, k \leq n$ be distinct integers, with $p + 1 \leq k \leq n$. Let $1 \leq j_1, \ldots, j_m \leq p$ and $p + 1 \leq k_1, \ldots, k_m \leq n$ be integers different from i and k; as above, we consider the m and $(m-1)$-tuples (7.11). Let (μ, λ) be an element of $S_1 \times S_2$. According to (7.20) and (7.22), we have the relation

$$\mathcal{N}(g, tu_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \tilde{k}) = t\mathcal{N}(g, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \tilde{k}), \tag{7.25}$$

and by (7.19) and (7.21) we see that

$$\{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{N}(g, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \tilde{k}) \neq 0 \,\}$$

is a Zariski-open subset of $S_1 \times S_2$. Let h be an element of $S^2 H_x$ satisfying $\mathrm{Tr}_g\, h = 0$ and $\sigma(\chi)_g h = 0$. If $1 \leq i \leq p$, according to (7.17), (7.20) and (7.24), we obtain the relations

$$\begin{aligned} \mathcal{M}(g, tu_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \bar{k}) &= t\mathcal{M}(g, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \bar{k}), \\ \mathcal{N}(g, tu_{\mu,\lambda}, \phi; i, k, \bar{\jmath}, \bar{k}, h) &= t\mathcal{N}(g, u_{\mu,\lambda}, \phi; i, k, \bar{\jmath}, \bar{k}, h), \end{aligned} \tag{7.26}$$

for $t \in \mathbb{R}$; moreover by (7.16), (7.19) and (7.23) we see that

$$\begin{gathered} \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{M}(g, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \bar{k}) \neq 0 \,\}, \\ \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{N}(g, u_{\mu,\lambda}, \phi; i, k, \bar{\jmath}, \bar{k}, h) \neq 0 \,\} \end{gathered}$$

are Zariski-open subsets of $S_1 \times S_2$. We denote by $[m/2]$ the integral part of $m/2$. In the following formulas, we shall identify the element $a\beta_1 \wedge \ldots \wedge \beta_m$

of $\bigwedge^m L_x^*$ with the real number a. If $p+1 \leq i \leq n$, for all elements μ of S_1 satisfying $\mu_{ik_q} = 0$ for $1 \leq q \leq m$, according to (7.21) we have

$$\det \mathcal{N}(g, u_\mu, \phi; i, k, \tilde{\jmath}, \tilde{k}) = (-1)^{[m/2]} \big(E(\mu_{kk_1})^\phi_{j_1} \wedge \ldots \wedge E(\mu_{kk_m})^\phi_{j_m} \big)^2 \tag{7.27}$$

If $1 \leq i \leq p$ and $\lambda \in S_2$, according to (7.16) and (7.19) we verify that

$$\begin{aligned} \det \mathcal{M}(g, u_\lambda, \phi; i, k, \tilde{\jmath}, \bar{k}) \\ &= c\, \big(F(\lambda_k)^\phi_{ij_1} \wedge F_1(\lambda_{k_2})^\phi_{ij_2} \wedge \ldots \wedge F_1(\lambda_{k_m})^\phi_{ij_m} \big) \\ &\quad \cdot \big(F_3(\lambda_k)^\phi_{ij_1} \wedge F_1(\lambda_{k_2})^\phi_{ij_2} \wedge \ldots \wedge F_1(\lambda_{k_m})^\phi_{ij_m} \big) \end{aligned} \tag{7.28}$$

where $c = (-1)^{[m/2]}\, 4^m\, 3$. On the other hand, for $\lambda \in S_2$ and for $1 \leq i \leq p$, by (7.19) and (7.23) we see that

$$\begin{aligned} \det \mathcal{N}(g, u_{\mu,\lambda}, \phi; i, k, \bar{\jmath}, \bar{k}, h) \\ &= c' \big(\langle F(\lambda_k), h \rangle_g \wedge F_1(\lambda_{k_2})^\phi_{ij_2} \wedge \ldots \wedge F_1(\lambda_{k_m})^\phi_{ij_m} \big)^2, \end{aligned} \tag{7.29}$$

where $c' = (-1)^{[m/2]}\, 4^m$.

If A and B are $q \times q$ matrices, then

$$\det(tA + B),$$

with $t \in \mathbb{R}$, is a polynomial of degree q in t whose leading coefficient is equal to $\det A$. From this remark and the relations (7.25) and (7.26), if $(\mu, \lambda) \in S_1 \times S_2$ and if g_t are Riemannian metrics on X satisfying

$$j_1(g_t)(x) = j_1(g)(x) + \varepsilon(t u_{\mu,\lambda}), \tag{7.30}$$

for $t \in \mathbb{R}$, we infer that the determinant

$$\det \mathcal{N}(g_t, \phi; i, k, \tilde{\jmath}, \tilde{k})$$

and, when $1 \leq i \leq p$, the determinants

$$\det \mathcal{M}(g_t, \phi; i, k, \tilde{\jmath}, \bar{k}), \qquad \det \mathcal{N}(g_t, \phi; i, k, \tilde{\jmath}, \bar{k}, h)$$

are polynomials of degree $2m$ in t whose leading coefficients are equal to

$$\det \mathcal{N}(g_t, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \tilde{k}),$$

$$\det \mathcal{M}(g_t, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \bar{k}), \qquad \det \mathcal{N}(g_t, u_{\mu,\lambda}, \phi; i, k, \tilde{\jmath}, \bar{k}, h),$$

respectively.

PROPOSITION 7.1. *Suppose that $m \geq 2$ and $p \geq 3$. Let η be a unit vector of T''_x which possesses property* (A) *of* §5 *with respect to the element $\widetilde{Q}_{g,x}$ of $S^2T''^*_x \otimes L^*_x$. Let $p+1 \leq k \leq n$ be a given integer. If α is the unit vector $g^\flat(\eta)$ of H_x, then there exist an element λ of S_2, an orthogonal transformation ϕ of T''_x satisfying $\phi(t_1) = \eta$ and $\alpha_1^\phi = \alpha$, integers*

$$2 \leq j_1, \ldots, j_m \leq p, \qquad p+1 \leq k_2, \ldots, k_m \leq n$$

distinct from k such that the relations (7.10) *hold for all $1 \leq l \leq m$ and such that the matrix $\mathcal{M}(g, u_\lambda, \phi; 1, k, \tilde{\jmath}, \bar{k})$ is invertible, where*

$$\tilde{\jmath} = (j_1, \ldots, j_m), \qquad \bar{k} = (k_2, \ldots, k_m).$$

PROOF: Assume that η possesses property (A). Let $\{t'_1, \ldots, t'_p\}$ be an orthonormal basis of T''_x, with $t'_1 = \eta$, for which $\{t'_2, \ldots, t'_{p_0}\}$ is a basis for W_η, where $2 \leq p_0 \leq p$. By (7.14), there exist integers $2 \leq j_1, \ldots, j_m \leq p_0$ and elements $\lambda'_1, \ldots, \lambda'_m$ of $\bigwedge^2 H_x$ such that the two elements

$$\Phi' = F(\lambda'_1)(t'_1, t'_{j_1}) \wedge F_1(\lambda'_2)(t'_1, t'_{j_2}) \wedge \ldots \wedge F_1(\lambda'_m)(t'_1, t'_{j_m}),$$
$$\Phi'' = F_3(\lambda'_1)(t'_1, t'_{j_1}) \wedge F_1(\lambda'_2)(t'_1, t'_{j_2}) \wedge \ldots \wedge F_1(\lambda'_m)(t'_1, t'_{j_m})$$

of $\bigwedge^m L^*_x$ do not vanish. Let ϕ be the unique orthogonal transformation of T''_x determined by the relations $\phi(t_i) = t'_i$, for $1 \leq i \leq p$; then we have $\phi^*\alpha = \alpha_1$. Since t'_{j_l} belongs to W_η, we know that (7.10) holds for all $1 \leq l \leq m$. Now, since $n - p \geq m$, we may choose distinct integers $p+1 \leq k_2, \ldots, k_m \leq n$ different from k; we set $\bar{k} = (k_2, \ldots, k_m)$. We then select an element λ of S_2 satisfying $\lambda_k = \lambda'_1$ and $\lambda_{k_l} = \lambda'_l$, for $2 \leq l \leq m$. By (7.28), if we identify the element $a\beta_1 \wedge \ldots \wedge \beta_m$ of $\bigwedge^m L^*_x$ with the real number a and we set $\tilde{\jmath} = (j_1, \ldots, j_m)$, then we have the equalities

