MATH 360 –Homework 9

Due Monday, November 20, 2017

To discuss in recitation:

1. Textbook page 240, problems 2,3.

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2. Textbook page 240, problems 4,5.

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3. Textbook page 244, problems 1, 3.

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4. Textbook page 248, problems 1, 3, 5.

To be handed in on November 20:

1. Textbook page 240, problems 7, 8.

Problem 7: Since $|a_k|^{1/k} < r$ (for all $k > N$), it is true that $|a_k| < r^k$, therefore, for all $n > N$ we have that s_n, the partial sum of the series $\sum_{k=1}^{\infty} |a_k|$, satisfies

$$s_n = \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{n} |a_k| < \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{n} r^k < \sum_{k=1}^{N} |a_k| + \frac{1}{1-r} < \infty$$

since the sum of r^k from $N+1$ to n is part of a convergent geometric series. But then the s_n are bounded so the series of $|a_k|$'s converges, and hence $\sum a_k$ converges absolutely.

Problem 8: Suppose $\sum b_k$ converges. Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \ell > 0,$$

we have that there exits a $K > 0$ such that $a_k/b_k < 2\ell$ for $k > K$. But then $a_k < 2\ell b_k$, so the partial sums s_n of the a_k series are bounded by

$$s_n < \sum_{k=1}^{K} a_k + \sum_{k=K+1}^{n} 2\ell b_k.$$

Therefore

$$s_n < \sum_{k=1}^{K} a_k + 2\ell \sum_{k=1}^{\infty} b_k$$

which is finite, so the a_k series converges as well.

If we suppose $\sum a_k$ converges, then note that

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \frac{1}{\ell},$$

so we can run the proof in the last paragraph again, switching a for b and $1/\ell$ for ℓ.

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2. Textbook page 244, problems 2, 4.

Problem 2: For $x \geqslant 2$ (in fact for $x > 1$) we have that $x^n \to \infty$ as $n \to \infty$, therefore

$$\lim_{n\to\infty} \frac{1}{1+x^n} = 0$$

for all of these values of x. Therefore $f_n(x) \to 0$ for $x \in [2, \infty)$.

Problem 4: We have $f_n(x) \to 0$ on $[0, 1]$, but the reason is not the same for every x. For $x = 0$, we have $f_n(0) = 0$ for all n. For $x > 0$, we have

$$f_n(x) = \frac{x}{nx+1} < \frac{x}{nx} = \frac{1}{n}$$

so $f_n(x) \to 0$ as $n \to \infty$ for these values of n as well.

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3. Textbook page 248, problems 2, 4, 7.

Problem 2: In problem 2 page 244 we showed that $f_n \to 0$ pointwise. To prove uniform convergence, first observe that for $n > 0$ the function $1 + x^2$ is increasing on the interval $[2, \infty)$, so the function $f_n(x)$ is decreasing (and $\lim_{x\to\infty} f_n(x) = 0$). Therefore, for all $n > 0$ and $x \in [2, \infty)$,

$$\frac{1}{1+2^n} = f_n(2) \geqslant f_n(x) > 0.$$

If we choose N so that $1/(1 + 2^N) < \varepsilon$, then $|f_n(x)| < \varepsilon$ for all $n > N$ and $x \in [2, \infty)$. So $f_n \to 0$ uniformly.

Problem 4: In problem 4 on page 244 we showed that $f_n \to 0$ pointwise. To prove uniform convergence, first observe that the function $x/(nx+1)$ is increasing on the interval $[0,1]$ for all $n \geqslant 0$ (its derivative is $1/(nx+1)^2 > 0$). The function is also non-negative on $[0,1]$, so we have $0 \leqslant f_n(x) < f_n(1) = 1/(n+1)$ for all $x \in [0,1]$ and $n \geqslant 0$. This is enough to prove uniform convergence.

Problem 7:
Say $f_n(x) < M_n$ for all $x \in \mathbb{R}$. Since $f_n \to f$ uniformly, it must be the case that for any $\varepsilon > 0$¡ say $\varepsilon = 1$, there is an N such that $|f_n(x) - f(x)| < 1$ for all $x \in \mathbb{R}$ and all $n \geqslant N$. But then

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leqslant |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N$$

for all x, so $f(x)$ is bounded.

4. Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$$

converges uniformly on $[0,C]$ where C is any number with $0 < C < 1$. Show that the convergence is *not* uniform on $[0,1)$.

If $0 \leqslant x \leqslant C$ then $0 \leqslant x^n \leqslant C^n$ and $1 + x^n \geqslant 1$, therefore

$$0 \leqslant \frac{x^n}{1+x^n} \leqslant C^n$$

And if $0 < C < 1$, then $\sum_{n=1}^{\infty} C^n$ converges, so that for any $\varepsilon > 0$ there is an N such that

$$c_{m+1} + c_{m+2} + \cdots + c_n < \varepsilon$$

provided $n > m > N$. Thus

$$\sum_{k=m+1}^{n} \frac{x^k}{1+x^k} < \varepsilon \quad \text{if } n > m > N.$$

Therefore the series is uniformly Cauchy and so converges uniformly on $[0,C]$.

If we consider the entire interval $[0,1)$, then the convergence is not uniform because for any fixed N, the Nth term of the series will not be small if x is too close to 1. To see this, choose $\varepsilon = \frac{1}{4}$ then because

$$\frac{x^k}{1+x^k} > \frac{x^k}{2}$$

for $x > 0$, given any $N > 0$ we can choose x so that $\frac{1}{2^{1/(2N)}} < x < 1$ so that

$$\frac{x^N}{1+x^N} > \frac{x^N}{2} > \frac{1}{2^{1+1/N}} > \frac{1}{4} = \varepsilon$$

so the series does not converge uniformly on $[0,1)$.

(Another argument would be to notice that since the given series is larger than the geometric series $\sum x^n$, and since the sum of the geometric series $1/(1-x)$ goes to ∞ as $x \to 1^-$, the given series does this as well — but each term of the given series is less than 1. This means that as x gets closer and closer to 1, more and more terms have to be relatively large (bigger than ε for relatively small ε).

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5. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

Show that the series converges uniformly for $x \geqslant C > 0$. Determine all points x where f is defined, and also where f is continuous.

For $x \geqslant C > 0$, the nth term of the series is less than $1/(Cn^2)$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{Cn^2}$ converges (by the integral test), given $\varepsilon > 0$ there is an N such that for $n > m > N$ we have

$$\sum_{k=m+1}^{n} \frac{1}{1+k^2x} < \sum_{k=m+1}^{n} \frac{1}{Ck^2} < \varepsilon.$$

Since N is chosen independently from x, the series for $f(x)$ is uniformly convergent on the interval $[C, \infty)$

In fact, the series will converge (by comparison with a multiple of the series $\sum 1/n^2$) uniformly on any closed interval that does not contain 0 or a number of the form $-1/n^2$ (where the nth term of the series will fail to exist). Therefore $f(x)$ is continuous on the complement of the set $\{0, -1, -1/4, -1/9, \ldots\}$.