MATH 360 –Practice Problems for Exam 1

Due Monday, October 4, 2017

1. (a) Show that $x^n \to 0$ for all $x \in (-1,1)$

(b) Show that $\dfrac{x^n}{n!} \to 0$ for all $x \in \mathbb{R}$.

(a) If $|x| < 1$, then $\{|x^n|\}$ is a decreasing sequence (because $|x^{n+1}| = |x||x^n| < 1 \cdot |x^n| = |x^n|$ – strict inequality unless $x = 0$ in which case the sequence x^n is the constant sequence $0, 0, 0, \ldots$), bounded below by 0 (since absolute values are non-negative). Therefore the sequence of absolute values converges, say to L. If $x \neq 0$, then

$$L = \lim_{n\to\infty} |x^n| = \lim_{n\to\infty} |x^{n+1}| = \lim_{n\to\infty} |x||x^n| = |x|L.$$

so $(1 - |x|)L = 0$ and $0 < |x| < 1$, so $L = 0$. Since $|x| \to 0$, then $x \to 0$ is immediate.

(b) For any x, we know that there is a positive integer N such that $N > 2|x|$. Therefore, if $n > N$ then

$$\left|\frac{x^{n+1}}{(n+1)!}\right| < \frac{|x|}{n+1}\left|\frac{x^n}{n!}\right| < \frac{1}{2}\left|\frac{x^n}{n!}\right|.$$

So the sequence $\left\{\left|\dfrac{x^n}{n!}\right|\right\}$ is decreasing after position N, and in fact the $N+n$th term

$$\left|\frac{x^{N+n}}{(N+n)!}\right| < \frac{1}{2^n}\left|\frac{x^N}{N!}\right|.$$

The sequence on the right side is a constant times $(\frac{1}{2})^n$ and so approaches zero by part (a), so the absolute value of $x^n/n! \to 0$, and we're done.

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2. Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences and suppose $x_n < y_n$ for every $n \geqslant 0$. Prove that $\lim\limits_{n\to\infty} x_n \leqslant \lim\limits_{n\to\infty} y_n$.

Suppose $x_n \to L$ and $y_n \to M$. If $L > M$ then choose $\varepsilon < \frac{1}{2}(L - M)$. Then there is an integer N such that $|x_n - L| < \varepsilon$ and $|y_n - M| < \varepsilon$ for all $n > N$. Therefore, for $n < N$ we have

$$x_n > L - \varepsilon > L - \frac{L-M}{2} = \frac{L+M}{2} = M + \frac{L-M}{2} > M + \varepsilon > y_n,$$

contradicting the hypothesis that $x_n < y_n$ for all n. Therefore $L \leqslant M$.

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3. Let $S \subset \mathbb{R}$. If $S \neq \varnothing$, and $c = \sup S$ and $c \notin S$, prove that there is a sequence $\{x_n\}$ such that $x_n \in S$ and $x_n \to c$.

If $c = \sup S$ then for every $\varepsilon > 0$ there is an $x \in S$ such that $0 \leqslant c - x < \varepsilon$, and $c - x \neq 0$ because $c \notin S$. . Define the sequence $\{x_n\}$ as follows: for each n, let x_n be an element of S such that $c - x_n < \frac{1}{n}$. Since, given any $\varepsilon > 0$ we will have $\frac{1}{n} < \varepsilon$ for n sufficiently large (by the Archimedean property), we get $|c - x_n| < \frac{1}{n} < \varepsilon$, so the sequence $\{x_n\}$ converges to c.

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4. Let $f := [a, b] \to \mathbb{R}$ be monotonically increasing, i.e., if $x \geqslant y$ then $f(x) \geqslant f(y)$. Show that $\inf\{f(x) \mid x > c\} \geqslant f(c)$. Then state and prove the corresponding statement for sup. Together, those two statements imply that

$$\inf\{f(x) \mid x > c\} \geqslant \sup\{f(x) \mid x < c\}.$$

Finally, show that f is continuous at c if and only if $\inf\{f(x) \mid x > c\} = \sup\{f(x) \mid x < c\}$.

