9. Applications of Fourier transforms

In this section, we’ll present a number of applications of the Fourier transform. For many applications, it is useful to have a table of Fourier transforms handy. Many advanced calculus books and engineering math books provide such tables, and there are also extensive books of such tables. Computer programs like Maple and Mathematica know about Fourier transforms, too.

We’ll start with the most basic kind of application, to ordinary differential equations. The most important formal property of the Fourier transform is that it maps differential operators with constant coefficients to multiplication by polynomials. This is because of the fundamental property:

\[ \hat{f}' - 2\pi i \xi \hat{f}. \]

A simple example that will illustrate this is provided by the problem

\[ u'' - u = f, \]

in which \( f \) is known and we have to find \( u \). We proceed by taking the Fourier transform of both sides, using the property above, and get

\[ (4\pi^2 \xi^2 + 1)\hat{u} = \hat{f}, \]

which we can solve for \( \hat{u} \):

\[ \hat{u} = \frac{1}{1 + 4\pi^2 \xi^2} \hat{f}. \]

We would be able to recover \( u \) if we knew what function has Fourier transform \( 1/(1 + 4\pi^2 \xi^2) \).

Exercise 1: Show that if \( g(x) = \frac{1}{2} e^{-|x|} \) then

\[ \hat{g}(\xi) = \frac{1}{1 + 4\pi^2 \xi^2}. \]

Now we can use the fact that Fourier transforms turn convolutions into products to conclude that

\[ u = \frac{1}{2} e^{-|x|} \ast f = \frac{1}{2} \int e^{-|x-y|} f(y) \, dy. \]

You can check this by formal differentiation.

**THE HEAT EQUATION**

Consider the problem of heat flow in an infinite rod:

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \]
for \( t > 0 \) and all \( x \), with
\[
\lim_{t \to 0^-} u(x, t) = f(x).
\]
We proceed as above by taking the Fourier transform in the \( x \) variable (the derivative with respect to \( t \) commutes with taking the Fourier transform) to get:
\[
\frac{\partial \hat{u}}{\partial t} = -2\pi^2 \xi^2 \hat{u},
\]
which is an ordinary differential equation for \( \hat{u} \). Note that \( \hat{u} \) is a function of \( \xi \) and \( t \), and we can take the Fourier transform of the initial condition of the heat equation to get an initial condition for the ordinary differential equation for \( \hat{u} \):
\[
\hat{u}(\xi, 0) = \hat{f}(\xi).
\]
The solution of this initial-value problem is
\[
\hat{u}(\xi, t) = \hat{f}(\xi)e^{-2\pi^2 \xi^2 t}.
\]
The inverse Fourier transform of this is the convolution of \( f \) with the inverse Fourier transform of \( e^{-2\pi^2 \xi^2 t} \). As you can check, you get:
\[
u(x, t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2t} f(y) dy.
\]

**Exercise 2**: Find the temperature in the “semi-infinite” rod \( x \geq 0 \) if the initial data \( u(x, 0) = f(x) \) is known and the left end \( (x = 0) \) is held at temperature 0, imposing upon \( f \) whatever technical conditions you require (i.e., Kelvin’s method of images). The answer is
\[
u(x, t) = \frac{1}{\sqrt{2\pi t}} \int_{0}^{\infty} \left( e^{-(x-y)^2/2t} - e^{(x+y)^2/2t} \right) f(y) dy.
\]

**THE WAVE EQUATION**

Next let’s consider the wave equation
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}
\]
with initial data \( u(x, 0) = f(x) \) and \( \partial u/\partial t(x, 0) = g(x) \). We take the transform:
\[
\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2 \xi^2 \hat{u},
\]
and solve for \( \hat{u} \):
\[
\hat{u}(\xi, t) = \hat{f}(\xi) \cos 2\pi \xi t + \frac{\sin 2\pi \xi t}{2\pi x i} \hat{g}(\xi)
\]
\[
= \frac{1}{2} \left( e^{2\pi i t} + e^{-2\pi i t} \right) \hat{f}(\xi) + \frac{1}{2} \int_{-t}^{t} e^{2\pi i y} \hat{g}(\xi) dy.
\]
and then do the inverse transform:
\[
u(x, t) = \frac{1}{2} f(x + t) + f(x - t) + \int_{x-t}^{x+t} g(y) \, dy.
\]
This is precisely the familiar formula of d’Alembert.

