Introduction to Fourier analysis

This semester, we're going to study various aspects of Fourier analysis. In particular, we'll spend some time reviewing and strengthening the results from Math 425 on Fourier series and then looking at various applications to partial differential equations and other parts of mathematics and science. Then we'll look at Fourier transforms, and see their similarities to and differences from Fourier series. Again, we'll look at applications to PDEs and to other parts of mathematics and physics. Then, as time permits in the semester, we'll look at Fourier analysis in more general contexts. You'll be surprised at how many different parts of mathematics will come together here.

To get started, let's review a little bit from last semester and think about the early history of Fourier analysis. Recall that the idea of Fourier series is that "any" periodic function of t can be expressed as an infinite trigonometric sum of sines and cosines of the same period T:

$$f(t) = \sum_{n=0}^{\infty} [\hat{f}_+(n) \cos(\frac{2\pi n t}{T}) + \hat{f}_-(n) \sin(\frac{2\pi n t}{T})].$$

The idea of using these expansions first came up in the work of D'Alembert and Euler on the wave equation (for oscillations of a string). The displacement $u(x,t)$ of a string that is stretched over the x-interval $[0,1]$ from its equilibrium configuration, as a function of the time $t \geq 0$ and the place x on the string, is a solution of

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < 1$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0$$

and the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \qquad 0 < x < 1.$$

A solution of this problem is given by D'Alembert's formula:

$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t),$$

where it is understood that f has been extended to be an odd function of period 2 (so it must vanish at $0, \pm 1, \pm 2, \ldots$). In 1748, Euler suggested that the odd periodic extension of f could be expanded as a sine series:

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n) \sin n\pi x,$$

so that the solution of the problem would be

$$u(x,t) = \sum_{n=1}^{\infty} \hat{f}(n) \cos n\pi t \, \sin n\pi x.$$

The formula

$$\hat{f}(n) = 2\int_0^1 f(x) \sin n\pi x \, dx$$

for the coefficients, later associated with Fourier's name, first appeared in a paper of Euler in 1777.

Fourier came into the picture with his study of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

In his *Théorie analytique de la chaleur* in 1822, Fourier was the first to make a serious attempt to prove the covergence of Fourier series for some wide class of functions. Soon after Dirichlet found the proof we discussed last semester, that the Fourier series of a piecewise smooth function converges pointwise. Through the rest of the 1800s, various further studies and improvements of Dirichlet's theorem were pursued by many mathematicians.

At the beginning of the 20th century, the key to understanding Fourier series was developed by Henri Lebesgue. His notion of measurability and the "Lebesgue integral" enabled the generalization of Fourier series to the class of Lebesgue measurable functions f of period 1 for which

$$\|f\|^2 = \int_0^1 |f(x)|^2 \, dx < \infty.$$

The main result of Fourier analysis, which we will prove, is the Riesz-Fischer Theorem of 1907:

Theorem: *If f is a Lebesgue-measurable function of period 1 for which*

$$\|f\|^2 = \int_0^1 |f(x)|^2 \, dx < \infty,$$

then the formula for the Fourier coefficients

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx \quad n \in \mathbb{Z}$$

provides a one-to-one map of the space of functions onto the space of sequences $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ for which

$$\|\hat{f}\|^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 < \infty.$$

Moreover, this map is norm-preserving:

$$\|f\| = \|\hat{f}\|$$

and the associated Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

actually converges to f in the sense of these norms:

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}|^2 dx = 0.$$

A similar development was carried out for the Fourier transform (or Fourier integral)

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\, dx$$

for (non-periodic) f that decay rapidly at infinity. For these we have Plancharel's theorem of 1910:

Theorem: *If f is Lebesgue measurable and*

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty,$$

then the same is true of $\hat{f}$, and

$$\|\hat{f}\| = \|f\|.$$

Moreover, f can be recovered from $\hat{f}$ via the inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi.$$

So our first order of business will be to explore the Lebesgue theory of integration.

1. The Lebesgue integral

We're going to go through half a semester's material in a graduate real analysis course in a day or two, so we won't do too many of the proofs.

