

MATH 360 – Additional problems

“Due” Monday, November 6, 2017

1. Prove that the composition of uniformly continuous functions is uniformly continuous.

2. Suppose $f(x)$ is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow \infty$. Let $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow \infty$.

3. Suppose $f'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon$$

whenever $|y - x| < \delta$ and $x, y \in [a, b]$. (You could say that f is *uniformly differentiable* on $[a, b]$).

4. Let $a, b \in \mathbb{R}$, with $b > 0$ and define $f(x)$ on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Prove the following assertions:

- (a) f is continuous if and only if $a > 0$.
 - (b) $f'(0)$ exists if and only if $a > 1$.
 - (c) f' is bounded if and only if $a \geq 1 + b$.
 - (d) f' is continuous if and only if $a > 1 + b$.
 - (e) $f''(0)$ exists if and only if $a \geq 2 + b$.
 - (f) f'' is bounded if and only if $a \geq 2 + 2b$.
 - (g) f'' is continuous if and only if $a > 2 + 2b$.
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5. Suppose f is a twice-differentiable function on (a, ∞) , and that M_0 , M_1 and M_2 are the suprema of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$, respectively, on (a, ∞) . Prove that

$$M_1^2 \leq 4M_0M_2$$

(*Hint*: Consider the first order Taylor polynomial of f , with remainder, centered at x and evaluated at $x + 2h$.) Can you find a function for which equality is achieved?

6. (a) Show that you can obtain Simpson's rule, including the error formula, by integrating the polynomial that interpolates f (from practice problem 7) at $x - h$, x and $x + h$.

(b) Do the corresponding thing for the trapezoidal rule.

(c) Now discover a new rule by using the polynomial that interpolates f at four equally-spaced points.