**Exercise 3:** Check the various inverse transform formulas used in this derivation.

**THE POISSON SUMMATION FORMULA**

**THE EULER-MACLAURIN FORMULA**

**THE CENTRAL LIMIT THEOREM**

The mathematical content of the central limit theorem of probability theory is that if \(f\) is a non-negative summable (i.e., \(L^1\)) function with \(\int f(x) \, dx = 1, \int xf(x) \, dx = 0\) and \(\int x^2f(x) = 1\), and if \(f^n\) is the \(n\)-fold convolution \(f * \cdots * f\), then
\[
\lim_{n \to \infty} \int_{a\sqrt{n}}^{b\sqrt{n}} f^n = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-x^2/2} \, dx
\]
for any \(-\infty < a < b < \infty\). The probabilistic interpretation is as follows: Think of an infinite number of statistically independent copies \(q_1, q_2, \ldots\) of a statistical quantity \(q\) that are distributed according to the rule
\[
P(a \leq q < b) = \int_{a}^{b} f(x) \, dx,
\]
in other words, \(f\) is the probability density function of each event \(q_i\). The adjective “independent” means that the probabilities multiply:
\[
P(a_1 \leq q_1 < b_1, a_2 \leq q_2 < b_2, \ldots) = \int_{a_1}^{b_1} f(x) \, dx \int_{a_2}^{b_2} f(x) \, dx \cdots
\]
and so you can infer that the sum of \(s_n = e_1 + \cdots + e_n\) is distributed according to the rule:
\[
P(a \leq s_n < b) = \int_{a}^{b} f^n(x) \, dx,
\]
which you can verify by making the substitution \(x_1 + \cdots + x_n = y_k\) for \(k \leq n\) and then by putting \(y_n = x\). The content of the central limit theorem can now be seen to be that the scaled sum \(s_n/\sqrt{n}\) is nearly Gaussian-distributed for large \(n\):
\[
P(a \leq \sqrt{n} s_n < b) = \sqrt{n} \int_{a}^{b} f^n(x\sqrt{n}) \, dx
\]
is approximately
\[ \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx. \]

This fact (though not the proof given here) goes back to de Moivre and Laplace in the eighteenth century.

The key step in the proof of the central limit theorem is to use that fact that
\[ \lim_{n \to \infty} \sqrt{n \hat{f}^* n(x \sqrt{n})} = \lim_{n \to \infty} \left( \hat{f}(\xi / \sqrt{n}) \right)^n = e^{-2\pi^2 \xi^2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]
to check that
\[ \lim_{n \to \infty} \sqrt{n f^n(x \sqrt{n})} k(x) dx = \frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} k(x) dx \]
for every function \( k \in S(\mathbb{R}) \). To do this, use the Fourier inversion formula to write
\[
\int \sqrt{n} f^n(x \sqrt{n}) k(x) dx = \int \sqrt{n} f^n(x \sqrt{n}) \left( \int k(\xi) e^{-2\pi i \xi x} d\xi \right) dx \\
= \int k(\xi) \left( \int \sqrt{n} f^n(x \sqrt{n}) e^{-2\pi i \xi x} dx \right) d\xi \\
= \int \hat{k}(\xi) (\hat{f}(\xi / \sqrt{n}))^n d\xi.
\]