$$\begin{aligned}
\det \mathcal{M}(g, \phi, u_\lambda; 1, k, \tilde{\jmath}, \bar{k}) \\
&= c\,(F(\lambda_k, \phi)_{1j_1} \wedge F_1(\lambda_{k_2}, \phi)_{1j_2} \wedge \ldots \wedge F_1(\lambda_{k_m}, \phi)_{1j_m}) \\
&\qquad \cdot (F_3(\lambda_k, \phi)_{1j_1} \wedge F_1(\lambda_{k_2}, \phi)_{1j_2} \wedge \ldots \wedge F_1(\lambda_{k_m}, \phi)_{1j_m}) \\
&= c\,\Phi' \cdot \Phi'',
\end{aligned}$$

where $c = (-1)^{[m/2]}\, 4^m\, 3$, and so this determinant is non-zero. Thus the desired conclusion holds.

Let $0 \leq d \leq m$ be a given integer and, for each $1 \leq l \leq m$, let $1 \leq i_l, j_l, k_l \leq n$ be distinct integers. Assume that $1 \leq i_l \leq p$ and $p+1 \leq j_l, k_l \leq n$ for $1 \leq l \leq d$, and that $1 \leq j_q \leq p$ and $p+1 \leq i_q, k_q \leq n$ for $d+1 \leq q \leq m$. We set

$$\tilde{\imath} = (i_1, \dots, i_m), \qquad \tilde{\jmath} = (j_1, \dots, j_m), \qquad \tilde{k} = (k_1, \dots, k_m)$$

and we consider the $m \times m$ matrix $\mathcal{D}(g, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k})$, whose q-th column is equal to the $m \times 1$-matrix $\big(b(\phi)^s_{i_q j_q k_q}\big)$, with $1 \leq s \leq m$. We also consider the $m \times m$ matrix

$$\mathcal{D}(g, u, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) = \mathcal{D}(\tilde{g}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) - \mathcal{D}(g, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}).$$

By (7.1), if u is of the form (6.33), then we have

$$\mathcal{D}(g, u, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) = 0. \tag{7.31}$$

If $(\mu, \lambda) \in S_1 \times S_2$, according to (7.15) we obtain the relation

$$\mathcal{D}(g, tu_{\mu,\lambda}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) = t\mathcal{D}(g, u_{\mu,\lambda}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}), \tag{7.32}$$

for all $t \in \mathbb{R}$, and, if we identify the element $a\beta_1 \wedge \ldots \wedge \beta_m$ of $\bigwedge^m L^*_x$ with the real number a, we have the equality

$$\begin{aligned} &\det \mathcal{D}(g, u_{\mu,\lambda}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) \\ &\qquad = c'' E(\mu_{j_1 k_1})^\phi_{i_1} \wedge \ldots \wedge E(\mu_{j_d k_d})^\phi_{i_d} \wedge E(\mu_{i_{d+1} k_{d+1}})^\phi_{j_{d+1}} \\ &\qquad\qquad \wedge \ldots \wedge E(\mu_{i_m k_m})^\phi_{j_m}, \end{aligned} \tag{7.33}$$

where $c'' = (-1)^d\, 3^d$; moreover we see that

$$\{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{D}(g, u_{\mu,\lambda}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}) \neq 0 \,\}$$

is a Zariski-open subset of $S_1 \times S_2$. From the equality (7.32) and the remark preceding Proposition 7.1, if g_t are Riemannian metrics satisfying (7.30) for $t \in \mathbb{R}$, we infer that

$$\det \mathcal{D}(g_t, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k})$$

is a polynomial of degree m in t whose leading coefficient is equal to

$$\det \mathcal{D}(g, u_{\mu,\lambda}, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{k}).$$

Lemma 7.4. *Let $1 \le i \le p$ be a given integer. Let ϕ be an orthogonal transformation of T''_x. Let*

$$1 \le j_1, \dots, j_m \le p, \qquad p+1 \le k_1, l_1, \dots, k_m, l_m \le n$$

be integers different from i satisfying $k_q \ne l_q$, for $1 \le q \le m$. Set

$$\tilde{\jmath} = (j_1, \dots, j_m), \qquad \tilde{k} = (k_1, \dots, k_m), \qquad \tilde{l} = (l_1, \dots, l_m).$$

If the matrix $\mathcal{D}(g, \phi; \tilde{k}, \tilde{\jmath}, \tilde{l})$ is invertible, then we have the inclusion

$$\alpha_i'^2 \otimes L_x^* \subset V_x + (\alpha_i' \cdot T^*_{\{t_1,\dots,t_p\}} + S^2 T^*_{\{t_1,\dots,t_p\}}) \otimes L_x^*.$$

Proof: For $1 \le q \le m$, by (6.23) the elements $\theta'_{ik_q} \otimes \alpha'_{j_q} \cdot \alpha'_{l_q}$ of $(S^3T^* \otimes S^2T^*)_x$ belong to $g'_{3,x}$. We consider their images under $\sigma_2(\Phi)_g$ and then we see that our assertion follows from formula (7.4).

Lemma 7.5. *Let $1 \le i, j \le p$ be distinct integers. Let $0 \le d_0 \le d \le m$ be given integers; set $i_q = i$ for $1 \le q \le d_0$ and $i_s = j$ for $d_0 + 1 \le s \le d$. For each integer $1 \le q \le m$, let $p+1 \le k_q, l_q \le n$ be distinct integers, and, for each integer $d+1 \le s \le m$, let $1 \le j_s \le p$ be an integer different from i and j. If the matrix $\mathcal{D}(g, \phi; \tilde{\imath}, \tilde{\jmath}, \tilde{l})$ is invertible, where*

$$\tilde{\imath} = (i_1, \dots, i_d, k_{d+1}, \dots, k_m), \qquad \tilde{\jmath} = (k_1, \dots, k_d, j_{d+1}, \dots, j_m)$$

and $\tilde{l} = (l_1, \dots, l_m)$, then we have the inclusion

$$\alpha_i' \cdot \alpha_j' \otimes L_x^* \subset V_x + (\alpha_i'^2 - \alpha_j'^2) \otimes L_x^* + \{\alpha_i', \alpha_j'\} \cdot T^*_{\{t_1,\dots,t_p\}} \otimes L_x^*.$$

Proof: By (6.23), we see that the elements

$$\theta'_{ij} \otimes \alpha'_{k_a} \cdot \alpha'_{l_a}, \qquad \theta'_{ji} \otimes \alpha'_{k_q} \cdot \alpha'_{l_q}, \qquad \alpha'_i \cdot \alpha'_j \cdot \alpha'_{k_s} \otimes \alpha'_{j_s} \cdot \alpha'_{l_s}$$

of $(S^3T^* \otimes S^2T^*)_x$ belong to $g'_{3,x}$, with $1 \le a \le d_0$, and $d_0 + 1 \le q \le d$ and $d+1 \le s \le m$. We consider their images under $\sigma_2(\Phi)_g$, and obtain the desired result using the formulas (7.4) and (7.6).

Proposition 7.2. *Suppose that $m \ge 2$ and $p \ge 2$. Assume that $\mathrm{Tr}_g\, Q_\xi = 0$, for all $\xi \in K_x$. Suppose that either the following two conditions do not hold:*

(i) *$m = p = 2$ and $n = 4$ or 5;*

(ii) $m = p = 3$ *and* $n = 6$,
or that the following condition holds:

(iii) $m = p = 3$ *and there is an orthonormal basis* $\{\eta_1, \eta_2, \eta_3\}$ *of* T''_x *such that the spaces* W_{η_i} *are non-zero, for* $i = 1, 2, 3$.
Then there exist an element (μ, λ) *of* $S_1 \times S_2$ *and a non-empty Zariski-open subset* $\mathcal{U}$ *of* $\mathbb{R}$ *such that, for all elements* v *of* $(T^* \otimes S^2T^*)_x$ *of the form* (6.33) *and all* $t \in \mathcal{U}$, *and for all Riemannian metrics* $\tilde{g}$ *on* X *satisfying*

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon(tu_{\mu,\lambda} + v),$$

the equality

$$\sigma_2(\Phi)_{\tilde{g}}(g'_{3,x}) + (S^2T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x) = (S^2T^* \otimes L^*)_x \tag{7.34}$$

holds.

