Math 360: Series of constants and functions

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Series

A **series** is an infinite sum, i,e, the sum of a sequence. To make sense of this, given the sequence

$$\{a_n\} = a_1, a_2, a_3, \ldots$$

we form a new sequence – the **sequence of partial sums** $\{s_n\}$ where

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k
\end{aligned}$$

Sum/limit of a series

If the sequence of partial sums s_n has a limit, say $\lim_{n\to\infty} s_n = S$, then we say that "the series converges to S" or "the sum of the series is S, and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S.$$

Otherwise, the series **diverges**.

For instance, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n!} = e - 1$.

Cauchy criterion

We can bring to bear everything we already know about sequences to series via the sequence of partial sums.

For instance, a series is **Cauchy** if the sequence of partial sums is, in other words given any $\varepsilon > 0$ there is an $N > 0$ such that for all $m, n > N$ we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$$

A series is convergent if and only if it is Cauchy.

In particular (with $n = m + 1$): If a series is convergent, then $\lim_{n\to\infty} a_n = 0$. ($n$th term test). The harmonic series shows that the converse is not true.

Series with positive terms

If $a_k > 0$ for each k, then the sequence of partial sums s_n is monotonically increasing. By the fundamental axiom, the sequence $\{s_n\}$ and hence the series $\sum_{k=0}^{\infty} a_k$, will converge if there is an upper bound to the sequence of partial sums. This observation is at the root of many of the familiar "tests" for convergence:

Comparison test

Suppose $b_k \geq a_k \geq 0$ for every k.

(a) If $\sum b_k$ converges then so does $\sum a_k$

(b) If $\sum a_k$ diverges then so does $\sum b_k$

In both cases the partial sums for one series bound those of the other, either from above or below.

Integral test

Integral test

Suppose $f(x) \geq 0$ is a decreasing function from $[0, \infty)$ to $\mathbb{R}$. Then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if

$$\lim_{M \to \infty} \int_1^M f(x)\,dx$$

exists and is finite.

We usually write that limit as the "improper integral" $\displaystyle\int_1^{\infty} f(x)\,dx$.

Proof: Compare $\displaystyle\sum_{n=1}^{\infty} f(n)$ to $\displaystyle\sum_{n=1}^{\infty} \int_{n-1}^{n} f(x)\,dx$ (for convergence) or to $\displaystyle\sum_{n=1}^{\infty} \int_{n}^{n+1} f(x)\,dx$ (for divergence).

Ratio test

Ratio test

For the series $\sum_{k=1}^{\infty} a_k$ (assume $a_k > 0$ for the time being), suppose

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L.$$

(a) If $L < 1$ then the series converges.
(b) If $L > 1$ then the series diverges.
(c) If $L = 1$ then no conclusion can be drawn

Proof: Comparison with the geometric series $\sum r^n$ for r between L and 1.
Some other tests (limit comparison test, root test) will be discussed in the homework.

Alternating series

A series of the form $\pm\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ (with $a_k > 0$) is called **alternating**.

Alternating series test

If a_k is a positive monotonically decreasing sequence then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges if and only if $a_k \to 0$.

Proof: Consider the subsequence s_{2n} – it's monotonically increasing and bounded above by a_1. Since $a_n \to 0$, the subsequence s_{2n+1} converges to the same limit. But then so does all of s_n.

Absolute convergence

Say that a series $\sum a_n$ of arbitrary numbers **converges absolutely** if $\sum |a_n|$ converges.

Using the Cauchy criterion, an absolutely convergent series is itself convergent.

So we can extend the ratio test by considering $\lim |a_{n+1}/a_n|$.

Pointwise convergence

Now we'll begin consideration of sequences and series of functions (pay attention to whether it's a sequence or a series!).

Definition: (pointwise convergence)

A sequence $\{f_n\}$ or series $\sum f_n$ of functions **converges pointwise** to F in the interval from a to b (open or closed or otherwise) if for every x in the interval the numerical sequence $\{f_n(x)\}$ or series $\sum f_n(x)$ converges to $F(x)$.