Let $S = \{f(x) \mid x > c\}$ and $T = \{f(x) \mid x < c\}$.

Since all the elements of S are values of the function at points $x > c$, we have $s \geqslant f(c)$ for all $s \in S$. Thus $f(c)$ is a lower bound for S and by the definition of infimum, $\inf S$ is greater than or equal to any lower bound for S, thus $\inf S \geqslant f(c)$. Likewise, all the elements of T are values of the function at points $x < c$, so we have $t \leqslant f(c)$ for all $t \in T$. Thus $f(c)$ is an upper bound for T and by the definition of supremum, $\sup T$ is less than or equal to any upper bound for T. Thus $\sup T \leqslant f(c)$.

For the last sentence, first suppose f is continuous at c. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ provided $|x - c| < \delta$. By choosing $\varepsilon_n = 1/n$ and choosing x_n^+ such that $x{+}_n > c$ and $x_n^+ - c < \delta_n$ we get a sequence of points of S for which $f(x_n^+) - f(c) < \frac{1}{n}$. Therefore $f(c) = \inf S$.

Likewise, if we choose x_n^- such that $x_n^- < c$ and $c - x_n^- < \delta_n$, we get a sequence of points of T for which $f(c) - f(x_n^-) < \frac{1}{n}$. Therefore $f(c) = \sup T$. This shows that $\inf S = \sup T$.

Conversely, suppose that $\inf S = \sup T$. Since from the previous parts we have $\sup T \leqslant f(c) \leqslant \inf S$ it follows that $f(c) = \inf S$ and $f(c) = \sup T$. Now let $\varepsilon > 0$ be given. We know that there is a point $x_+ > c$ for which $f(x_+) - f(c) = f(x_+) - \inf S < \varepsilon$, and once you find one such point, this inequality will hold for all points between c and x_+ because the function is increasing. Likewise we know there is a point $x_- < c$ for which $f(c) - f(x_-) = \sup T - f(x_-) > \varepsilon$ and this inequality is true for all points between x_- and c. So let $\delta = \min\{x_+ - c\,,\, c - x_-\}$. We've shown that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows f is continuous at c.

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5. Let f be twice differentiable for all $x \in \mathbb{R}$, and suppose $f''(x) > 0$ for all x. Prove that $f(x) > f(0) + f'(0)x$ for all $x \neq 0$.

Let $g(x) = f(x) - (f(0) + f'(0)x)$. Then $g(0) = 0$ and $g'(0) = 0$ and $g''(x) = f''(x) > 0$ for all x. If we can show that $g(x) > 0$ for all $x \neq 0$ then this will imply the result.

If there is a point $b > 0$ where $g(b) \leqslant 0$ then by the mean value theorem there is a point c_1 between 0 and b (so $c_1 > 0$) where

$$g'(c_1) = \frac{g(b) - g(0)}{b - 0} = \frac{g(b)}{b} \leqslant 0.$$

But then, again by the mean value theorem, there is a point c_2 between 0 and c_1 where

$$g''(c_2) = \frac{g'(c_1) - g'(0)}{c_1 - 0} = \frac{g'(c_1)}{c_1} \leqslant 0$$

which contradicts the hypothesis that $g''(x) > 0$ for all x.

On the other hand if there is a point $b < 0$ where $g(b) \leqslant 0$ then by the mean value theorem there is a point c_1 between b and 0 (so $c_1 < 0$) where

$$g'(c_1) = \frac{g(0) - g(b)}{0 - b} = \frac{g(b)}{b} \geqslant 0$$

(numerator $\leqslant 0$ and denominator < 0). But then, again by the mean value theorem, there is a point c_2 between c_1 and 0 where

$$g''(c_2) = \frac{g'(0) - g'(c_1)}{0 - c_1} = \frac{g'(c_1)}{c_1} \leqslant 0$$

(numerator $\geqslant 0$ and denominator < 0) which contradicts the hypothesis that $g''(x) > 0$ for all x.