To begin, recall how the Riemann integral is defined. If we have a continuous function f defined on an interval $[a, b]$ (with a and b finite and $a < b$), then we pick a *partition* of the interval:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and form the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}),$$

where x_k^* is any point between x_{k-1} and x_k. One verifies that these Riemann sums approach a limit as $n \to \infty$ and the *norm* of the partition (i.e., the largest difference between consecutive x_k's) goes to zero. The limit is defined to be the Riemann integral

$$\int_a^b f(x)\,dx.$$

One way to look at Lebesgue's idea is simply to turn this on its side and subdivide the range instead of the domain of f. First, assuming f is continuous, and thus bounded, on $[a, b]$, we choose numbers $y_0 \le \min f$ and $y_n \ge \max f$, and then subdivide the vertical axis as

$$y_0 < y_1 < y_2 < \cdots < y_n.$$

We then form the sum

$$\sum_{k=1}^{n} y_{k-1} \cdot \mu\{x \,|\, y_{k-1} \le f(x) < y_k\},$$

where $\mu\{x \,|\, y_{k-1} \le f(x) < y_k\}$ is the sum of the lengths of the subintervals of $[a, b]$ where the inequality holds (this is called the *measure* of the set). One then verifies that these sums approach the same limit as the Riemann sums did as $n \to \infty$ and the biggest of the differences $y_k - y_{k-1}$ approaches zero. The new wrinkle in the Lebesgue theory is that we can extend the idea of measure from unions of disjoint intervals to a much larger class of "Lebesgue measurable" sets, and then we will be able to integrate a much larger class of functions.

So our first task is to define the notion of measure and explore some of its properties. First, let's start with an interval $J \subset \mathbb{R}$, which might be a bounded interval $[a, b]$, a half-line $(-\infty, b]$ or $[a, \infty)$, or the whole line. We are going to define measurable subsets of J. To begin, we'll agree that

- *the measure of a (countable) union of non-overlapping intervals is the sum of their lengths, whether or not this sum is finite.*

In particular, the measure of a single point, or a countable set of points, is zero. The notion of measure is then extended to the class of *Borel measurable* sets, which is the smallest collection of subsets of J that contains all subintervals of J and is closed under countable unions, countable intersections, and complementation. This extension is unique provided we define the measure of a countable collection of disjoint

Borel measurable sets to be the sum of their individual measures. To get from Borel to Lebesgue, we just throw in all subsets of sets of measure zero as additional sets of measure zero. Taking countable unions, intersections, and complements, we get the collection of all Lebesgue measurable sets. And when we say "measurable", we always will mean "Lebesgue measurable". A more constructuve way of expressing Lebesgue measure is to say that for any measurable set E,

$$\mu(E) = \inf \mu(C),$$

where C is a covering of E by a countable set of intervals, i.e., $C = \cup_{n=1}^{\infty} I_n \supset E$.

Example (the Cantor set): Take $J = [0,1]$. Any open subset $U \subset J$ is measurable, being the union of a countable number of non-overlapping open intervals (*Exercise:* Prove this). So the measure of a compact (closed) subset $K \subset J$ is to put $\mu(K) = 1 - \mu(U)$, where U is the complement of K in J. For example, if K is the closed set obtained from $[0,1]$ by first removing the middle third $(\frac{1}{3}, \frac{2}{3})$, and then removing the middle thirds of the two remaining pieces, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so forth, then you can show that the measure of K is zero, even though K is uncountable. K is called the Cantor set.

Exercise 1: Here are basic properties of Lebesgue measure. Assume $J = [a,b]$, and $A, B, \ldots$ are measurable subsets of J:

1. $0 \le \mu(A) \le \mu(J) = b - a$
2. $\mu(A) \le \mu(B)$ if $A \subset B$
3. $\mu(J - A) + \mu(A) = \mu(J)$
4. $\mu(\cup_{n=1}^{\infty} B_n) \le \sum_{n=1}^{\infty} \mu(B_n)$
5. $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ if $B_i \cap B_j = \emptyset$ for $i \ne j$
6. $\lim_{n\to\infty} \mu(B_n) = \mu(A)$ if $B_1 \subset B_2 \subset \cdots$ and $\cup_{n=1}^{\infty} B_n = A$
7. $\lim_{n\to\infty} \mu(B_n) = \mu(A)$ if $B_1 \supset B_2 \supset \cdots$ and $\cap_{n=1}^{\infty} B_n = A$, provided $\mu(B_1) < \infty$.