Think about $\{e^{-nx}\}$ on $[0, \infty)$ or $\{\arctan nx\}$ on all of $\mathbb{R}$. (Pointwise limits of continuous functions need not be continuous).

Or the sequence $\{(n+1)x^n\}$ on $[0, 1)$. (The integral of the limit is not the limit of the integrals).

Uniform convergence

We need a stronger notion of convergence to preserve desirable properties:

Definition: (uniform convergence)

A sequence f_n of functions on a domain D **converges uniformly** to F if for any $\varepsilon > 0$ there is an $N > 0$ (depending only on ε and not on $x \in D$) such that

$$|F(x) - f_n(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and all } x \in D$$

Define the notion of "uniformly Cauchy" and show that a uniformly Cauchy sequence is uniformly convergent.

As we shall see, the limit of a uniformly convergent sequence of continuous functions is continuous. Likewise for integrable, and the integral of the limit is equal to the limit of the integrals.

The Weierstrass M-test

A result that is often helpful for showing that a series of functions is uniformly convergent is

Theorem (Weierstrass M-test)

Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on some domain D. If there are constants $M_n > 0$ such that $|f_n(x)| < M_n$ for all $x \in D$ and all n, and if the series of numbers $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D

To prove this, note that the M_n series is Cauchy, so the f_n series is uniformly Cauchy.

Three fundamental theorems

We'll look at the relationship between uniform convergence and continuity, integrability and differentiability.

Theorem (uniform convergence and continuity)

Let $\{f_n(x)\}$ be a uniformly convergent sequence of continuous functions on the domain D. Then the limit function is continuous.

Proof: Let $f(x)$ be the limit function. To show that f is continuous at $x \in D$, let $\varepsilon > 0$ be given. Then there is an $N > 0$ so that $|f(x) - f_n(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$ and all $n > N$. For such an n, we have that there is a $\delta > 0$ so that $|f_n(x) - f_n(y)| < \frac{1}{3}\varepsilon$ for all $y \in D$ with $|x - y| < \delta$. Therefore for $y \in D$ with $|y - x| < \varepsilon$ we have

$$|f(x)-f(y)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Three fundamental theorems

Theorem (uniform convergence and integrability)

Let $\{f_n(x)\}$ be a uniformly convergent sequence of integrable functions on $[a, b]$. Then the limit function $f : [a, b] \to \mathbb{R}$ is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

To show: Given $\varepsilon > 0$, $U(f, P) < L(f, P) + \varepsilon$ for some partition P. By uniform convergence, there is $N > 0$ such that for $n > N$ and some partition P, $U(f_n, P) < L(f_n, P) + \frac{1}{3}\varepsilon$ and

$$f_n(x) - \frac{\varepsilon}{3(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{3(b-a)} \quad \text{for all } x \in [a, b]$$

But then $U(f, P) < U(f_n, P) + \frac{1}{3}\varepsilon < L(f_n, P) + \frac{2}{3}\varepsilon < L(f, P) + \varepsilon$.

Three fundamental theorems

Theorem (uniform continuity and differentiability)

Let $\{f_n(x)\}$ be a sequence of continuously differentiable functions on an interval I that converges pointless to $f(x)$ and such that the sequence of derivatives $\{f_n'(x)\}$ converges uniformly on I. Then $f(x)$ is continuously differentiable and $ff'(x) = \lim_{n\to\infty} f_n'(x)$ on I.

Choose $x_0 \in I$ and use the preceding theorem to get

$$f(x)-f(x_0) = \lim_{n\to\infty} f_n(x)-f_n(x_0) = \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)\,dt = \int_{x_0}^{x} \lim_{n\to\infty} f_n'(t)\,dt$$

which shows that $f'(x) = \lim_{n\to\infty} f_n'(x)$, and $f'(x)$ is continuous by uniform convergence of the f_n'.
(There is a version of this theorem that does not assume continuity of the derivatives.)

Corollaries for series

Each of the three preceding theorems has a version for series:

Corollary

Suppose the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ Then

1. If the f_n are continuous, then the sum is continuous.
2. If the f_n are integrable, then the sum is integrable and the series may be integrated term-by-term.
3. If the f_n are continuously differentiable, and the series of derivatives converges uniformly, then the sum is differentiable and the series may be differentiated term by term.

