

MATH 360 – Exam 2
Wednesday, November 8, 2017

Name _____

1. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function, and suppose that for every continuous function $g: [a, b] \rightarrow \mathbb{R}$,

$$\int_a^b f(x)g(x) dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

We can, in particular, choose $g = f$, and so we know that the integral of f^2 is zero. If there were a point x_0 where $f(x_0) \neq 0$ then (because of the continuity of f and hence of f^2) there would be a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x)^2 - f(x_0)^2| < \frac{1}{2}f(x_0)^2$, which implies that $f(x)^2 > \frac{1}{2}f(x_0)^2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But then

$$\int_a^b f(x)^2 dx = \int_a^{x_0-\delta} f(x)^2 dx + \int_{x_0-\delta}^{x_0+\delta} f(x)^2 dx + \int_{x_0+\delta}^b f(x)^2 dx > 0 + \delta f(x_0)^2 + 0 = \delta f(x_0)^2 > 0$$

which is a contradiction.

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2. Let $v(t)$ be the velocity of an object for $t \in [0, T]$. You know that $v(t)$ is the derivative with respect to t of $x(t)$, the position of the object at time t . Assume that $v(t) > 0$ for all t .

(a) What is the average velocity with respect to t ?

(b) Show that $x(t)$ is an invertible function of t . What are its domain and range?

(c) Since x is an invertible function of t , we can consider v as a function of x . Show that the average of v with respect to x is greater than or equal to the average of v with respect to t . At a key point, you will need to use one of those famous inequalities.

(a) The average velocity with respect to t is $\bar{v}_t = \frac{1}{T} \int_0^T v(t) dt$.

(b) Because $\frac{dx}{dt} = v(t) > 0$, we have that $x(t)$ is a (strictly) monotonically increasing function of t , i.e, $x(t_2) > x(t_1)$ if and only if $t_2 > t_1$ (for $t_1, t_2 \in [0, T]$). So $x(t)$ is a one-to-one map from $[0, T]$ to $[x(0), x(T)]$, and thus invertible (by the intermediate value theorem – for every $x_0 \in [x(0), x(T)]$ there is a unique $t_0 \in [0, T]$ such that $x(t) = x_0$ and we can define $x^{-1}(x_0) = t_0$).

(c) The average value of v with respect to x is

$$\bar{v}_x = \frac{1}{x(T) - x(0)} \int_{x(0)}^{x(T)} v(x) dx.$$

Make the change of variables $x = x(t)$ (so $dx = x'(t) dt = v(t) dt$) in the integral (being careful to change the limits of integration as well) and get

$$\bar{v}_x = \frac{1}{x(T) - x(0)} \int_0^T v(t)^2 dt = \frac{\int_0^T v(t)^2 dt}{\int_0^T v(t) dt}$$

since v is the derivative of x with respect to t . Now the temptation to use Cauchy-Schwarz is almost overwhelming – with $f = 1$ and $g = v$, the inequality $(\int fg)^2 \leq (\int f^2)(\int g^2)$ becomes

$$\left(\int_0^T v(t) dt \right)^2 \leq \int_0^T 1^2 dt \int_0^T v(t)^2 dt = T \int_0^T v(t)^2 dt.$$

and so

$$\bar{v}_x = \frac{\int_0^T v(t)^2 dt}{\int_0^T v(t) dt} \geq \frac{\left(\int_0^T v(t) dt \right)^2}{T \int_0^T v(t) dt} = \frac{1}{T} \int_0^T v(x) dx = \bar{v}_t$$

which is what we were trying to show.

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3. Let $f(x)$ and $g(x)$ be functions for which their Taylor series (centered at $x = 0$) converge to the functions themselves for $x \in [-r, r]$.

(a) Show that the Taylor series for $cf(x)$ converges to $cf(x)$ for $x \in [-r, r]$.

(b) Show that the Taylor series for $xf(x)$ converges to $xf(x)$ for $x \in [-r, r]$.

(c) Show that the Taylor series for $f(x) + g(x)$ converges to $f(x) + g(x)$ for $x \in [-r, r]$.

(d) Suppose the Taylor series for $\varphi(x)$ is

$$f_0 + f_1x + f_2x^2 + \dots$$

where $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$, so f_n is the n th Fibonacci number. What is $\varphi(x)$?

Let $p_n(x)$ be the Taylor polynomial of degree n (centered at $x = 0$) for $f(x)$ and let $q_n(x)$ be the Taylor polynomial for $g(x)$. For the first three parts, we know that for each $x \in [-r, r]$, we have that $p_n(x) \rightarrow f(x)$ and $q_n(x) \rightarrow g(x)$ as $n \rightarrow \infty$.

For parts (a)-(c), standard properties of limits of sequences (limit of the sum is the sum of the limits, and limit of a constant times a sequence is the constant times the limit of the sequence) shows that $cp_n(x) \rightarrow cf(x)$, $xp_n(x) \rightarrow xf(x)$ and $p_n(x) + q_n(x) \rightarrow f(x) + g(x)$. The only issue that remains is to show that $cp_n(x)$, $xp_{n-1}(x)$ and $p_n(x) + q_n(x)$ actually are the n th degree Taylor polynomials of $cf(x)$, $xf(x)$ and $f(x) + g(x)$, respectively.

(a) For $a(x) = cf(x)$, it is trivial that $a^{(n)}(x) = cf^{(n)}(x)$, so $cp_n(x)$ is the n th degree Taylor polynomial of $cf(x)$.