Proof: At various instances, in the course of this proof, we shall identify the element $a\beta_1 \wedge \ldots \wedge \beta_m$ of $\bigwedge^m L^*_x$ with the real number a. Let $\{t'_1, \dots, t'_p\}$ be an orthonormal basis of T''_x. According to our hypotheses, we may assume either that $p \geq 3$ and $\dim W_{t'_j} \geq p - 2$, for all $1 \leq j \leq p$, or that conditions (i) and (ii) of the proposition do not hold and that this basis diagonalizes the restriction of the quadratic form $Q_{\xi_1(x)}$ to T''_x. In all cases, we have $\dim W_{t'_j} \geq p - m$, for all $1 \leq j \leq p$. Let ϕ be the orthogonal transformation of T''_x satisfying $\phi(t_j) = t'_j$, for $1 \leq j \leq p$; we set $\alpha'_j = g^\flat(t'_j)$. Let $1 \leq i \leq p$ be a given integer. We shall apply various results of §5, with

$$a_1 = -b_3 = 2, \qquad a_2 = a_3 = b_2 = -b_1 = 1,$$

to the element $A = \widetilde{Q}_{g,x}$ of $S^2T''^*_x \otimes L^*_x$, the orthonormal basis $\{t'_1, \dots, t'_p\}$ and the unit vector t'_i of T''_x. We now proceed to construct a non-empty Zariski-open subset $\mathcal{V}_i$ of $S_1 \times S_2$. First, suppose that the vector t'_i possesses property (A) of §5 with respect to the element $\widetilde{Q}_{g,x}$ of $S^2T''^*_x \otimes L^*_x$. Let $p + 1 \leq k \leq n$ be arbitrary integer. Proposition 7.1 gives us an element $\tilde{\lambda}$ of S_2, an orthogonal transformation ϕ' of T''_x satisfying $\phi'(t_1) = t'_i$ and $\alpha_1^{\phi'} = \alpha'_i$, and integers

$$2 \leq j_1, \dots, j_m \leq p, \qquad p + 1 \leq k_2, \dots, k_m \leq n$$

distinct from k such that the relations

$$\sigma(\chi)_g(\alpha_1^{\phi'} \cdot \alpha_{j_l}^{\phi'}) = 0 \tag{7.35}$$

hold for all $1 \le l \le m$ and such that the matrix

$$\mathcal{M}(g, u_{\tilde{\lambda}}, \phi'; 1, k, \tilde{\jmath}, \bar{k})$$

is invertible, where

$$\tilde{\jmath} = (j_1, \dots, j_m), \qquad \bar{k} = (k_2, \dots, k_m).$$

Let $\tilde{g}$ be a Riemannian metric satisfying $\tilde{g}(x) = g(x)$ and u be an element of $(T^* \otimes S^2T^*)_x$. We write

$$\mathcal{M}(\tilde{g}; i, k) = \mathcal{M}(\tilde{g}, \phi'; 1, k, \tilde{\jmath}, \bar{k}), \qquad \mathcal{M}(\tilde{g}, u; i, k) = \mathcal{M}(\tilde{g}, u, \phi'; 1, k, \tilde{\jmath}, \bar{k}).$$

If the matrix $\mathcal{M}(\tilde{g}; i, k)$ is invertible, by Lemma 7.2 and (7.35) we obtain the inclusion

$$\{\alpha_i' \cdot \alpha_k, \alpha_i'^2 - \alpha_k^2\} \otimes L_x^* \subset \sigma_2(\Phi)_{\tilde{g}}(g'_{3,x}). \tag{7.36}$$

If v is an element of $(T^* \otimes S^2T^*)_x$ of the form (6.33), since the relations (7.35) hold, by (7.12) we have

$$\mathcal{M}(\tilde{g}, v; i, k) = 0. \tag{7.37}$$

We know that the Zariski-open subset

$$\mathcal{V}_{ik} = \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{M}(g, u_{\mu,\lambda}; i, k) \neq 0 \,\}$$

of $S_1 \times S_2$ is non-empty; therefore the subset

$$\mathcal{V}_i = \bigcap_{p+1 \le k \le n} \mathcal{V}_{ik}$$

of $S_1 \times S_2$ is also non-empty. Now suppose that t_i' does not possess property (A). Then if $p \ge 3$, according to Lemma 5.5 applied to $\widetilde{Q}_{g,x}$ and the unit vector t_i' of T_x'', the dimension of $W_{t_i'}$ is $< p - 2$. Thus according to our assumptions, the basis $\{t_1', \dots, t_p'\}$ diagonalizes the restriction of the quadratic form $Q_{\xi_1(x)}$ to T_x'' and conditions (i) and (ii) of the proposition do not hold. If $p \ge 3$, it also follows that $m \ge 3$. Hence we know that either $m \ge 4$, or that $m = 3$ and $p \ge 4$, or that $m = p = 3$ and $n \ge 7$, or that $m = p = 2$ and $n \ge 6$. If $m \ge 4$ (resp. $m = 2, 3$), according to Lemma 5.10 (resp. Lemmas 5.7 and 5.11), by (7.14) there exist integers $2 \le j_1, \dots, j_m \le p$ and a system $\{\mu_1', \dots, \mu_m'\}$ of vectors of H_x, where at most $m-2$ (resp. at most 2) of these vectors are distinct, such that the element

$$E(\mu'_1,\phi)_{j_1}\wedge\ldots\wedge E(\mu'_m,\phi)_{j_m}$$

of $\bigwedge^m L^*_x$ does not vanish. We set $\tilde\jmath=(j_1,\ldots,j_m)$. Let $p+1\leq k_1,k_2\leq n$ be given distinct integers. We first suppose that $m\geq 4$, or that $m=p=3$ and $n\geq 7$, or that $m=p=2$ and $n\geq 6$; then we always have $n-p\geq 4$. Thus we may choose integers $p+1\leq l_1,\ldots,l_m\leq n$ distinct from k_1 and k_2 with the following property: the equality $l_j=l_q$ holds if and only if $\mu'_j=\mu'_q$, for $1\leq j,q\leq m$. We write $\tilde l=(l_1,\ldots,l_m)$ and we select an element $\tilde\mu$ of S_1 satisfying $\tilde\mu_{k_2l_q}=\mu'_q$ and $\tilde\mu_{k_1l_q}=0$, for $1\leq q\leq m$. We set

$$\mathcal N(\tilde g;i,k_1,k_2)=\mathcal N(g,\phi;k_1,k_2,\tilde\jmath,\tilde l),$$
$$\mathcal N(\tilde g,u;i,k_1,k_2)=\mathcal N(\tilde g,\phi,u;k_1,k_2,\tilde\jmath,\tilde l).$$

Then if the matrix $\mathcal N(\tilde g;i,k_1,k_2)$ is invertible, by Lemma 7.3 we obtain the inclusion

$$\{\alpha'_i\cdot\alpha_{k_1},\alpha'_i\cdot\alpha_{k_2}\}\otimes L^*_x\subset\sigma_2(\Phi)_{\tilde g}(g'_{3,x})+\alpha_{k_1}\cdot\alpha_{k_2}\otimes L^*_x. \tag{7.38}$$

