MATH 360 –Homework 2

Due Monday, September 18, 2017

To discuss in recitation (the first two are from Homework 1):

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

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2. Use Theorem 8 to prove: If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) \geqslant C \geqslant f(b)$, then there exists $c \in [a,b]$ such that $f(c) = C$.

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3. (a) Give an "ε-N" definition of what it means for a sequence ***not*** to converge. Use your definition to show that $x_n = (-1)^n$ is not convergent.

(b) Give an "ε-δ" definition of what it means for a function ***not*** to be continuous at $x = a$. Use your definition to show that

$$f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is not continuous at $x = 0$.

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4. Suppose the function $f\colon (0,1) \to \mathbb{R}$ is continuous and satisfies $0 < f(x) < x$. Define the sequence of functions $f_n\colon (0,1) \to \mathbb{R}$ by

$$f_1(x) = f(x), \qquad f_n(x) = f(f_{n-1}(x)) \quad \text{for } n \geqslant 2.$$

Prove that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in (0,1)$.

To be handed in on September 18 (the first three are from Homework 1):

1. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

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2. (a) Suppose that $g\colon [0,1] \to [0,1]$ is a continuous function. Show that there exists a $c \in [0,1]$ with $g(c) = c$ (i.e., every continuous map of $[0,1]$ to itself has a "fixed point"). (*Hint*: consider $f(x) = g(x) - x$)

(b) Give an example of a bijective (one-to-one and onto) function $h\colon (0,1) \to (0,1)$ such that $h(x) \neq x$ for all $x \in (0,1)$.

This exercise shows that there is an essential difference between open and closed intervals.

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3. (a) Suppose $g\colon (A,B) \to \mathbb{R}$ is a differentiable function such that $g'(x) \geqslant 0$ for all $x \in (A,B)$. For $a, b \in (A,B)$ with $b > a$, show that $g(b) - g(a) \geqslant 0$.

(b) Prove the converse of the result it part (a) (you might want to prove that $g'(x) \geqslant -\varepsilon$ for all $\varepsilon > 0$).

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4. Define the function $f\colon [0,1] \to [0,1]$ via

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 1 - x & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

For each of the following statements, prove the ones that are true and show that the others are false.

1. $f(f(x)) = x$ for all $x \in [0,1]$.
2. $f(x) + f(1-x) = 1$ for all $x \in [0,1]$.
3. f is bijective (one-to-one and onto).
4. f is everywhere discontinuous on $[0,1]$.
5. $f(x+y) - f(x) - f(y)$ is rational for all $x, y \in [0,1]$ such that $x + y \in [0,1]$.

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5. Suppose $\{x_n\}$ is a bounded sequence that satisfies

$$x_n \leqslant \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geqslant 2$. Show that the sequence converges (*Hint*: Perhaps first show that the sequence $x_n - x_{n-1}$ is increasing).

What if instead you have

$$x_n \geqslant \frac{x_{n-1} + x_{n+1}}{2}$$

for all $n \geqslant 2$?