(b) For $b(x) = xf(x)$, we have $b(0) = 0$ and $b^{(n)}(x) = xb^{(n)}(x) + nb^{(n-1)}(x)$. Therefore $b^{(n)}(0) = nf^{(n-1)}(0)$, so the coefficient of x^n in the Taylor polynomial of $xf(x)$ is $nb^{(n-1)}(0)/n! = f^{(n-1)}(0)/(n-1)!$, which is the coefficient of x^{n-1} in the Taylor polynomial of $f(x)$. Therefore the n th degree Taylor polynomial of $xf(x)$ is $xp_{n-1}(x)$.

(c) Finally for $c(x) = f(x) + g(x)$ it is again trivial that $c^{(n)}(x) = f^{(n)}(x) + g^{(n)}(x)$ so $p_n(x) + q_n(x)$ is the n th degree Taylor polynomial of $f(x) + g(x)$.

(d) If we let $g(x) = xf(x) + x^2f(x)$ then the Taylor series for $g(x)$ is (using (b) and (c) above)

$$\begin{aligned}g(x) &= f_0x + (f_1 + f_0)x^2 + (f_2 + f_1)x^3 + (f_3 + f_2)x^4 + \dots \\&= 0x + f_2x^2 + f_3x^3 + f_4x^4 + \dots \\&= f(x) - x\end{aligned}$$

We conclude that $xf(x) + x^2f(x) = f(x) - x$, or

$$f(x) = \frac{x}{1 - x - x^2}.$$

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4. Let

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 3 - x & \text{for } 1 < x \leq 2 \end{cases}$$

Prove carefully (upper and lower sums etc., and “mind the gap”) that f is integrable on $[0, 2]$ and evaluate the integral.

This is overly clever, but it's probably the easiest way to do it without invoking bounded variation or something: Write $f(x) = g(x) + h(x)$ where

$$g(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 < x \leq 2 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } 1 < x \leq 2 \end{cases}$$

Then g is continuous, hence integrable (or if you must:

$$\begin{aligned} \int_0^2 g(x) dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^{2n} g\left(\frac{k}{n}\right) \frac{1}{n} = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{k}{n^2} + \sum_{k=n+1}^{2n} \left(1 - \frac{k-n}{n}\right) \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} 1 = 1 \end{aligned}$$

where in the last sum in the first line we split 2 up as $1 + n/n$ and noticed that the sum of $(k-n)/n$ for k going from $n+1$ to $2n$ is exactly the same as the sum of k/n for k going from 1 to n , so they cancel.) So g is integrable and its integral is 1.

For the integral of $h(x)$, use partitions $P_n = \{0, 1 - 1/n, 1 + 1/n, 2\}$ Then clearly

$$L(h, P_n) = 0 + 0 + \left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$$

and

$$U(h, P_n) = 0 + \frac{2}{n} + \left(1 - \frac{1}{n}\right) = 1 + \frac{1}{n}$$

and both approach 1 as $n \rightarrow \infty$. So h is also integrable and its integral is also 1.

Therefore f , being the sum of two integrable functions, is integrable and its integral is 2.

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5. Let

$$f(x) = \int_0^x \frac{1}{1+t^4} dt.$$

Prove that f is uniformly continuous on all of \mathbb{R} . **Hint:** Do not attempt to evaluate the integral!!

Since $t^4 \geq 0$ for all $t \in \mathbb{R}$, we have $\frac{1}{1+t^4} \leq 1$ for all $t \in \mathbb{R}$. So given $\varepsilon > 0$, choose $\delta = \varepsilon$ and so if $|x - y| < \delta$ then

$$|f(y) - f(x)| = \left| \int_x^y \frac{1}{1+t^4} dt \right| \leq \left| \int_x^y dt \right| = |y - x| < \delta = \varepsilon$$

Since the δ we chose does not depend on x or y , $f(x)$ is uniformly continuous (on all of \mathbb{R}).

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6. Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$ and define

$$f(x) = x + \varepsilon g(x).$$

Prove that f is one-to-one if ε is sufficiently small.

Take $\varepsilon < 1/(2M)$. Then $|\varepsilon g'(x)| < \frac{1}{2}$, in other words, $-\frac{1}{2} < \varepsilon g'(x) < \frac{1}{2}$. But then $f'(x) = 1 + \varepsilon g'(x)$ satisfies

$$\frac{1}{2} = 1 - \frac{1}{2} < 1 + \varepsilon g'(x) = f'(x) < 1 + \frac{1}{2}$$

The important thing is that $f'(x) > \frac{1}{2}$, so by the mean value theorem,

$$|f(x) - f(y)| = |f'(c)||x - y| > \frac{1}{2}|x - y|$$

which shows that if $x \neq y$ then $f(x) \neq f(y)$, in other words, f is one-to-one.

7, Prove that for every positive integer n ,

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

Recall the definition of the logarithm:

$$\ln(x) = \int_1^x \frac{1}{t} dt,$$

so (noting that $1 + \frac{1}{n} = \frac{n+1}{n}$)

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{(n+1)/n} \frac{1}{t} dt.$$

Since $1/t$ is a strictly decreasing function of t , we know that

$$1 > \frac{1}{t} > \frac{n}{n+1} \quad \text{for all } t \text{ satisfying } 1 < t < \frac{n+1}{n}$$

But then

$$\frac{n}{n+1} \int_1^{(n+1)/n} dt < \int_1^{(n+1)/n} \frac{1}{t} dt < \int_1^{(n+1)/n} dt$$

which evaluates to

$$\frac{n}{n+1} \left(1 + \frac{1}{n} - 1\right) < \ln\left(1 + \frac{1}{n}\right) < 1 + \frac{1}{n} - 1$$

or

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$