If v is an element of $(T^*\otimes S^2T^*)_x$ of the form (6.33), by (7.12) we have

$$\mathcal N(\tilde g,v;i,k_1,k_2)=0. \tag{7.39}$$

According to formula (7.27), we see that

$$\det\mathcal N(\tilde g,u_{\tilde\mu};i,k_1,k_2)=(-1)^{[m/2]}(E(\mu'_1,\phi)_{j_1}\wedge\ldots\wedge E(\mu'_m,\phi)_{j_m})^2,$$

and hence the Zariski-open subset

$$\mathcal V_{i,k_1k_2}=\{\,(\mu,\lambda)\in S_1\times S_2\mid\det\mathcal N(g,u_{\mu,\lambda};i,k_1,k_2)\neq 0\,\}$$

of $S_1\times S_2$ is non-empty. Therefore the subset

$$\mathcal V^1_i=\bigcap_{\substack{p+1\leq k_1,k_2\leq n\\ k_1\neq k_2}}\mathcal V_{i,k_1k_2}$$

of $S_1\times S_2$ is also non-empty. Since $n-p\geq 4$, we may also choose integers

$$p+1\leq k'_1,l'_1,\ldots,k'_m,l'_m\leq n$$

such that $k'_q \neq l'_q$ and such that the unordered pairs (k'_q, l'_q) are all distinct, for $1 \leq q \leq m$. We set

$$\tilde{k}' = (k'_1, \dots, k'_m), \qquad \tilde{l}' = (l'_1, \dots, l'_m)$$

and we select an element $\tilde{\mu}'$ of S_1 satisfying $\tilde{\mu}'_{k'_q l'_q} = \mu'_q$, for $1 \leq q \leq m$. We write

$$\mathcal{N}(\tilde{g}; i) = \mathcal{D}(g, \phi; \tilde{k}', \tilde{\jmath}, \tilde{l}'), \qquad \mathcal{N}(\tilde{g}, u; i) = \mathcal{D}(\tilde{g}, \phi, u; \tilde{k}', \tilde{\jmath}, \tilde{l}').$$

If the matrix $\mathcal{N}(\tilde{g}; i) \neq 0$ is invertible, by Lemma 7.4 we obtain the inclusion

$$\alpha_i'^2 \otimes L^*_x \subset \sigma_2(\Phi)_{\tilde{g}}(g'_{3,x}) + (\alpha'_i \cdot T^*_{\{t_1,\dots,t_p\}} + S^2T^*_{\{t_1,\dots,t_p\}}) \otimes L^*_x. \tag{7.40}$$

If v is an element of $(T^* \otimes S^2T^*)_x$ of the form (6.33), by (7.31) we have

$$\mathcal{N}(\tilde{g}, v; i) = 0. \tag{7.41}$$

According to formula (7.33) with $d = 0$, we see that

$$\det \mathcal{N}(\tilde{g}, u_{\tilde{\mu}'}; i) = E(\mu'_1, \phi)_{j_1} \wedge \ldots \wedge E(\mu'_m, \phi)_{j_m},$$

and hence the Zariski-open subset

$$\mathcal{V}_i^2 = \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{N}(g, u_{\mu,\lambda}; i) \neq 0 \,\}$$

of $S_1 \times S_2$ is non-empty. Therefore the subset $\mathcal{V}_i = \mathcal{V}_i^1 \cap \mathcal{V}_i^2$ of $S_1 \times S_2$ is also non-empty. Finally, we suppose that $m = 3$ and $p \geq 4$; we know that $W_{t'_i}$ is non-zero. Let $p+1 \leq k \leq n$ be a given integer; choose distinct integers $p+1 \leq k_2, k_3 \leq n$ different from k. We apply Lemmas 5.13 and 5.15 to $Q_{g,x}$, the orthonormal basis $\{t'_1, \dots, t'_p\}$ and the unit vector $\eta = t'_i$ of T''_x. By (7.14), there exist a vector ξ of W_η, vectors $\lambda'_1, \lambda'_2, \lambda'$ of $\bigwedge^2 H_x$ and an element h of $\mathcal{H}_\eta$ such that the element

$$\langle F(\lambda'), h\rangle_g \wedge F_1(\lambda'_1)(\eta, \xi) \wedge F_1(\lambda'_2)(\eta, \xi)$$

of $\bigwedge^3 L^*_x$ does not vanish. Indeed, when $p \geq 5$, according to Lemma 5.13 and (7.13), the element h of $\mathcal{H}_\eta$ can be written in the form $\alpha'_j \cdot g^\flat(\zeta)$, where $1 \leq j \leq p$ is an integer different from i and ζ is a vector of $W_{t'_j}$ orthogonal to η; in this case, we have $\langle F(\lambda'), h\rangle_g = F_2(\lambda')(t'_j, \zeta)$. We choose an orthogonal transformation ϕ' of T''_x satisfying $\phi'(t_1) = t'_i$, $\phi'(t_2) = \xi$, $\alpha_1^{\phi'} = \alpha'_i$ and $\alpha_2^{\phi'} = g^\flat(\xi)$. Since ξ belong to W_η, we see that

$$\sigma(\chi)_g(\alpha_1^{\phi'} \cdot \alpha_2^{\phi'}) = \sigma(\chi)_g(\alpha'_i \cdot g^\flat(\xi)) = 0.$$

We write

$$\mathcal{M}(\tilde{g}; i, k) = \mathcal{M}(\tilde{g}, \phi'; 1, k, \bar{\imath}, \bar{k}, h), \quad \mathcal{M}(\tilde{g}, u; i, k) = \mathcal{M}(\tilde{g}, u, \phi'; 1, k, \bar{\imath}, \bar{k}, h),$$

where $\bar{\imath} = (2,2)$ and $\bar{k} = (k_2, k_3)$. If the matrix $\mathcal{M}(\tilde{g}; i, k)$ is invertible, by Lemma 7.2 we have the inclusion (7.36). If v is an element of $(T^* \otimes S^2T^*)_x$ of the form (6.33), by (7.12) we see that the relation (7.37) holds. We choose an element $\tilde{\lambda}$ of S_2 satisfying $\tilde{\lambda}_{k_2} = \lambda'_1$, $\tilde{\lambda}_{k_3} = \lambda'_2$ and $\tilde{\lambda}_k = \lambda'$. According to formula (7.29), we see that

$$\det \mathcal{M}(\tilde{g}, u_{\tilde{\lambda}}; i, k) = -64\big(\langle F(\lambda'), h\rangle_g \wedge F_1(\lambda'_1)(\eta, \xi) \wedge F_1(\lambda'_2)(\eta, \xi)\big)^2,$$

and hence the Zariski-open subset

$$\mathcal{V}_{ik} = \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{N}(g, u_{\mu,\lambda}; i, k) \neq 0 \,\}$$

of $S_1 \times S_2$ is non-empty. Therefore the subset

$$\mathcal{V}_i = \bigcap_{p+1 \leq k \leq n} \mathcal{V}_{ik}$$

of $S_1 \times S_2$ is also non-empty. We no longer make any assumption on the vector t'_i. Now, let $1 \leq j \leq p$ be an integer different from i. We proceed to construct a non-empty Zariski-open subset $\mathcal{W}_{ij}$ of $S_1 \times S_2$. According to Lemmas 5.7 and 5.9 applied to $\widetilde{Q}_{g,x}$ and the basis $\{t'_1, \dots, t'_p\}$ of T''_x, there are integers $1 \leq i_1, \dots, i_m \leq p$ and elements $\mu''_1, \dots, \mu''_m$ of H_x, which satisfy $\mu''_1 = \mu''_2$ when $m = 2$, such that the element

$$E(\mu''_1, \phi)_{i_1} \wedge \ldots \wedge E(\mu''_m, \phi)_{i_m}$$

of $\bigwedge^m L^*_x$ does not vanish. Without loss of generality, we may suppose that $i_a = i$ for $1 \leq a \leq d_0$, that $i_q = j$ for $d_0 + 1 \leq q \leq d$ and that i_s is different from i and j for $d + 1 \leq s \leq m$. When $m = 2$, we choose two distinct integers $p + 1 \leq k, l \leq n$, and we set $k_1 = k_2 = k$ and $l_1 = l_2 = l$. When $m \geq 3$, we choose integers

$$p + 1 \leq k_1, \dots, k_m, l_1, \dots, l_m \leq n$$

such that $k_q \neq l_q$ and such that the unordered pairs (k_q, l_q) are all distinct, for $1 \leq q \leq m$. We consider the $m \times m$ matrices

$$\mathcal{D}(\tilde{g}; i, j) = \mathcal{D}(\tilde{g}, \phi; \tilde{\imath}, \tilde{k}, \tilde{l}), \qquad \mathcal{D}(g, u; i, j) = \mathcal{D}(g, u, \phi; \tilde{\imath}, \tilde{k}, \tilde{l}),$$

where

$$\tilde{\imath} = (i_1, \dots, i_d, k_{d+1}, \dots, k_m), \qquad \tilde{\jmath} = (k_1, \dots, k_d, i_{d+1}, \dots, i_m)$$

and $\tilde{l} = (l_1, \dots, l_m)$. If the matrix $\mathcal{D}(\tilde{g}; i, j)$ is invertible, by Lemma 7.5 we obtain the inclusion

$$\alpha'_i \cdot \alpha'_j \otimes L^*_x \subset \sigma_2(\Phi)_{\tilde{g}}(g'_{3,x}) + (\alpha'^2_i - \alpha'^2_j) \otimes L^*_x + \{\alpha'_i, \alpha'_j\} \cdot T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x. \tag{7.42}$$

