MATH 360 – Final Exam Sample Problems
Thursday, December 14, 2017

1. Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the closed interval $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

---

2. Let $f(x)$ be a continuous function on the closed interval $0 \leqslant x \leqslant 1$, and such that $f(0) = 1$, $f(\frac{1}{2}) = 2$ and $f(1) = 3$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx$$

exists, and compute the limit.

---

3. Let $\{b_n\}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |b_n| = 1$$

and let $f(x)$ be the function given by

$$\sum_{n=1}^{\infty} b_n \cos(nx).$$

Prove that the series converges and that f is continuous on all of $\mathbb{R}$. Is f uniformly continuous?

---

4. Let $f\colon X \to Y$ be a continuous mapping from the metric space (X, d_X) to the metric space (Y_d, Y). For $A \subset X$ be a subset of X we write $f(A)$ for $\{y \in Y \mid y = f(a) \text{ for some } a \in A\}$, and for $B \subset Y$ a subset of Y we write $f^{-1}(B)$ for $\{x \in X \mid f(x) \in B\}$. For each of the following, give a proof or a counterexample:

(a) If $A \subset X$ is connected, then $f(A) \subset Y$ is also connected.

(b) If $B \subset Y$ is connected, then $f^{-1}(B) \subset X$ is also connected.

(c) If $A \subset X$ is sequentially compact, then $f(A) \subset Y$ is also sequentially compact.

(d) If $B \subset Y$ is sequentially compact, then $f^{-1}(B) \subset X$ is also sequentially compact.

5. Consider the set $\mathbb{Q}$ of rational numbers with its usual metric.

(a) Is every closed, bounded subset of $\mathbb{Q}$ sequentially compact?

(b) Show that every continuous function $f\colon \mathbb{R} \to \mathbb{Q}$ is constant.

Justify your assertions.

---

6. Suppose f is a twice-differentiable function which satisfies the differential equation

$$\frac{d^2 f}{dx^2} = -(2 + e^{-x}) f(x)^2$$

for $x \geqslant 0$. Suppose $f(0) = 1$ and $f'(0) = 0$. (Do not attempt to solve the equation.) Sketch the graph of f and show that $f(x) = 0$ for one and only one positive value of x.

---

7. Suppose f is a continuous function defined on the whole real line which is periodic with period one (so $f(x + 1) = f(x)$ for all real x). Suppose

$$\int_0^1 f(x)\, dx = 1 \qquad \text{and} \qquad f(0) = 2.$$

Compute the limits

$$\lim_{c\to\infty} \int_0^1 f(cx)\, dx \qquad \text{and} \qquad \lim_{c\to 0} \int_0^1 f(cx)\, dx$$

and justify your answers.

---

8. Say $a_n > 0$ are a sequence of positive real numbers and $a_n \to A$ as $n \to \infty$. Either prove that $A \geqslant 0$ or provide a counterexample.

---

9. For each of the following, give either a proof or a counterexample.

(a) Let f be a continuous real-valued function on the open interval $0 < x < 3$. Must f be uniformly continuous on the open interval $1 < x < 2$?

(b) Suppose instead that f is only assumed to be continuous on the open interval $0 < x < 2$. Must f be uniformly continuous on the open interval $1 < x < 2$?

---

10. Write the equivalent integral equation formulation for the initial-value problem $y' = y$, $y(0) = 1$. Then carry out the first few iterations of the contraction mapping proof of existence and show explicitly that they converge to the solution of the initial-value problem.