Power series

A series of the form $\sum_{n=0}^{\infty} a_n x^n$ or $\sum_{n=0}^{\infty} a_n(x-b)^n$ is called a **power series** (Taylor series are power series).

Working with the first form: if $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_0 \neq 0$ then $a_n x_0^n \to 0$ as $n \to \infty$, and so $|a_n x_0^n| < M$ for all n. But then for any $0 < r < |x_0|$ and all $x \in [-r, r]$, $|a_n x^n| < M(r/|x_0|)^k$, and the latter are the terms of a convergent geometric series. Therefore the power series converges uniformly on $[-r, r]$.

Choosing r' between r and x_0 and noting that $|a_n r^n| < K(r')^k$ for some K shows that the derivative of the series also converges uniformly on $[-r, r]$, so power series can be differentiated (and integrated) term-by-term (infinitely often) on their open intervals of convergence.

A continuous, nowhere-differentiable function

Let $\varphi_\ell(x)$ be the periodic extension with period 2ℓ of $|x|$ from $[-\ell, \ell]$ to the whole line. Then $\sum_{k=1}^{\infty} \varphi_{1/4^k}(x)$ is continuous on $\mathbb{R}$ but is not differentiable at any point. (Continuity is easy by the M-test $(0 \leq \varphi_k(x) \leq k$ for all $x)$.

Let $x_0 \in \mathbb{R}$. For each k, $\varphi_{1/4^k}(x)$ is monotonic either on the closed interval of length $\varepsilon_\ell = \dfrac{1}{2 \cdot 4^k}$ to the left or the right of x_0 (and $\varphi_{1/4^j}(x)$ is monotonic on the same interval for all $j \leq k$). And if $j > k$, then $\varphi_{1/4^j}(x_0 \pm \varepsilon_\ell) = \varphi_{1/4^k}(x_0)$.

But then choosing $x_k = x_0 \pm \varepsilon_k$, $\dfrac{f(x_k) - f(x_0)}{x_k - x_0}$ is either an even integer or an odd integer (as k is odd or even) and $x_k \to x_0$. So the derivative limit cannot exist for x_0.

Convolutions

A new operation on functions is **convolution**. Given two functions f and g "of compact support" on $\mathbb{R}$ (zero outside a bounded interval), define

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

You can show that $f*g = g*f$ and that $(f*g)*h = f*(g*h)$ so reasonable functions comprise a(n) commutative *algebra* under the operations of addition and convolution.

But there is no identity element. Instead, we have "approximate identities", or "kernels" or "Dirac sequences".

Approximate identities

A sequence of (compactly supported) functions K_n is an approximate identity if

1. $K_n(x) \geq 0$ for all n and all x.
2. Each K_n is continuous and
$$\int_{-\infty}^{\infty} K_n(x)\,dx = 1$$
3. Given any $\varepsilon > 0$ and $\delta > 0$, there is an N so that if $n \geq N$ then
$$\int_{-\infty}^{-\delta} K_n(x)\,dx + \int_{\delta}^{\infty} K_n(x)\,dx < \varepsilon.$$

Usually, K_n is chosen to be an even function, that peaks higher and higher at zero and such that almost all of the integral is concentrated near zero.

Dirac sequences

We will show that for any bounded, piecewise continuous function f, we have

$$f_n(x) = (K_n * f)(x) = \int_{-\infty}^{\infty} f(t) K_n(x - t)\, dt$$

approximates f uniformly on closed intervals where f is continuous.