If v is an element of $(T^* \otimes S^2T^*)_x$ of the form (6.33), by (7.31) we have

$$\mathcal{D}(g, v; i, j) = 0. \tag{7.43}$$

When $m = 2$, we choose an element $\tilde{\mu}$ of S_1 satisfying $\tilde{\mu}_{kl} = \mu''_1 = \mu''_2$. If $m \geq 3$, we choose an element $\tilde{\mu}$ of S_1 satisfying $\tilde{\mu}_{k_q l_q} = \mu''_q$, for $1 \leq q \leq m$. According to (7.33), we see that

$$\det \mathcal{D}(g, u_{\tilde{\mu}}; i, j) = (-1)^d\, 3^d\, E(\mu'_1, \phi)_{i_1} \wedge \ldots \wedge E(\mu'_m, \phi)_{i_m},$$

and hence that the subset

$$\mathcal{W}_{ij} = \{\, (\mu, \lambda) \in S_1 \times S_2 \mid \det \mathcal{N}(g, u_{\mu,\lambda}; i, j) \neq 0 \,\}$$

of $S_1 \times S_2$ is non-empty. Therefore the subsets

$$\mathcal{V} = \bigcap_{1 \leq i \leq p} \mathcal{V}_i, \qquad \mathcal{W} = \bigcap_{\substack{1 \leq i, j \leq p \\ i \neq j}} \mathcal{W}_{ij}$$

of $S_1 \times S_2$ are non-empty, and hence so is the subset $\mathcal{V} \cap \mathcal{W}$. Thus we may choose an element (μ, λ) of $\mathcal{V} \cap \mathcal{W}$. Let v be an arbitrary element of $(T^* \otimes S^2T^*)_x$ of the form (6.33). For $t \in \mathbb{R}$, let g_t, $\tilde{g}_t$ be Riemannian metrics on X satisfying

$$j_1(g_t)(x) = j_1(g)(x) + \varepsilon(t u_{\mu,\lambda}), \qquad j_1(\tilde{g}_t)(x) = j_1(g)(x) + \varepsilon(t u_{\mu,\lambda} + v).$$

By the remark preceding Lemma 7.4 and by (7.32) and (7.43), for all distinct integers $1 \leq i, j \leq p$ we know that the function

$$P_{ij}(t) = \det \mathcal{D}(g_t; i, j) = \det \mathcal{D}(\tilde{g}_t; i, j)$$

is a polynomial in t of degree m, whose leading coefficient is equal to

$$\det \mathcal{D}(g, u_{\mu,\lambda}; i, j).$$

Since (μ, λ) belongs to $\mathcal{V} \cap \mathcal{W}$, this coefficient is non-zero, and hence the polynomial P_{ij} is non-zero and

$$\mathcal{U}_{ij} = \{\, t \in \mathbb{R} \mid P_{ij}(t) \neq 0 \,\}$$

is a non-empty Zariski-open subset of $\mathbb{R}$. For all integers $1 \leq i \leq p$ and $p{+}1 \leq k, k_1, k_2 \leq n$, with $k_1 \neq k_2$, by the remarks preceding Proposition 7.1 and Lemma 7.4, according to (7.25), (7.26), (7.37), (7.39) and (7.41), when t'_i satisfies condition (A) or when $m = 3$ and $p \geq 4$, the function

$$P_{ik}(t) = \det \mathcal{M}(g_t; i, k) = \det \mathcal{M}(\tilde{g}_t; i, k)$$

is a polynomial in t of degree $2m$, whose leading coefficient is equal to

$$\det \mathcal{M}(g, u_{\mu,\lambda}; i, k),$$

while, in all the other cases, the functions

$$\begin{aligned} P_{i,k_1k_2}(t) &= \det \mathcal{N}(g_t; i, k_1, k_2) = \det \mathcal{N}(\tilde{g}_t; i, k_1, k_2), \\ P_i(t) &= \det \mathcal{N}(g_t; i) = \det \mathcal{N}(\tilde{g}_t; i) \end{aligned}$$

are polynomials in t of degree $2m$ and m, whose leading coefficients are the real numbers

$$\det \mathcal{N}(g, u_{\mu,\lambda}; i, k_1, k_2), \qquad \det \mathcal{N}(g, u_{\mu,\lambda}; i), \tag{7.44}$$

respectively. Therefore, since (μ, λ) belongs to $\mathcal{V} \cap \mathcal{W}$, in the first cases, the coefficient $\det \mathcal{M}(g, u_{\mu,\lambda}; i, k)$ is non-zero, and hence the polynomial P_{ik} is non-zero; therefore

$$\mathcal{U}_i = \{\, t \in \mathbb{R} \mid P_{ik}(t) \neq 0, \ \text{for all } p+1 \leq k \leq n \,\}$$

is a non-empty Zariski-open subset of $\mathbb{R}$. In the other cases, the coefficients (7.44) do not vanish and the polynomials P_{i,k_1k_2} and P_i are non-zero; hence

$$\mathcal{U}_i = \{\, t \in \mathbb{R} \mid P_{i,k_1k_2}(t) \neq 0,\ P_i(t) \neq 0 \text{ for all } p+1 \leq k_1 < k_2 \leq n \,\}$$

is a non-empty Zariski-open subset of $\mathbb{R}$. We then consider the non-empty Zariski-open subsets

$$\mathcal{U}^1 = \bigcap_{\substack{1\leq i\leq p\\ p+1\leq k\leq n}} \mathcal{U}_{ik}, \qquad \mathcal{U}^2 = \bigcap_{1\leq i,j\leq p} \mathcal{U}_{ij}$$

and $\mathcal{U} = \mathcal{U}^1 \cap \mathcal{U}^2$ of $\mathbb{R}$. The inclusions (7.36), (7.38) and (7.40) now imply that

$$\sigma_2(\Phi)_{\tilde{g}_t}(g'_{3,x}) + (S^2T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x) = (S^2T^* \otimes L^*)_x,$$

for all $t \in \mathcal{U}$.

Since $u_{t\mu,t\lambda} = tu_{\mu,\lambda}$, for $t \in \mathbb{R}$ and $(\mu,\lambda) \in S_1 \times S_2$, we see that, under the assumptions of Proposition 7.2, this proposition give us an element (μ,λ) of $S_1 \times S_2$ with the following property: for any Riemannian metric $\tilde{g}$ on X and any element v of $(T^* \otimes S^2T^*)_x$ of the form (6.33) satisfying

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon(u_{\mu,\lambda} + v),$$

the equality (7.34) holds.

For the remainder of this section, we shall assume that $m = 1$ and that the basis $\{t_1,\dots,t_p\}$ diagonalizes the restriction of the quadratic form $Q_{\xi_1(x)}$ to T''_x. Then we have $\sigma(\chi)_g(\alpha_i \cdot \alpha_j) = 0$, for $1 \leq i < j \leq n$. We write $\beta = \beta^1$ and $b_{ijk} = b^1_{ijk}$, for $1 \leq i,j,k \leq n$.