Now consider \((\hat{f}(\xi / \sqrt{n}))^n\) for large \( n \). This function is bounded by
\[ \| \hat{f} \|_\infty \leq \| f \|_1^a = 1, \]
and for any fixed \( \xi \in \mathbb{R} \),
\[
\hat{f}(\xi / \sqrt{n}) = \int e^{-2\pi i \xi x / \sqrt{n}} f(x) dx \\
= \int \left( 1 - \frac{2\pi i \xi x}{\sqrt{n}} - \frac{2\pi^2 \xi^2 x^2}{n} (1 + \delta_n(x)) \right) f(x) dx,
\]
by the Maclaurin expansion for the exponential, in which \( \delta_n \) is bounded and approaches zero pointwise as \( n \to \infty \). Because \( \hat{f} = 1 \), \( f x f = 0 \), and \( f x^2 f = 1 \), we find
\[
\hat{f}(\xi / \sqrt{n}) = 1 - \frac{2\pi^2 \xi^2}{n} \left( 1 + \int x^2 \delta_n(x) f(x) dx \right) \\
= 1 - \frac{2\pi^2 \xi^2}{n} (1 + o(1)),
\]
where the error \( o(1) \) tends to zero as \( n \to \infty \). But then
\[
\left( \hat{f}(\xi / \sqrt{n}) \right)^n = \left( 1 - \frac{2\pi^2 \xi^2}{n} (1 + o(1)) \right)^n
\]
is bounded by 1 and tends to
\[ e^{-2\pi^2 \xi^2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]
as \( n \to \infty \), so
\[
\lim_{n \to \infty} \int \sqrt{n} f^n(x) k(x) \, dx = \lim_{n \to \infty} \hat{k}(\xi/\sqrt{n})^n \, d\xi = \frac{1}{\sqrt{2\pi}} \int \hat{k}(\xi) e^{-x^2/2} d\xi = \frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} k(x) \, dx,
\]
by the dominated convergence theorem. The last line comes from Parseval’s formula, and makes use of the fact that \( e^{-x^2/2} \) is real and is left fixed by the Fourier transform. Technically, we must also approximate the indicator function of the interval \( a \leq x \leq b \) by functions \( k \) in \( S(\mathbb{R}) \), but we have done proofs like this before.

**THE HEISENBERG INEQUALITY**

An important theme in the study of the Fourier transform is the local/global duality between \( f \) and \( \hat{f} \). One instance of this is the operational formula
\[
(2\pi i \xi)^p \frac{d^q \hat{f}}{d\xi^q} = (-2\pi i)^q f,
\]
which relates the decay of a function \( f \in S(\mathbb{R}) \) (or perhaps of \( \hat{f} \)) to the smoothness of \( \hat{f} \) (or \( f \)). Another striking example of it is Heisenberg’s inequality, which states that for \( f \in L^2(\mathbb{R}) \),
\[
\int x^2 |f(x)|^2 \, dx \times \int \xi^2 |\hat{f}(\xi)|^2 \, d\xi \geq \frac{1}{16\pi^2} \|f\|_2^4.
\]
Moreover, the lower bound is attained exactly by constant multiples of \( f = e^{-kx^2} \). We present a proof of this due to Hermann Weyl which is very simple mathematically. The original proof by Heisenberg was entirely different, being based upon a quantum-mechanical picture. For the sake of general culture, it is important to understand what he had in mind. To do this, we need an outline of the formulation of quantum mechanics for a one-dimensional particle.

A “state” of such a particle is a function \( \psi \in L^2(\mathbb{R}) \) such that \( \|\psi\|_2 = 1 \). Then we interpret the integral
\[
\int_a^b \psi \overline{\psi} = \int_a^b |\psi|^2
\]
as the probability of finding the particle in the interval \( a \leq x \leq b \).
An “observable” is a symmetric operator $A$ acting on an appropriate domain $D(A) \subset L^2(\mathbb{R})$. The “average” (or expectation) of $A$ in the state $\psi$ is declared to be

$$\text{average}(A) = \int (A\psi)\overline{\psi}$$

for $\psi$ belonging to the domain of $A$. For example, the “position” of the particle is associated with the operation of “multiplication by $x$” ($A\psi = x\psi$), the domain of which is

$$D(A) = \{ \psi \in L^2(\mathbb{R}) | \| x\psi \|_2 < \infty \},$$

and

$$\text{average}(A) = \int x\psi\overline{\psi} = \int x|\psi|^2.$$

The adjective “symmetric” means the usual thing: $\langle A\psi, \phi \rangle = \langle \psi, A\phi \rangle$, so

$$\text{average}(A) = \int (A\psi)\overline{\psi} = \int \psi(A\psi) = \text{average}(A),$$

so the average is always a real number.

