MATH 360 – Exam 2

Wednesday, November 8, 2017

Name ______________________________

1. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, and suppose that for every continuous function $g\colon [a,b] \to \mathbb{R}$,

$$\int_a^b f(x)g(x)\,dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a,b]$.

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2. Let $v(t)$ be the velocity of an object for $t \in [0, T]$ You know that $v(t)$ is the derivative with respect to t of $x(t)$, the position of the object at time t. Assume that $v(t) > 0$ for all t.

(a) What is the average velocity with respect to t?

(b) Show that $x(t)$ is an invertible function of t. What are its domain and range?

(c) Since x is an invertible function of t, we can consider v as a function of x. Show that the average of v with respect to x is greater than or equal to the average of v with respect to t. At a key point, you will need to use one of those famous inequalities.

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3. Let $f(x)$ and $g(x)$ be functions for which their Taylor series (centered at $x = 0$) converge to the functions themselves for $x \in [-r, r]$.

(a) Show that the Taylor series for $cf(x)$ converges to $cf(x)$ for $x \in [-r, r]$.

(b) Show that the Taylor series for $xf(x)$ converges to $xf(x)$ for $x \in [-r, r]$.

(c) Show that the Taylor series for $f(x) + g(x)$ converges to $f(x) + g(x)$ for $x \in [-r, r]$.

(d) Suppose the Taylor series for $\varphi(x)$ is

$$f_0 + f_1 x + f_2 x^2 + \cdots$$

where $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$, so f_n is the nth Fibonacci number. What is $\varphi(x)$?

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4. Let

$$f(x) = \begin{cases} x & \text{for} \quad 0 \leqslant x \leqslant 1 \\ 3 - x & \text{for} \quad 1 < x \leqslant 2 \end{cases}$$

Prove carefully (upper and lower sums etc., and "mind the gap") that f is integrable on $[0, 2]$ and evaluate the integral.

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5. Let

$$f(x) = \int_0^x \frac{1}{1+t^4}\, dt.$$

Prove that f is uniformly continuous on all of $\mathbb{R}$. **Hint**: Do not attempt to evaluate the integral!!

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6. Suppose $g\colon \mathbb{R} \to \mathbb{R}$, with bounded derivative (say $|g'| \leqslant M$). Fix $\varepsilon > 0$ and define

$$f(x) = x + \varepsilon g(x).$$

Prove that f is one-to-one if ε is sufficiently small.

7, Prove that for every positive integer n,

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$