Lemma 7.6. *Suppose that $m = 1$. Let $1 \leq i_1, i_2, j, k \leq n$ be distinct integers. If either $b_{i_1jk} \neq 0$ or $b_{i_2jk} \neq 0$, then $\alpha_{i_1} \cdot \alpha_{i_2} \otimes \beta$ and $(\alpha^2_{i_1} - \alpha^2_{i_2}) \otimes \beta$ belong to V_x.*

Proof: The elements $\theta_{i_1i_2} \otimes \alpha_j \cdot \alpha_k$ and $\theta_{i_2i_1} \otimes \alpha_j \cdot \alpha_k$ belong to $g'_{3,x}$. We compare the expressions for their images under $\sigma_2(\Phi)_g$ given by formula (7.4) to obtain the desired result.

Lemma 7.7. *Assume that $m = 1$ and $p \geq 3$. If*

(i) $b_{i1p} \neq 0$, *for* $2 \leq i < p$,
(ii) $b_{1pn} \neq 0$, $b_{p1n} \neq 0$,
(iii) $b_{np2} \neq 0$,

(iv) $b_{12n} \neq 0$, $b_{n12} \neq 0$,
then we have

$$V_x + (S^2T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x) = (S^2T^* \otimes L^*)_x.$$

PROOF: According to Lemma 7.6, we see that (i) implies that $\alpha_i \cdot \alpha_j \otimes \beta$ and $(\alpha_i^2 - \alpha_n^2) \otimes \beta$ belong to V_x, for all $2 \leq i < p$ and $2 \leq j \leq n$, with $j \neq i, p$. From the relation $b_{n12} \neq 0$, it follows that $(\alpha_p^2 - \alpha_n^2) \otimes \beta$ and $\alpha_p \cdot \alpha_n \otimes \beta$ are elements of V_x. On the other hand, we infer from the inequality $b_{12n} \neq 0$ that $(\alpha_1^2 - \alpha_p^2) \otimes \beta$ and $\alpha_1 \cdot \alpha_j \otimes \beta$ belong to V_x, for $3 \leq j < n$, and from the inequality $b_{1pn} \neq 0$ that $\alpha_1 \cdot \alpha_2 \otimes \beta$ is an element of V_x. Finally, since $b_{p1n} \neq 0$, we see that $\alpha_p \cdot \alpha_j \otimes \beta \in V_x$, for all $2 \leq j \leq n-1$, with $j \neq p$, and, from condition (iii), we deduce that $\alpha_1 \cdot \alpha_n \otimes \beta \in V_x$.

LEMMA 7.8. *Assume that $m = 1$ and $p = 2$. If*
(i) $b_{12n} \neq 0$, $b_{21n} \neq 0$,
(ii) $b_{12n} \neq b_{21n}$,
(iii) $\det M(g; 1, n, 2) \neq 0$, $\det M(g; 2, n, 1) \neq 0$,
then we have

$$V_x + (S^2T^*_{\{t_1,\dots,t_p\}} \otimes L^*_x) = (S^2T^* \otimes L^*)_x.$$

PROOF: By Lemma 7.6, condition (i) implies that $\alpha_1 \cdot \alpha_i \otimes \beta$ and $\alpha_2 \cdot \alpha_i \otimes \beta$ are elements of V_x for $3 \leq i \leq n-1$. Condition (iii) and Lemma 7.2 tell us that

$$\alpha_1 \cdot \alpha_n \otimes \beta, \quad \alpha_2 \cdot \alpha_n \otimes \beta, \quad (\alpha_1^2 - \alpha_n^2) \otimes \beta, \quad (\alpha_2^2 - \alpha_n^2) \otimes \beta$$

belong to V_x. We then consider the expression for the image of the element w_{12n} of $g'_{3,x}$ given by (7.5), with ϕ equal to the identity mapping of T''_x. From condition (ii), we deduce that $\alpha_1 \cdot \alpha_2 \otimes \beta$ is an element of

$$V_x + (\alpha_1^2 - \alpha_2^2) \otimes L^*_x.$$

8. Construction of strongly prolongable jets

In this section, we suppose that K is integrable and that $m \geq 1$. Let x be a point of X and let g be a Riemannian metric on X. We denote by T''_x the orthogonal complement of K''_x in T_x; we write $p = n - \dim K''_x = \dim T''_x$. For $\eta \in T_x$, we set

$$W_\eta = \{\, \xi \in T''_x \mid g(\xi, \eta) = Q_\zeta(\xi, \eta) = 0, \text{ for all } \zeta \in K_x \,\}.$$

We assume that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. Suppose that either the following two conditions do not hold:

(i) $m = p = 2$ and $n = 4$ or 5;

(ii) $m = p = 3$ and $n = 6$,

or that the following condition holds:

(iii) $m = p = 3$ and there is an orthonormal basis $\{\eta_1, \eta_2, \eta_3\}$ of T''_x such that the spaces W_{η_i} are non-zero, for $i = 1, 2, 3$.

As in §6, without loss of generality we may suppose that X is $\mathbb{R}^n$ with a system of linear coordinates $(x^1, \dots, x^n)$ which satisfy the conditions of §6 relative to K and K', and we may also assume that (6.3) holds. We consider the objects associated to R and x in §6 and §7; these include the complement T' of K and the connection ∇ determined by the coordinate system $(x^1, \dots, x^n)$. We shall use the notations introduced there. Throughout this section, the morphisms ψ and Φ and the subset N'_k of $J_k(S^2_+T^*)$ are the ones corresponding to this sub-bundle T' and this connection ∇. Let $\{\xi_1, \dots, \xi_m\}$ be a system of linearly independent sections of K over X, where each ξ_j is a linear combination (with real coefficients) of $\partial_{r+1}, \dots, \partial_n$, such that $\{\pi(\xi_1), \dots, \pi(\xi_m)\}$ is a frame for L over X. Let $\{\beta_1, \dots, \beta_m\}$ be the basis of L^*_x dual to the basis $\{\pi(\xi_1)(x), \dots, \pi(\xi_m)(x)\}$ of L_x.

By performing a linear change of the coordinates $(x^1, \dots, x^r)$, without loss of generality we may further suppose that $\{t_{p+1}, \dots, t_r\}$ is a basis for $T'_x \cap K''_x$. The subspace T''_x of T_x is generated by the vectors $\{t_1, \dots, t_p\}$. Then performing a linear change of the coordinates $(x^{r+1}, \dots, x^n)$ and replacing the metric g if necessary, according to Lemma 6.4 we may also assume that g satisfies conditions (i) and (ii) of Lemma 6.2. Indeed, any new metric given by Lemma 6.4 will verify the previous assumptions on g.

Proposition 8.1. *Suppose that $m \geq 2$ and $p \geq 2$. Assume that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$, and that g satisfies conditions* (i) *and* (ii) *of Lemma* 6.2. *Suppose that either conditions* (i) *and* (ii) *do not hold or that condition* (iii) *holds. Then there exists a Riemannian metric $\tilde{g}$ on X, satisfying conditions* (i) *and* (ii) *of Lemma* 6.2 *and*

$$\tilde{g}(x) = g(x), \qquad j_1(\tilde{g})(x) \in N'_1,$$

for which the sequences (6.4) *are exact at x for all $k \geq 0$.*

Proof: According to Proposition 7.2, there exists an element (μ, λ) of $S_1 \times S_2$ such that, for any Riemannian metric $\tilde{g}$ on X and any element w of $(T^* \otimes S^2T^*)_x$ of the form (6.33) related by

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon(u_{\mu,\lambda} + w),$$

the equality (7.34) holds. The element $u_{\mu,\lambda}$ of $(T^* \otimes S^2T^*)_x$ is of the form (6.40) and so, according to Lemma 6.2, a Riemannian metric g' on X, whose 1-jet at x is equal to

$$j_1(g')(x) = j_1(g)(x) + \varepsilon u_{\mu,\lambda},$$

also satisfies conditions (i) and (ii) of Lemma 6.2. Then by Proposition 6.7,(iii), there exists an element v of $(T^* \otimes S^2T^*)_x$ of the form (6.33) such that $j_1(g')(x) + \varepsilon v$ belongs to N'_1. By Lemma 6.2, a Riemannian metric $\tilde{g}$ on X, whose 1-jet at x is equal to

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon(u_{\mu,\lambda} + v),$$

also satisfies conditions (i) and (ii) of Lemma 6.2. Since the equality (7.34) and condition (ii) of Lemma 6.2 hold for $\tilde{g}$, it follows that $\tilde{g}$ satisfies conditions (i) and (ii) of Proposition 6.3; the exactness of all the sequences (6.4) at x corresponding to the metric $\tilde{g}$ is a consequence of Proposition 6.3.