A second important observable is the “momentum” which is associated with the operator

$$B = \frac{1}{2\pi i} \frac{d}{dx},$$

so $B\psi = \psi'(2\pi i)$, which acts on the domain

$$D(B) = \{ \psi \in L^2(\mathbb{R}) | \| \psi' \|_2 = \| 2\pi \xi \hat{\psi} \|_2 < \infty \}.$$

The average of a power of the momentum is

$$\int (B^n\psi)\overline{\psi} = \int \left( \frac{1}{(2\pi i)^n} \frac{d^n\psi}{dx^n} \right)\overline{\psi} = \int \xi^n \hat{\psi}\overline{\psi} = \int \xi^n |\psi|^2$$

so it is natural to interpret

$$\int_a^b \hat{\psi}\overline{\psi} = \int_a^b |\hat{\psi}|^2$$

as the probability that the momentum is in the interval $a \leq \xi \leq b$. Of course, $|\hat{\psi}|^2$ is a probability density function by the Plancherel identity: $\|\hat{\psi}\|_2 = \|\psi\|_2 = 1$.

The position and momentum operators do not commute with one another:

$$AB - BA = \frac{xD - Dx}{2\pi i} = \frac{i}{2\pi},$$

and in quantum mechanics this means that they cannot be measured to an arbitrary degree of precision simultaneously. Heisenberg’s inequality reflects this fact by saying that in any state:

$$\text{average}(A - \text{average}(A))^2 \times \text{average}(B - \text{average}(B))^2 \geq \frac{1}{16\pi^2}.$$
To see that this is actually the same as the first form of Heisenberg’s inequality we stated above, notice that in the state $\psi$ the left side of this inequality can be expressed as:

$$\int (x - \text{average}(A))^2 |\psi(x)|^2 \, dx \times \int (\xi - \text{average}(B))^2 |\hat{\psi}(\xi)|^2 \, d\xi,$$

where $\psi_1(x) = \psi(x + \text{average}(A))$, and you can take advantage of the fact that $\|\psi_1\|_2 = \|\psi\|_2$. A second application of the same trick with $\hat{\psi}_2(\xi) = \hat{\psi}_1(\xi + \text{average}(B))$ reduces this new expression to

$$\int x^2 |\psi_2(x)|^2 \, dx \times \int \xi^2 |\hat{\psi}_2(\xi)|^2 \, d\xi,$$

and the latter is greater than or equal to $1/(16\pi^2)$ since $\|\psi_1\|_2 = \|\psi_2\|_2 = \|\psi\|_2 = 1$.

**Exercise 4:** Check that

$$\text{average}(A - \text{average}(A))^2 \times \text{average}(B - \text{average}(B))^2 \geq \frac{1}{4} \left| \text{average}(AB - BA) \right|^2$$

for any observables $A$ and $B$. Don’t worry about domains and such, just give a formal proof. This is the most general form of Heisenberg’s inequality, which reduces to the previous one if $A = \text{position}$ and $B = \text{momentum}$. Hint: The average of $AB - BA$ is $2i$ times the imaginary part of $\langle B\psi, A\psi \rangle$. Use Schwarz’s inequality to check the bound if average($A$) = average($B$) = 0, and then reduce the general case to this one.

**Proof of Heisenberg’s inequality.** We’ll just do the proof if $f \in S(\mathbb{R})$. For general $f \in L^2$, the standard approximation argument works.

$$4\pi^2 \int x^2 |f(x)|^2 \, dx \int |\xi^2 |\hat{f}(\xi)|^2 \, d\xi = \int |xf(x)|^2 \, dx \int |2\pi i \hat{f}(\xi)|^2 \, d\xi$$

$$= \int |xf(x)|^2 \, dx \int |f'(x)|^2 \, dx \quad \text{Plancherel}$$

$$\geq \left( \int |xf f'| \, dx \right)^2 \quad \text{Schwarz}$$

$$\geq \left( \int \frac{1}{2}(f f' + f' f) \, dx \right)^2$$

$$= \frac{1}{4} \left( \int |f|^2 \, dx \right)^2$$

$$= \frac{1}{4} \|f\|_2^4.$$

Finally, we can explore when the lower bound is actually attained. The key step in the proof of Heisenberg’s inequality is the application of Schwarz’s inequality to the
integral $\int xf'f$, so the bound is attained if and only if $f' = kx\bar{f}$ (almost everywhere) for some complex constant $k$. Since we can approximate of $L^2$ functions by $C^\infty$ ones, we may as well assume that the differential equation is actually satisfied. But then
\[
\frac{1}{x} \left( \frac{f'}{x} \right)' = \frac{1}{x} (k\bar{f})' = |k|^2 f,
\]
and constant multiples of $f = e^{-|k|^2 x/2}$ are the only solutions of this equation that belong to $L^2(\mathbb{R})$. 