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6. Let f be twice differentiable on $(-1, 1)$ and continuous on $[-1, 1]$ and suppose $f(1) = f(-1) = 1$ and $f(0) = 0$. Show that there is a point $c \in (-1, 1)$ where $f''(x) \geqslant 2$.

I really liked this proof from class (Jeffrey Y?): Let $g(x) = f(x) - x^2$. Then $g(1) = g(0) = g(-1) = 0$, and $g''(x) = f''(x) - 2$. So if we can show there is a point c where $g''(c) \geqslant 0$, then we will have $f''(c) = g''(c) + 2 \geqslant 2$. But the mean value theorem guarantees us points a and b such that $-1 < a < -0$ and $0 < b < 1$ where $g'(a) = g'(b) = 0$. And using the mean value theorem again between a and b gives us a c such that $a < c < b$ and $g''(c) = 0$, and we are done.

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7. (a) Let f be differentiable and suppose $f'(x)$ is continuous on some open interval containing a point c where $f(c) = c$. Furthermore, suppose $|f'(c)| < 1$. Show that if x_0 is sufficiently close to c then the sequence defined by $x_n = f(x_{n-1})$ for all $n \geqslant 1$ will converge to c. Give an estimate of how far away x_0 can be from c and still have this work.

(b) Let $f(x) = \dfrac{1}{2}\left(x + \dfrac{17}{x}\right)$, let $x_0 = 4$ and create the sequence described in part (a).

Prove that it converges and find its limit. For which values of x_0 will this work? What if you replace 17 by another (positive) number?

(c) What happens in part (a) if $|f'(c)| > 1$?

(a) If $|f'(c)| < 1$, then in fact there is a number $\eta < 1$ such that $|f'(c)| \leqslant \eta$. This means that there is a $\delta > 0$ such that for any x such that $|x - c| < \delta$ we will have $|f(x) - f(c)| < \eta|x - c|$. Since $f(c) = c$ this means that $|f(x) - c| < \eta|x - c|$. So if we choose x_0 so that $|x_0 - c| < \delta$, then we'll have $|x_1 - c| = |f(x_0) - c| < \eta|x_0 - c|$. And inductively we'll have

$$|x_n - c| < \eta^n|x_0 - c|$$

and since $\eta < 1$, we'll get $\eta^n \to 0$ (by problem 1a), so $x_n \to c$. We can choose x_0 in the interval around c where $|f'(x)| < 1$.

(b) Check:

$$f'(x) = \frac{1}{2}\left(1 - \frac{17}{x^2}\right)$$

which is in fact less than $\frac{1}{2}$ in absolute value if $x > \sqrt{8.5}$. But in fact for any positive value of x_0, we'll have

$$\begin{aligned} x_1^2 - 17 &= \frac{1}{4}\left(x_0^2 + 34 + \frac{17^2}{x_0^2}\right) - 17 \\ &= \frac{1}{4}\left(x_0^2 - 34 + \frac{17^2}{x_0^2}\right) \\ &= \frac{1}{4}\left(x_0 - \frac{17}{x_0}\right)^2 > 0 \end{aligned}$$

so it follows that x_1 is larger than $\sqrt{17}$ in any case (so the sequence converges – and it will converge to $\sqrt{17}$ by calculating the fixed point of f). And then we have inductively that $x_n > \sqrt{17}$ for all $n > 0$, and it is interesting to note that

$$x_{n+1} - \sqrt{17} = \frac{(x_n - \sqrt{17})^2}{2x_n} < \frac{(x_n - \sqrt{17})^2}{2\sqrt{17}}$$

which shows that the convergence is *quadratic* – each subsequent x_{n+1} agrees with $\sqrt{17}$ to twice as many decimal places that x_n.

(c) If $|f'(c)| > 1$, the analysis of part (a) shows that $|f(x) - c| > |x - c|$, so the sequence of points moves farther and farther away from c.