Check that each of statements 4, 5 and 7 implies the other two.

Exercise 2: There are sets that are not measurable. Consider the set obtained by choosing one element from each coset of the rational points in J.

Exercise 3: Suppose $f_1, f_2, \ldots$ are continuous functions on $\mathbb{R}$, and that $f = \lim_{n\to\infty} f_n$ exists pointwise, then $\{x \mid 0 \le f(x) < 1\}$ is measurable. *Hint*: Note that

$$\{x \mid f(x) < 1\} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \mid f_n(x) \le 1 - 1/k\}.$$

A real-valued function f on $J = [0, 1]$ is *measurable* if the set

$$\{x \mid \alpha \le f(x) < \beta\}$$

is measurable for every choice of α and β. The integral of a non-negative measurable function, which can take on the value $+\infty$ at some points, is defined by forming the "Lebesgue sums"

$$\sum_{k=0}^{\infty} \frac{k}{2^n} \cdot \mu(\{x \mid \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\}) + \infty \cdot \mu(\{x \mid f(x) = \infty\})$$

and letting $n \to \infty$, where it is understood that $0 \cdot \infty = 0$. As n increases, the subdivision of the y-axis becomes finer and finer, and the sums increase either to a finite or an infinite limit, which is declared to be the Lebesgue integral of f:

$$\int_J f = \int_0^1 f = \int_0^1 f(x)\, dx.$$

It is important to note that *the Lebesgue integral of a non-negative function always exists, although it may be* $+\infty$.

The Lebesgue integral of a general measurable function f (which may take on positive and negative values) can be obtained by splitting f into its positive part $f^+ = \max(f, 0)$ and its negative part $f^- = \min(f, 0)$ and then declaring the integral to be

$$\int_J f = \int_J f^+ - \int_J (-f^-),$$

provided that at least one of the integrals on the right is less than ∞. If both are less than ∞, the function f is called *summable*. Note that f is summable if and only if $|f| = f^+ - f^-$ is so.

If B is a subset of J and f is a measurable function on J, then the integral of f times the "indicator function" 1_B is said to be the *integral of f over B*:

$$\int_B f = \int_J f \cdot 1_B.$$

Exercise 3. Check that the Lebesgue integral $\int_0^1 f$ of the indicator function f of the rational numbers exists (and equals zero), but that f is *not* integrable in the Riemann sense. Also, check that the class of measurable functions is closed under:

- multiplication by real constants
- addition (and subtraction)
- multiplication (and division) of functions (with obvious precautions about multiplying infinities)

- (countable) infima, suprema, as well as limits superior and inferior.

Exercise 4. Check that any (piecewise) continuous function is measurable.

Exercise 5. Suppose $f_1, f_2, \ldots$ is a sequence of measurable functions. Prove that $\inf f_n$, $\sup f_n$, $\liminf f_n$ and $\limsup f_n$ are measurable, too. (*Hint*: For $f = \liminf f_n$, note that

$$\{x \mid f(x) > a\} = \bigcup_{b>a} \bigcup_{m\geq 1} \bigcap_{n\geq m} \{x \mid f_n(x) \geq b\},$$

where the first union is over the *rational* numbers b that are bigger than a. Explain why this is true.)

Properties of measurable functions: (Assume $f, f_1, f_2, \ldots$ are measurable functions):

1. $\mu(B) = \int_J f$ if f is the indicator function of a measurable subset B of J.
2. $a \cdot \mu(A) \leq \int_A f \leq b \cdot \mu(A)$ if $a \leq f \leq b$.
3. $\int_{A\cup B} f = \int_A f + \int_B f$ if A and B are disjoint.
4. $\int_A cf = c \int_A f$.
5. $\int_A f_1 + f_2 = \int_A f_1 + \int_A f_2$.

