MATH 360 – Additional problems

"Due" Monday, November 6, 2017

1. Prove that the composition of uniformly continuous functions is uniformly continuous.

Let f be uniformly continuous on its domain, and g be uniformly continuous on the range of f. Given $\varepsilon > 0$, there is a $\eta > 0$ such that if $|p - q| < \eta$ then $|g(p) - g(q)| < \varepsilon$ (independent of p and q). But then given $\eta > 0$ there is a $\delta > 0$ such that if $|x - y| < \delta$ the $|f(x) - f(y)| < \eta$. But then

$$|g(f(x)) - g(f(y))| < \varepsilon$$

independent of x and y, so $g \circ f$ s uniformly continuous.

2. Suppose $f(x)$ is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

The definition of $\lim_{x\to\infty} f'(x) = 0$ is: Given $\varepsilon > 0$ there is an M such that $|f'(x)| < \varepsilon$ for all $x > M$. Now, given ε, choose the x as in the preceding sentence. If $x > M$, then

$$|g(x)| = |f(x+1) - f(x)| = |f'(\xi)| < \varepsilon$$

because $x < \xi < x + 1$ (so in particular $\xi > M$) by the mean-value theorem. Thus $g(x) \to 0$ as $x \to \infty$.

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3. Suppose $f'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \varepsilon$$

whenever $|y - x| < \delta$ and $x, y \in [a, b]$. (You could say that f is *uniformly differentiable* on $[a, b]$).

This is a bit like the last problem. Since $f'(x)$ is continuous on (the closed, bounded interval) $[a, b]$, it is uniformly continuous there. Therefore there is a $\delta > 0$ such that if $x, y \in [a, b]$ and $|y - x| < \delta$, then $|f'(y) - f'(x)| < \varepsilon$. Now, if $x, y \in [a, b]$ and $|y - x| < \delta$, then

$$\frac{f(y) - f(x)}{y - x} = f'(\xi),$$

with ξ between x and y, so in particular $|\xi - x| < \delta$. Therefore $|f'(\xi) - f'(x)| < \varepsilon$ and we have:

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| = |f'(\xi) - f'(x)| < \varepsilon.$$

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4. Let $a, b \in \mathbb{R}$, with $b > 0$ and define $f(x)$ on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

Prove the following assertions:

(a) f is continuous if and only if $a > 0$.

Since $b > 0$ the $\sin(x^{-b})$ factor oscillates infinitely often between -1 and 1 as $x \to 0$. So f willl be continuous at $x = 0$ if and only if $x^a \to 0$ as $x \to 0$, that is, if and only if $a > 0$.

(b) $f'(0)$ exists if and only if $a > 1$.

The limit that defines $f'(0)$ is

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^a \sin(h^{-b})}{h}$$

since $f(0) = 0$. So now we need $h^{a-1} \to 0$ as $x \to 0$, i,e, $a - 1 > 0$ or $a > 1$.

(c) f' is bounded if and only if $a \geqslant 1 + b$.

For $x \neq 0$ we have

$$f'(x) = ax^{a-1} \sin(x^{-b}) - bx^{a-b-1} \cos(x^{-b})$$

The first term is bounded if and only if $a \geqslant 1$, and the second term if and only if $a - b - 1 \geqslant 0$. Since $b > 0$ the second of these is a stronger condition, so we need $a \geqslant b + 1$.

(d) f' is continuous if and only if $a > 1 + b$.

In f', we'll have that same infinite oscillation as $x \to 0$ unless x^{a-1} and x^{a-b-1} both go to zero as $x \to 0$, i,e, $a > b + 1$.

(e) $f''(0)$ exists if and only if $a > 2 + b$.

If $f''(0)$ exists, then f' must be continuous, so by (d) we already have $a > 1 + b$, which also shows that $f'(0) = 0$. Then the limit that defines the second derivative is

$$\lim_{h\to 0} \frac{f'(h)}{h} = \lim_{h\to 0} \frac{ah^{a-1} \sin(h^{-b}) - bh^{a-b-1} \cos(h^{-b})}{h} = \lim_{h\to 0} ah^{a-2} \sin(h^{-b}) - bh^{a-b-2} \cos(h^{-b})$$

which will exist (and be zero) if and only if $a > b + 2$.