We now suppose that $m = 1$ and $p \geq 2$ and proceed to show that the analogue of Proposition 8.1 holds in this case. Let $\{t'_1, \dots, t'_p\}$ be an orthonormal basis for the orthogonal complement T''_x of K''_x in T_x for which the $p \times p$ matrix

$$\big(Q_{\xi_1}(t'_i, t'_j)\big)_{1 \leq i,j \leq p}$$

is diagonal. By performing a linear change of the coordinates $(x^1, \dots, x^p)$ if necessary, we may assume without loss of generality that $t_i = t'_i$, for $1 \leq i \leq p$; thus we have

$$Q_{\xi_1}(t_i, t_j) = q_i \delta_{ij}, \tag{8.1}$$

with $q_i \in \mathbb{R}$. We write

$$\beta = \beta_1, \qquad b_{ijk} = b^1_{ijk}, \qquad \theta'(u)_{ijk} = 2\theta(u)^1_{ijk},$$

for $u \in (T^* \otimes S^2T^*)_x$ and $1 \leq i, j, k \leq n$. Let U be the subspace of $(T^* \otimes S^2T^*)_x$ consisting of those elements u of $(T^* \otimes S^2T^*)_x$ satisfying $u_{ijj} = u_{jji} = 0$, for all $1 \leq i, j \leq n$.

Let u be an element of $(T^* \otimes S^2T^*)_x$ and let $\tilde{g}$ be a Riemannian metric on X satisfying (6.32); we write

$$\tilde{b}^s_{ijk} = (\sigma(\Phi)_{\tilde{g}}(\alpha_i \otimes \alpha_j \cdot \alpha_k))_s,$$

for $1 \leq i, j, k \leq n$. If u belongs to U, by (8.1), (2.14) and (4.1), we see that u satisfies (6.16). Hence by Proposition 6.7,(ii), condition (i) of Lemma 6.2 holds for $\tilde{g}$ and we have $\omega(\tilde{g})(x) = \omega(g)(x)$. By (6.25), we know that the restriction of $\sigma(\Phi)_{\tilde{g}}$ to $T^*_{\{t_1,\ldots,t_p\}} \otimes S^2T^*_{\{t_1,\ldots,t_p\}}$ is equal to the restriction of $\sigma(\Phi)_g$. Since the element $\omega(\tilde{g})(x)$ of $(K^* \otimes L^*)_x$ is of maximal rank, by Lemma 6.1 condition (ii) of Lemma 6.2 is also true for $\tilde{g}$.

If u is of the form (6.33), according to (8.1) and (6.34) the equality

$$\tilde{b}^s_{ijk} = b^s_{ijk} \tag{8.2}$$

holds whenever $1 \leq i, j, k \leq n$ are distinct integers; moreover if $1 \leq i, j \leq p$ are distinct integers and $p + 1 \leq k \leq n$, by (7.7) we see that

$$M(g, u; i, k, j) = 0. \tag{8.3}$$

Let $1 \leq i, j, k \leq n$ be distinct integers; assume that at least one of these integers is $\leq p$. Consider the element

$$v = A\alpha_k \otimes \alpha_i \cdot \alpha_j + B\alpha_j \otimes \alpha_i \cdot \alpha_k + C\alpha_i \otimes \alpha_j \cdot \alpha_k$$

of U, where $A, B, C \in \mathbb{R}$; then we have

$$\theta'(v)_{ijk} = (q_i - q_j + q_k)A + (q_i + q_j - q_k)B - (3q_i + q_j + q_k)C.$$

Since our assumption implies that $q_i^2 + q_j^2 + q_k^2 \neq 0$, if y is a given real number, we can choose the coefficients $A, B, C \in \mathbb{R}$ in such a way that $\theta'(v)_{ijk} \neq y$.

We first consider the case when $p \geq 3$. By the preceding remark, we see that there exists an element u of U such that

$$\begin{aligned} &\theta'(u)_{i1p} \neq -b_{i1p}, \qquad &&\theta'(u)_{1pn} \neq -b_{1pn}, \qquad &&\theta'(u)_{p1n} \neq -b_{p1n}, \\ &\theta'(u)_{np2} \neq -b_{np2}, \qquad &&\theta'(u)_{12n} \neq -b_{12n}, \qquad &&\theta'(u)_{n12} \neq -b_{n12}, \end{aligned}$$

for $2 \leq i < p$. By (6.37), a Riemannian metric g' on X, whose 1-jet at x is equal to

$$j_1(g')(x) = j_1(g)(x) + \varepsilon u, \tag{8.4}$$

satisfies both the hypotheses of Lemma 7.7 and the conditions (i) and (ii) of Lemma 6.2.

Next, we suppose that $p = 2$. We know that $q_2 = -q_1$ and that $q_1 \neq 0$. We consider the element

$$v = \frac{1}{2q_1}(A\alpha_1 \otimes \alpha_2 \cdot \alpha_n + B\alpha_2 \otimes \alpha_1 \cdot \alpha_n)$$

of U, where $A, B \in \mathbb{R}$. Then by (6.9), we have

$$\theta'(v)_{12n} = -A, \qquad \theta'(v)_{21n} = B, \qquad \theta'(v)_{n12} = A - B. \tag{8.5}$$

By (6.37), (6.39) and (8.5), we easily verify that

$$M(g, v; 1, n, 2) = \begin{pmatrix} -B & 0 \\ 0 & -2A + B \end{pmatrix}, \quad M(g, v; 2, n, 1) = \begin{pmatrix} A & 0 \\ 0 & 2B - A \end{pmatrix}.$$

If g_t are Riemannian metrics on X satisfying

$$j_1(g_t)(x) = j_1(g)(x) + \varepsilon(tv)$$

for $t \in \mathbb{R}$, according to remarks preceding Proposition 7.1, we see that the determinants

$$P_1(t) = \det M(g_t; 1, n, 2), \qquad P_2(t) = \det M(g_t; 2, n, 1)$$

are polynomials in t of degree 2 whose leading coefficients are equal to $B(2A - B)$ and $A(2B - A)$, respectively. We choose non-zero real numbers A, B satisfying $A + B \neq 0$, $2A - B \neq 0$ and $2B - A \neq 0$ and consider the corresponding element v of U. Then we have

$$\theta'(v)_{12n} \neq 0, \qquad \theta'(v)_{21n} \neq 0, \qquad \theta'(v)_{12n} - \theta'(v)_{21n} \neq 0$$

and P_1 and P_2 are non-zero polynomials. Hence we are able to choose a real number t_0 such that, if u is the element $t_0 v$ of U, the inequalities

$$\theta'(u)_{12n} \neq -b_{12n}, \quad \theta'(u)_{21n} \neq -b_{21n}, \quad \theta'(u)_{12n} - \theta'(v)_{21n} \neq b_{21n} - b_{12n}$$
$$P_1(t_0) \neq 0, \qquad P_2(t_0) \neq 0$$

hold. Then by (6.37) and by our choice of t_0, it is easily seen that the metric $g' = g_{t_0}$ satisfies both the hypotheses of Lemma 7.8 and conditions (i) and (ii) of Lemma 6.2.