The three big convergence theorems:

Monotone convergence theorem: If $0 \leq f_1 \leq f_2 \leq \cdots$ then

$$\lim_{n\to\infty} \int_A f_n = \int_A \lim_{n\to\infty} f_n.$$

The integrals exist automatically, because we allow both sides to be $+\infty$. The condition $0 \leq f_1$ can be replaced by the condition that f_1^- be summable (i.e., that $\int f_1^- > -\infty$).

Exercise 6: If $f_1 \geq f_2 \geq \cdots$ and $\int f_1^+ < \infty$, then then

$$\lim_{n\to\infty} \int_A f_n = \int_A \lim_{n\to\infty} f_n.$$

Exercise 7: If $f_n \geq 0$, then

$$\int_A \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A f_n.$$

Dominated convergence theorem: If for every $n \geq 1$, $|f_n| < h$ where h is a summable function, then

$$\lim_{n\to\infty} \int_A f_n = \int_A \lim_{n\to\infty} f_n.$$

The special case where h is a constant function and $\mu(A)$ is finite is sometimes called the "bounded convergence theorem".

Exercise 8: Check that each of the dominated and monotone convergence theorems implies the other. For instance, consider the function

$$\inf_{k\geq n}(h \pm f_k)$$

to go from monotone to dominated.

Exercise 9: Let $f_n = ne^{-nx}$ and show that

$$\lim_{n\to\infty} \int_A f_n \neq \int_A \lim_{n\to\infty} f_n.$$

Why doesn't this contradict the theorems?

Fatou's Lemma: If $f_n \geq 0$ for all n, then

$$\lim \inf_{n\to\infty} \int_A f_n \geq \int_A \lim \inf_{n\to\infty} f_n.$$

Many of the above properties and all three theorems can be strengthened by permitting the hypotheses to fail on a set of measure zero. In this context, we use the phrase "almost everywhere", and the abbreviation a.e., to indicate this. For example, $\lim_{n\to\infty} f_n = f$ a.e. means that the set of points at which $\lim_{n\to\infty} f_n(x)$ fails to exist, or else exists but fails to agree with $f(x)$, is of measure zero.

To integrate complex-valued functions, we proceed in a pretty natural way. We'll say that f is summable over A if $\int_A |f| < \infty$, where $|f|$ means the complex absolute value. Then the functions $\operatorname{Re} f$ and $\operatorname{Im} f$ will be summable, and we put

$$\int_A f = \int_A \operatorname{Re} f + i \int_A \operatorname{Im} f.$$

Most of the preceding rules hold without change, except property 2 can be replaced by:

Exercise 10: $|\int_A f| \leq \int_A |f|$ (Hint: Use that $|\int f| = \int \operatorname{Re}(fe^{-i\varphi})$ where φ is the argument (complex angle) of $\int f$.)

Because we will need it later in our approach to the Riesz-Fischer theorem, we'll prove

The Borel-Cantelli Lemma: If

$$\sum_{n=1}^{n} B_n < \infty,$$

then almost no point belongs to an infinite number of the sets B_n. In other words,

$$\mu\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} B_k\right) = 0.$$

Exercise 11: Check that this last set is really the set of points x that belong to B_n for infinitely many values of n. (Probabilists sometimes denote this set by B_n i.o., for “infinitely often”.)

Proof of Borel-Cantelli:

$$\mu\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} B_k\right) \leq \mu\left(\bigcup_{k\geq m} B_k\right) \leq \sum_{k\geq m} \mu(B_k)$$

for any $m \geq 1$, and this is the tail of a convergent series and so can be made arbitrarily small by taking m large.

Exercise 12: Prove Chebyshev's inequality:

$$\mu(\{x \in A \,|\, |f(x)| \geq a\}) \leq \frac{1}{a^2}\int_A |f|^2.$$

(Hint: $a^{-2}|f^2| \geq 1$ on the set where $|f| \geq a$.)

Exercise 13: Check that if $f \geq 0$ and if $\int_A f = 0$, then $f = 0$ a.e. (Hint:

$$\{x \,|\, f(x) > 0\} = \bigcup_{n\geq 1}\{x \,|\, f(x) \geq \frac{1}{n}\}.)$$