(f) f'' is bounded if and only if $a \geqslant 2 + 2b$.

For $x \neq 0$ we have

$$f''(x) = a(a-1)x^{a-2} \sin(x^{-b}) - abx^{a-b-2} \cos(x^{-b}) - b(a-b-1)x^{a-b-2} \cos(x^{-b}) - b^2 x^{a-2b-2} \sin(x^{-b})$$

The smallest exponent here is $a-2b-2$, and we need it to be non-negative for f'' to remain bounded, i.e., $a \geqslant 2b + 2$.

(g) f'' is continuous if and only if $a > 2 + 2b$.

For f'' continuous, since $f''(0) = 0$ from (e), we need all the exponents in the second derivative expression in (e) to be positive, in particular $a > 2 + 2b$.

5. Suppose f is a twice-differentiable function on (a, ∞), and that M_0, M_1 and M_2 are the suprema of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leqslant 4M_0M_2$$

(*Hint*: Consider the first order Taylor polynomial of f, with remainder, centered at x and evaluated at $x + 2h$.) Can you find a function for which equality is achieved?

For $h > 0$, write $f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(\xi)h^2$, where $x < \xi < x + h$. Solve this for $f'(x)$ and get

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(\xi)h}{2}$$

Therefore

$$|f'(x)| \leqslant (|f(x+h)| + |f(x)|)\frac{1}{h} + \frac{|f''(\xi)}{2}h \leqslant \frac{2M_0}{h} + \frac{M_2}{2}h$$

since $h > 0$ and using the triangle inequality and the definitions of M_0 and M_2. Since this is true for all x, we can conclude that

$$M_1 \leqslant \frac{2M_0}{h} + \frac{M_2}{2}h$$

Multiply by h and move everything to the right to get

$$0 \leqslant \frac{1}{2}M_2h^2 - M_1h + 2M_0.$$

Since this quadratic polynomial in h is always positive (we've only demonstrated this for $h > 0$, but because M_0, M_1 and M_2 are positive the quadratic cannot have a negative root), its discriminant must be non-positive, so

$$M_1^2 - 4M_0M_2 \leqslant 0.$$

6. (a) Show that you can obtain Simpson's rule, including the error formula, by integrating the polynomial that interpolates f (from practice problem 7) at $x - h$, x and $x + h$.

$\square$

(b) Do the corresponding thing for the trapezoidal rule.

For the trapezoidal rule, we want the linear polynomial that interpolates $(c, f(c))$ and $(d, f(d))$ to approximate the integral

$$\int_c^d f(x)\,dx$$

But after the change of variables $x = c + \frac{d-c}{h}t$, the integral becomes

$$\int_0^h g(t)\,dt$$

where $g(t) = \frac{d-c}{h} f\left(c + \frac{d-c}{h}t\right)$ so we can instead work with g and interpolate it at 0 and h, so

$$p(t) = g(0) + \frac{g(h) - g(0)}{h}t$$

and from the other problem, we know that the difference between g and p is less that $M_2 t(h-t)/2$ for all $t \in [0, h]$, where M_2 is the maximum value of the second derivative of g on the interval. The integral of p is

$$\int_0^h p(t)\,dt = \int_0^h g(0) + \frac{g(h)-g(0)}{h}t\,dt = g(0)h + \frac{g(h)-g(0)}{h}\frac{h^2}{2} = \frac{h}{2}(g(0) + g(h))$$

which is the trapezoidal rule, and the integral of the error is

$$\int_0^h \frac{M_2 t(h-t)}{2}\,dt = \frac{M_2}{2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{M_2 h^3}{12}$$

which is the error term.

(c) Now discover a new rule by using the polynomial that interpolates f at four equally-spaced points.

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