When $p \geq 2$, we now consider the Riemannian metric g' on X satisfying conditions (i) and (ii) of Lemma 6.2 and whose 1-jet at x is given by (8.4), where u is the element of $(T^* \otimes S^2T^*)_x$ constructed above. Then by Proposition 6.7,(iii), there exists an element v of $(T^* \otimes S^2T^*)_x$ of the form (6.33) such that $j_1(g')(x) + \varepsilon v$ belongs to N'_1. By Lemma 6.2, a Riemannian metric $\tilde{g}$ on X, whose 1-jet at x is equal to

$$j_1(\tilde{g})(x) = j_1(g)(x) + \varepsilon(u + v),$$

also satisfies conditions (i) and (ii) of Lemma 6.2. When $p \geq 3$ (resp. $p = 2$), by (8.2) and (8.3) the Riemannian metric $\tilde{g}$ also satisfies the hypotheses of Lemma 7.7 (resp. Lemma 7.8) and hence the equality (7.34) holds. Since condition (ii) of Lemma 6.2 holds for $\tilde{g}$, it follows that $\tilde{g}$ satisfies conditions (i) and (ii) of Proposition 6.3; the exactness of all the sequences (6.4) at x corresponding to the metric $\tilde{g}$ is a consequence of Proposition 6.3. We have therefore proved the following result:

Proposition 8.2. *Suppose that $m = 1$. Assume that $\mathrm{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$, and that g satisfies conditions* (i) *and* (ii) *of Lemma* 6.4. *When $p \geq 3$* (*resp.* $p = 2$), *there exists a Riemannian metric $\tilde{g}$ on X, satisfying conditions* (i) *and* (ii) *of Lemma* 6.4 *and*

$$\tilde{g}(x) = g(x), \qquad j_1(\tilde{g})(x) \in N'_1,$$

for which the hypotheses of Lemma 7.7 (*resp. Lemma* 7.8) *hold and for which the sequences* (6.4) *are exact at x for all $k \geq 0$.*

From Propositions 8.1, 8.2 and 6.5, we obtain the following two propositions:

Proposition 8.3. *Assume that K is integrable and that L is a vector bundle of rank ≥ 1. Suppose that there exists a Riemannian metric g on X such that $\mathrm{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. Suppose that either conditions* (i) *and* (ii) *do not hold or that condition* (iii) *holds. Then there exists a Riemannian metric $\tilde{g}$ on X satisfying $j_1(\tilde{g})(x) \in N'_1$ and for which the sequences* (6.4) *are exact at x for all $k \geq 0$. The element $q = j_1(\tilde{g})(x)$ of $N'_{1,x}$ has the following property: for all $q' \in N'_{k,x}$, with $k \geq 1$, satisfying $\pi_1 q' = q$, there exists $q'' \in N'_{k+1,x}$ such that $\pi_k q'' = q'$.*

Proposition 8.4. *Assume that K is integrable and that L is a vector bundle of rank ≥ 1. Suppose that there exists a Riemannian metric g on X such that $\mathrm{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. Suppose that either conditions*

(i) *and* (ii) *do not hold or that condition* (iii) *holds. Then there exists a Riemannian metric* $\tilde{g}$ *on* X *such that* $j_2(\tilde{g})(x)$ *is an element of* $N'_{2,x}$ *which is strongly prolongable* (*for* Ψ *and* s').

From Proposition 8.4, using Malgrange's strong prolongability theorem (see [6]), we obtain the following result:

THEOREM 8.1. *Assume that* X *is a real-analytic manifold of dimension* $n \geq 3$ *and that* R *is a real-analytic symmetric* 2*-form on* X. *Suppose that* $R^\flat$ *has constant rank, that* K *is integrable and that* L *is a vector bundle of rank* ≥ 1. *Let* $x \in X$ *and suppose that there exists a Riemannian metric* g *on* X *such that* $\operatorname{Tr}_g Q_\xi = 0$, *for all* $\xi \in K_x$. *Suppose that either conditions* (i) *and* (ii) *do not hold or that condition* (iii) *holds. Then there exists a real-analytic Riemannian metric* $\tilde{g}$ *on a neighborhood of* x *such that*

$$\operatorname{Ric}(\tilde{g}) = R.$$

9. Existence results

Let R be a section of S^2T^* over X. We suppose that the morphism $R^\flat$ has constant rank and that its kernel K is integrable. We also assume that the mapping $\iota : K \to S^2T^*$, sending $\xi \in K$ into Q_ξ has constant rank. If K' is the kernel of ι, we consider the vector bundle $L = K/K'$, whose rank we denote by m.

Let x be a point of X. We know that the subspace

$$K''_x = \{\, \xi \in T_x \mid Q_\zeta(\xi, \eta) = 0, \text{ for all } \eta \in T_x,\ \zeta \in K_x \,\}$$

of T_x contains K_x. We write $\nu = \dim K''_x$ and $p = n - \nu$; so we have $\nu \geq m$.

Let g be a Riemannian metric on X. We denote by T''_x the orthogonal complement of K''_x in T_x and, for $\eta \in T_x$, we set

$$W_\eta = \{\, \xi \in T''_x \mid g(\xi, \eta) = Q_\zeta(\xi, \eta) = 0, \text{ for all } \zeta \in K_x \,\}.$$

We assume that $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_x$. We suppose moreover that either conditions (i) and (ii) of §8 do not hold or that condition (iii) of §8 holds. We may rephrase these conditions as follows:

(i) we have $m = 2$ and $n = 4$, or $m = 2$, $n = 5$ and $\nu = 3$;

(ii) we have $m = 3$, $n = 6$;

(iii) we have $m = 3$ and $n = 6$ and there exists an orthonormal basis $\{\eta_1, \eta_2, \eta_3\}$ of T''_x satisfying $\dim W_{\eta_i} \geq p - 2$, for all $i = 1, 2, 3$.

From Theorems 6.1 and 8.1, we obtain the following result:

THEOREM 9.1. *Assume that X is a real-analytic manifold of dimension $n \geq 3$ and that R is a real-analytic symmetric 2-form on X. Suppose that $R^\flat$ has constant rank, that K is integrable and that L is a vector bundle. Let $x \in X$ and suppose that there exists a Riemannian metric g on X such that $\mathrm{Tr}_g\, Q_\xi = 0$, for all $\xi \in K_x$. Suppose that either conditions* (i) *and* (ii) *do not hold or that condition* (iii) *holds. Then there exists a real-analytic Riemannian metric $\tilde{g}$ on a neighborhood of x such that*

$$\mathrm{Ric}(\tilde{g}) = R.$$

By Lemma 2.3 or Lemma 4.1, from the preceding theorem we obtain:

THEOREM 9.2. *Assume that X is a real-analytic manifold of dimension $n \geq 3$ and that R is a real-analytic symmetric 2-form on X. Suppose that $R^\flat$ has constant rank, that K is integrable and that L is a vector bundle of rank* 1. *Let $x \in X$ and $\xi \in K_x$. If Q_ξ is not semi-definite, then there exists a real-analytic Riemannian metric g on a neighborhood of x such that*

$$\mathrm{Ric}(g) = R.$$

We remark that, if $R^\flat$ has corank 1, then K is integrable. Thus from Theorem 9.2, we deduce the important special case:

THEOREM 9.3. *Assume that X is a real-analytic manifold of dimension $n \geq 3$ and that R is a real-analytic symmetric 2-form on X. Suppose that $R^\flat$ has corank* 1. *Let $x \in X$ and $\xi \in K_x$. If Q_ξ is not semi-definite, then there exists a real-analytic Riemannian metric g on a neighborhood of x such that*

$$\mathrm{Ric}(g) = R.$$

References

[1] A. L. Besse, "Einstein manifolds," Ergeb. Math. Grenzgeb., Vol. (3) 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[2] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, "Exterior differential systems," Math. Sci. Res. Inst. Publ., Vol. 18, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[3] D. DeTurck, Existence of metrics with prescribed Ricci curvature: local theory, Invent. Math., **65** (1981), 179–207.

[4] ——————, Metrics with prescribed Ricci curvature, in "Seminar on differential geometry," edited by S. T. Yau, Ann. of Math. Studies, No. 102, Princeton University Press, University of Tokyo Press, Princeton, N.J., 1982, 525–537.

[5] J. Gasqui, Sur la résolubilité locale des équations d'Einstein, Compositio Math. **47** (1982), 43–69.

[6] H. Goldschmidt, Existence theorems for analytic linear partial differential equations, Ann. of Math., **86** (1967), 246–270.

[7] ———, Integrability criteria for systems of non-linear partial differential equations, J. Differential Geom., **1** (1967), 269–307.

[8] B. Malgrange, Équations de Lie. II, J. Differential Geom., **7** (1972), 117–141.

[9] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. in Math., **21** (1976), 293–329.

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