Proof: First, rewrite the convolution as

$$f_n(x) = (f * K_n)(x) = \int_{-\infty}^{\infty} f(x - t) K_n(t)\, dt$$

and note that

$$f(x) = f(x) \int_{-\infty}^{\infty} K_n(t)\, dt = \int_{-\infty}^{\infty} f(x) K_n(t)\, dt$$

Approximations by Dirac sequences

Therefore $f_n(x) - f(x) = \displaystyle\int_{-\infty}^{\infty} \Big(f(x-t) - f(x)\Big) K_n(t)\, dt.$

Now let $\varepsilon > 0$ be given. On the closed, bounded interval there is a δ such that

$$|f(x-t) - f(x)| < \varepsilon \quad \text{if} \quad |t| < \delta.$$

Next, if M is a bound for f then choose N so that for $n \geq N$,

$$\int_{-\infty}^{-\delta} K_n(t)\, dt + \int_{\delta}^{\infty} K_n(t)\, dt < \frac{\varepsilon}{2M}.$$

Now

$$|f_n(x) - f(x)| \leq \int_{-\infty}^{\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} |f(x-t) - f(x)| K_n(t)\, dt.$$

The first and third of these add up to less than ε because of the above bound (using triangle inequality on $|f(x-t) - f(x)|$ as well as the bound on f).

Approximations by Dirac sequences (cont'd)

For the middle integral in

$$|f_n(x) - f(x)| \leq \int_{-\infty}^{\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} |f(x-t) - f(x)| K_n(t)\, dt.$$

use that if $|x - t| < \delta$ then $|f(x-t) - f(x)| < \varepsilon$ and so

$$\int_{-\delta}^{\delta} |f(x-t) - f(x)| K_n(t)\, dt \leq \varepsilon \int_{-\delta}^{\delta} K_n(t)\, dt < \varepsilon.$$

So we're done.

The point of all this is that the f_n often have better properties than f (differentiability, etc..)

Weierstrass approximation theorem

An application of this Dirac idea is:

Theorem (Weierstrass approximation theorem)

Let f be a continuous function on the closed interval $[a, b]$ Then f can be uniformly approximated by polynomials on $[a, b]$, that is, there is a sequence of polynomials $\{p_n(x)\}$ such that $p_n \to f_n$ uniformly on $[a, b]$.

By the change of variables $u = (x - a)/(b - a)$ we can reduce to the case $[a, b] = [0, 1]$, and by subtracting a linear function from f we can further assume that $f(0) = f(1) = 0$. Then extend f to be zero outside of $[0, 1]$, so f is bounded and continuous on all of $\mathbb{R}$.

Weierstrass approximation theorem: Kernel functions

The Dirac sequence we will use is

$$K_n(t) = \begin{cases} \dfrac{1}{c_n}(1 - t^2)^n & \text{if } -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{where } c_n = \int_{-1}^{1} (1-t^2)^n \, dt$$

Since $c_n > 0$ we'll have $K_n(t) \ge 0$ and continuous for all t and $\displaystyle\int K_n(t)\, dt = 1$. It is easy to see that

$$f_n(x) = (K_n * f)(x) = \int_{-\infty}^{\infty} f(t) K_n(x-t)\, dt = \int_0^1 f(t) K_n(x-t)\, dt$$

is a polynomial for each n. So we have only to show that given any $\varepsilon > 0$ and $\delta > 0$, there is an N so that if $n \ge N$ then

$$\int_{-\infty}^{-\delta} K_n(x)\, dx + \int_{\delta}^{\infty} K_n(x)\, dx < \varepsilon.$$

Estimating c_n

FIrst,

$$\frac{c_n}{2} = \int_0^1 (1-t^2)^n\,dt = \int_0^1 (1+t)^n(1-t)^n\,dt \ge \int_0^1 (1-t)^n\,dt = \frac{1}{n+1}$$

so $c_n \ge 2/(n+1)$. Give $\delta > 0$ we have

$$\begin{aligned}\int_\delta^1 K_n(t)\,dt &= \int_\delta^1 \frac{(1-t^2)^n}{c_n}\,dt \le \int_\delta^1 \frac{(n+1)}{2}(1-\delta^2)^n\,dt \\ &\le \frac{n+1}{2}(1-\delta^2)^n(1-\delta)\end{aligned}$$

Since $1-\delta^2 < 1$, we have that this last quantity $\to 0$ as $n \to \infty$, so it will eventually be $< \varepsilon$ for any $\varepsilon > 0$, concluding the proof.