MATH 360 – Exam 1

Wednesday, October 4, 2017

Name ____

1. Let $x_n = \frac{1}{n}\left(\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right)^2$. Prove that $\lim_{n\to\infty} x_n$ exists (without finding the limit).

Name ____

2. Let S be the set consisting of all the points $x_n = (-1)^n \left(2 - \frac{4}{2^n}\right)$ for $n = 1, 2, 3, \ldots$. Find $\inf S$ and $\sup S$ (with proof, meaning show that the number you claim is $\sup S$ is actually an upper bound for S and that there is no smaller number that is an upper bound; likewise for $\inf$).

Name ____

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leqslant (x - y)^2$$

for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Prove that f is constant. (*Hint*: What must $f'(x)$ be?)

Name ____

4. Assume that f is twice differentiable on (a, b) and that

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that $f''(x) \geqslant 0$ for all $x \in (a, b)$.

Name ____________________

5. Let f be a function which is defined and continuous for all $x \in \mathbb{R}$, such that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = Cx$ where $C = f(1)$. (*Hint*: First think about $f(x)$ for $x = p/q \in \mathbb{Q}$.)

Name ____________________

6. Let $\{x_n\}$ be a sequence with the property that for every $\varepsilon > 0$ there is an N such that if m and n are both greater than N, then

$$|x_n - x_m| < \varepsilon.$$

(Such a sequence is called a *Cauchy sequence*.)

(a) Prove that if $\{x_n\}$ is a convergent sequence, then it is a Cauchy sequence.

(b) Prove that if $\{x_n\}$ is a Cauchy sequence then it is bounded (i.e., there exists a number M such that $|x_n| < M$ for all n).

(c) Now prove that the Cauchy sequence $\{x_n\}$ has a convergent subsequence.

(d) Let L be the limit of the subsequence whose existence you proved in part (c). Now prove that the entire sequence converges to L, i.e., that $\lim_{n\to\infty} x_n = L$.